Revisiting Chromatic Polynomials of Some Sequences of Graphs

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Abstract

The chromatic polynomial of a graph Γ, $C(\Gamma;\lambda)$, is the polynomial in λ which counts the number of distinct proper vertex λ-colorings of Γ, given λ colors. By applying the addition-contraction method, chromatic polynomials of some sequences of 2-connected graphs satisfy a number of recursive relations. We will show that by knowing chromatic polynomial of a few small graphs, chromatic polynomial of each of these sequences can be computed by utilizing either matrices or generating functions.

1 Introduction

Much information about the chromatic polynomials can be found in [1] and [2]. In [3], the authors compute the chromatic polynomial of some strip graphs with their asymptotic limits. We feel that although our results may be similar to some of the results in [3], they have been obtained independently and differently. Let's assume that $\{\Psi_n\}_{n\in\mathbf{N}}$, $\{\Psi'_n\}_{n\in\mathbf{N}}$, and $\{\Psi''_n\}_{n\in\mathbf{N}}$ are sequences of graphs such that for $n \geq 2$, the following relations hold

$$C(\Psi_n;\lambda) = \frac{C(\Psi'_{n-1};\lambda)C(\Omega_1;\lambda)}{\lambda(\lambda-1)} + \frac{C(\Psi''_{n-1};\lambda)C(\Omega_2;\lambda)}{\lambda},$$

$$C(\Psi'_n;\lambda) = \frac{C(\Psi'_{n-1};\lambda)\,C(\Omega_3;\lambda)}{\lambda(\lambda-1)} + \frac{C(\Psi''_{n-1};\lambda)C(\Omega_4;\lambda)}{\lambda},$$

$$C(\Psi''_n;\lambda) = \frac{C(\Psi'_{n-1};\lambda)\,C(\Omega_5;\lambda)}{\lambda(\lambda-1)} + \frac{C(\Psi''_{n-1};\lambda)C(\Omega_6;\lambda)}{\lambda},$$

in which Ω_i, for $1 \leq i \leq 6$, are given graphs. Clearly, the above relations can be rewritten in matrix form as

$$C(\Psi_n;\lambda) = \begin{bmatrix} \frac{C(\Omega_1;\lambda)}{\lambda(\lambda-1)} & \frac{C(\Omega_2;\lambda)}{\lambda} \end{bmatrix} \begin{bmatrix} C(\Psi'_{n-1};\lambda) \\ C(\Psi''_{n-1};\lambda) \end{bmatrix},$$

$$\begin{bmatrix} C(\Psi'_n;\lambda) \\ C(\Psi''_n;\lambda) \end{bmatrix} = \begin{bmatrix} \frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)} & \frac{C(\Omega_4;\lambda)}{\lambda} \\ \frac{C(\Omega_5;\lambda)}{\lambda(\lambda-1)} & \frac{C(\Omega_6;\lambda)}{\lambda} \end{bmatrix} \begin{bmatrix} C(\Psi'_{n-1};\lambda) \\ C(\Psi''_{n-1};\lambda) \end{bmatrix}.$$

One may notice that the second matrix relation can also be rewritten as

$$\begin{bmatrix} C(\Psi'_n;\lambda) \\ C(\Psi''_n;\lambda) \end{bmatrix} = \begin{bmatrix} \frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)} & \frac{C(\Omega_4;\lambda)}{\lambda} \\ \frac{C(\Omega_5;\lambda)}{\lambda(\lambda-1)} & \frac{C(\Omega_6;\lambda)}{\lambda} \end{bmatrix}^{n-1} \begin{bmatrix} C(\Psi'_1;\lambda) \\ C(\Psi''_1;\lambda) \end{bmatrix};$$

