GLOBAL BEHAVIOR OF A RATIONAL RECURSIVE SEQUENCE

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ABSTRACT

In this paper we study the global behavior of the nonnegative equilibrium points of the difference equation

$$x_{n+1} = \frac{ax_{n-2l}}{b + c\prod_{i=0}^{k+1} x_{n-2i}}, \quad n = 0, 1, ...,$$

where a, b, c are nonnegative parameters, initial conditions are nonnegative real numbers and k, l are nonnegative integers, $l \leq k+1$.

Keywords: Difference Equation, Globally Asymptotically, Periodicity.

1. INTRODUCTION

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, etc. [18].

Recently there has been an increasing interest in the study of nonlinear difference equations. Although difference equations' forms are very simple, it is extremely difficult to understand thorougly the global behaviors of their solutions. There has been a lot of work concerning the global asymptotic behavior of solutions of rational difference equations. For example see Refs. [1-18].

Hamza et al. [11] studied the asymptotic stability of the nonnegative equilibrium point of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + C\prod_{i=l}^{k} x_{n-2i}}.$$

Elabbasy et al. [7] investigated some qualitative behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elsayed [9] investigated the qualitative behavior of the solution of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + dx_{n-1}}.$$

Also Elsayed [10] studied the behavior of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}}.$$

Andruch et al. [2] studied the asymtotic behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_n x_{n-1}}.$$

Cinar [4] investigated the global asymptotic stability of all positive solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Yalcinkaya [16] investigated the global behaviour of the rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

El-Owaidy et al. [8] studied the dynamics of the recurcive sequence

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}.$$

Battaloğlu [3] discussed the global asymptotic behavior and periodicity character of the following difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x_{n-(k+1)}^p},$$

by generalizing the results due to El-Owaidy et al.

Our aim in this paper is to investigate the dynamics of the solution of the difference equation

$$x_{n+1} = \frac{ax_{n-2l}}{b + c \prod_{i=0}^{k+1} x_{n-2i}}, \quad n = 0, 1, ... \tag{1.1}$$

where a, b, c are nonnegative real numbers, initial conditions are nonnegative and l, k are nonnegative integers, $l \leq k+1$.

2. Preliminaries

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-(k+1)}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. *An equilibrium point for Eq.(2.1) is a point $\overline{x} \in I$ such that $\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x})$.*

Definition 2. *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

Definition 3. *(i) The equilibrium point $\overline{x}$ of Eq.(2.1) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-(k-1)} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta$, we have $|x_n - \overline{x}| < \varepsilon$ for all $n \geq -k$.*

(ii) The equilibrium point $\overline{x}$ of Eq.(2.1) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2.1) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-(k-1)} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma$, we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of Eq.(2.1) is global attractor if for all $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$, we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ of Eq.