Thus, for $n \geq 2$, we have

$$C(\Psi_n;\lambda) = \begin{bmatrix} \frac{C(\Omega_1;\lambda)}{\lambda(\lambda-1)} & \frac{C(\Omega_2;\lambda)}{\lambda} \end{bmatrix} \begin{bmatrix} \frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)} & \frac{C(\Omega_4;\lambda)}{\lambda} \\ \frac{C(\Omega_5;\lambda)}{\lambda(\lambda-1)} & \frac{C(\Omega_6;\lambda)}{\lambda} \end{bmatrix}^{n-2} \begin{bmatrix} C(\Psi'_1;\lambda) \\ C(\Psi''_1;\lambda) \end{bmatrix}.$$

These observations show us that by computing the chromatic polynomials of Ψ'_1, Ψ''_1 and Ω_i's, chromatic polynomials of Ψ_n can be computed for $n \geq 2$. In section 2, we will give three examples of $\{\Psi_n\}_{n\in\mathbf{N}}$ whose chromatic polynomials will be computed by defining appropriate $\{\Psi'_n\}_{n\in\mathbf{N}}$ and $\{\Psi''_n\}_{n\in\mathbf{N}}$ and applying the above method.

An alternative method is to find generating functions for chromatic polynomial of each sequence. Let $\mathfrak{f}_1(x)$, $\mathfrak{f}_2(x)$, and $\mathfrak{f}_3(x)$ denote the generating functions for chromatic polynomials of Ψ_n, Ψ'_n, and Ψ''_n, respectively. To be more precise, we have $\mathfrak{f}_1(x) = \sum_{n=1}^{\infty} C(\Psi_n;\lambda)x^n$, $\mathfrak{f}_2(x) = \sum_{n=1}^{\infty} C(\Psi'_n;\lambda)x^n$, and $\mathfrak{f}_3(x) = \sum_{n=1}^{\infty} C(\Psi''_n;\lambda)x^n$. By applying the so-called "snake oil" method to the recursive relations we had for $C(\Psi_n;\lambda)$, $C(\Psi'_n;\lambda)$, and $C(\Psi''_n;\lambda)$, these generating functions can be computed as follows:

$$\mathfrak{f}_1(x) = \left(C(\Psi_1;\lambda) + \frac{C(\Omega_1;\lambda)}{\lambda(\lambda-1)}\mathfrak{f}_2(x) + \frac{C(\Omega_2;\lambda)}{\lambda}\mathfrak{f}_3(x)\right)x,$$

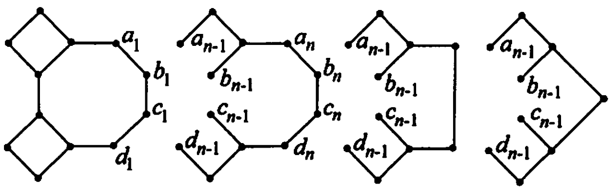
$$\mathfrak{f}_2(x) = \frac{C(\Psi'_1;\lambda)\,x + \left(\frac{C(\Omega_4;\lambda)}{\lambda}C(\Psi''_1;\lambda) - \frac{C(\Omega_6;\lambda)}{\lambda}C(\Psi'_1;\lambda)\right)x^2}{1 - \left(\frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)} + \frac{C(\Omega_6;\lambda)}{\lambda}\right)x + \left(\frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)}\frac{C(\Omega_6;\lambda)}{\lambda} - \frac{C(\Omega_4;\lambda)}{\lambda}\frac{C(\Omega_5;\lambda)}{\lambda(\lambda-1)}\right)x^2},$$

$$\mathfrak{f}_3(x) = \frac{C(\Psi''_1;\lambda)\,x + \left(\frac{C(\Omega_5;\lambda)}{\lambda(\lambda-1)}C(\Psi'_1;\lambda) - \frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)}C(\Psi''_1;\lambda)\right)x^2}{1 - \left(\frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)} + \frac{C(\Omega_6;\lambda)}{\lambda}\right)x + \left(\frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)}\frac{C(\Omega_6;\lambda)}{\lambda} - \frac{C(\Omega_4;\lambda)}{\lambda}\frac{C(\Omega_5;\lambda)}{\lambda(\lambda-1)}\right)x^2}.$$

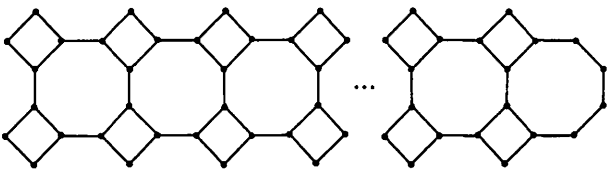
2 Main Results

Assume Ψ_1 is the graph in Figure 2.1(a) and Ψ_n is recursively built from Ψ_{n-1} by identifying the vertices a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1} in Ψ_{n-1} with vertices a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1} of $\tilde{\Psi}_n$, respectively. Ψ_n is depicted in Figure 2.1(b). We know that

$$C(\Psi_1;\lambda) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)^2(\lambda^6-7\lambda^5+21\lambda^4-35\lambda^3+35\lambda^2-21\lambda+7).$$



(a) Ψ_1, $\tilde{\Psi}_n$, $\tilde{\Psi}'_n$, $\tilde{\Psi}''_n$



(b) Ψ_n

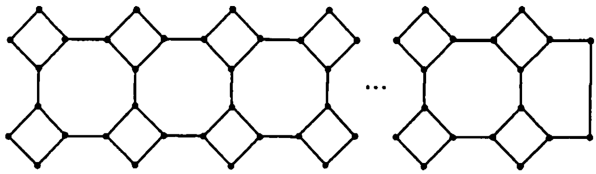
Figure 2.1

In order to find the chromatic polynomial of Ψ_n for $n \geq 2$, we will use the addition-contraction method. By adding the edge between a_{n-1} and d_{n-1} and contracting it, one may notice that that $C(\Psi_n;\lambda)$ is equal to chromatic polynomial of a vertex-gluing of Ψ'_{n-1} and Ω_1 plus chromatic polynomial of an edge-gluing of Ψ''_{n-1} and Ω_2. Clearly, this satisfies the first of the relations we introduced in previous section. Ψ'_n and Ψ''_n are graphs built from Ψ_{n-1} by identifying vertices a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1} with vertices a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1} of $\tilde{\Psi}'_n$ and $\tilde{\Psi}''_n$ (see Figure 2.1(a)), respectively. In Figure 2.2, we have drawn Ψ'_n and Ψ''_n and in Figure 2.3, Ω_1 and Ω_2.

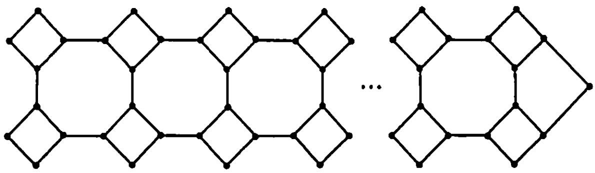
$$\frac{C(\Omega_1;\lambda)}{\lambda(\lambda-1)} = \lambda^{10} - 14\lambda^9 + 91\lambda^8 - 361\lambda^7 + 968\lambda^6 - 1837\lambda^5 +$$

$$2511\lambda^4 - 2465\lambda^3 + 1694\lambda^2 - 759\lambda + 175;$$

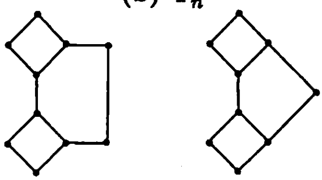
$$\frac{C(\Omega_2;\lambda)}{\lambda} = (\lambda-1)(\lambda-2)^2(\lambda^7-9\lambda^6+37\lambda^5-89\lambda^4+136\lambda^3-134\lambda^2+83\lambda-28).$$



(a) Ψ'_n



(b) Ψ''_n



(c) Ψ'_1, Ψ''_1

Figure 2.2

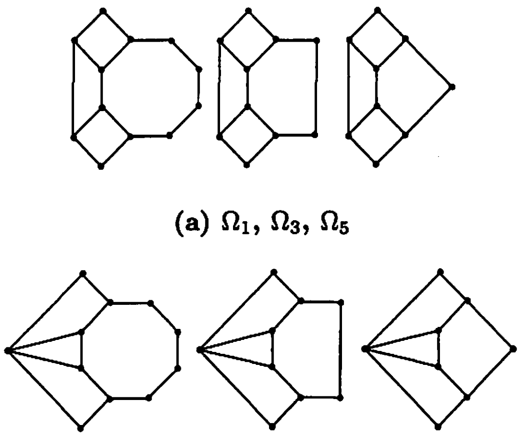
By applying the addition-contraction method to Ψ'_n and Ψ''_n in a similar way we did to Ψ_n, clearly chromatic polynomials of Ψ'_n and Ψ''_n satisfy recursive relations we had in previous section with Ω_2-Ω_6 graphs given in Figure 2.3. Chromatic polynomials of these graphs are as follows:

$$\frac{C(\Omega_3;\lambda)}{\lambda(\lambda-1)} = \lambda^8 - 12\lambda^7 + 66\lambda^6 - 217\lambda^5 + 468\lambda^4 - 683\lambda^3 + 668\lambda^2 - 408\lambda + 121,$$

$$\frac{C(\Omega_4;\lambda)}{\lambda} = (\lambda-1)(\lambda-2)^2(\lambda^5 - 7\lambda^4 + 22\lambda^3 - 38\lambda^2 + 38\lambda - 19),$$

$$\frac{C(\Omega_5;\lambda)}{\lambda(\lambda-1)} = (\lambda-2)(\lambda^6 - 9\lambda^5 + 37\lambda^4 - 88\lambda^3 + 129\lambda^2 - 133\lambda + 47),$$

$$\frac{C(\Omega_6;\lambda)}{\lambda} = (\lambda-1)(\lambda-2)(\lambda^5 - 8\lambda^4 + 28\lambda^3 - 54\lambda^2 + 59\lambda - 29).$$



(a) Ω_1, Ω_3, Ω_5

(b) Ω_2, Ω_4, Ω_6

Figure 2.3

Finally, in order to compute $C(\Psi_n;\lambda)$ using either methods we discussed in previous section, we need the chromatic polynomials of Ψ_1' and Ψ_1'' which are given below.

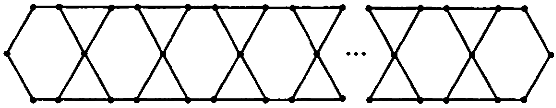
$$C(\Psi_1') = \lambda(\lambda-1)(\lambda^2-3\lambda+3)^2(\lambda^4-5\lambda^3+10\lambda^2-10\lambda+5);$$

$$C(\Psi_1'';\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+3)^2(\lambda^2-3\lambda+2).$$

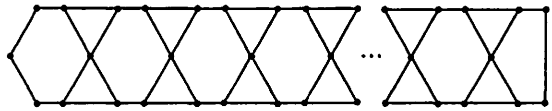
For our next example see Figure 2.4 (a)-(c) (Please note that $\Psi_1 = C_6$, $\Psi_1' = C_5$, and $\Psi_1'' = C_4$). By applying a similar method we used for our previous example, chromatic polynomials of Ψ_n for $n \geq 2$ can be written in terms of those of Ψ_{n-1}', Ψ_{n-1}'', Ω_1, and Ω_2, as stated in previous section. These two graphs can be found in Figure 2.4 (d)-(e). In addition to that, chromatic polynomials of Ψ_n' and Ψ_n'' satisfy the relations in introduction with Ω_3-Ω_6 being graphs in Figure 2.4 (d)-(e).

$$C(\Omega_1;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^5-8\lambda^4+26\lambda^3-44\lambda^2+41\lambda-19);$$

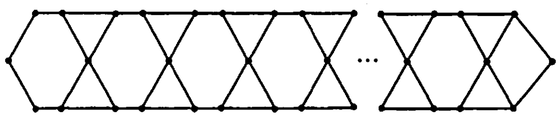
$$C(\Omega_2;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^4-6\lambda^3+14\lambda^2-16\lambda+9);$$
$$C(\Omega_3;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^4-7\lambda^3+19\lambda^2-25\lambda+15);$$
$$C(\Omega_4;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^3-5\lambda^2+9\lambda-7);$$
$$C(\Omega_5;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^3-6\lambda^2+13\lambda-11);$$
$$C(\Omega_6;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+5).$$



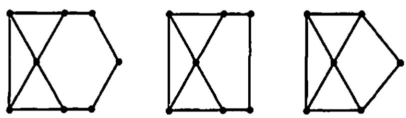
(a) Ψ_n



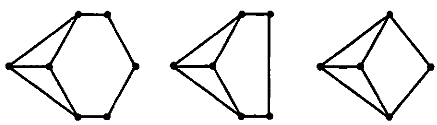
(b) Ψ'_n



(c) Ψ''_n



(d) Ω_1, Ω_3, Ω_5



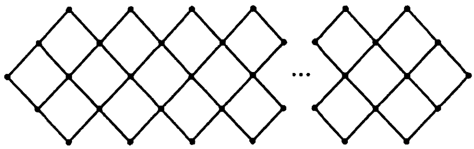
(e) Ω_2, Ω_4, Ω_6

Figure 2.4

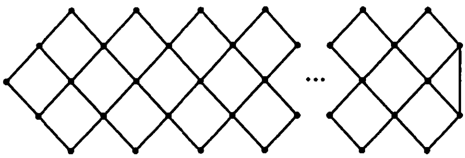
Finally, our next example can be found in Figure 2.5 (a)-(c) (Please note that $\Psi_1 = C_4$, $\Psi'_1 = K_3$, and $\Psi''_1 = K_2$). Similarly, chromatic polynomial of Ψ_n, for $n \geq 2$, can be written in terms of those of Ψ'_{n-1}, Ψ''_{n-1}, Ω_1, and Ω_2 with these two latter graphs drawn in Figure 2.5 (d)-(e). It also can be checked that chromatic polynomials of Ψ'_n and Ψ''_n satisfy recursive relations we had in our introduction with Ω_3-Ω_6 being graphs in Figure 2.5 (d)-(e).

$$C(\Omega_1; \lambda) = \lambda(\lambda-1)(\lambda-2)^2(\lambda^4 - 6\lambda^3 + 16\lambda^2 - 21\lambda + 13);$$

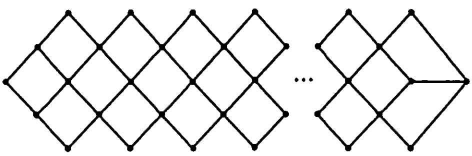
$$C(\Omega_2; \lambda) = \lambda(\lambda - 1)(\lambda^2 - 4\lambda + 5)(\lambda^3 - 4\lambda^2 + 7\lambda - 5).$$



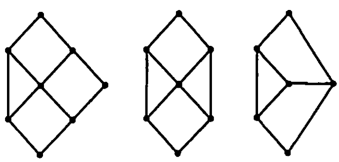
(a) Ψ_n



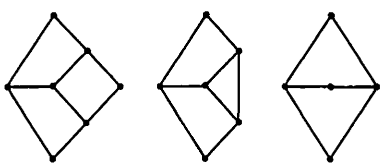
(b) Ψ'_n



(c) Ψ''_n



(d) Ω_1, Ω_3, Ω_5



(e) Ω_2, Ω_4, Ω_6

Figure 2.5

We will finish this paper by computing the chromatic polynomial of Ω_3-Ω_6 as one may see below.

$$C(\Omega_3;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^4 - 7\lambda^3 + 20\lambda^2 - 28\lambda + 17);$$

$$C(\Omega_4;\lambda) = \lambda(\lambda-1)(\lambda-2)^2(\lambda^2 - 3\lambda + 4);$$

$$C(\Omega_5;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 4\lambda + 5)(\lambda^2 - 2\lambda + 2);$$

$$C(\Omega_6;\lambda) = \lambda(\lambda-1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7).$$

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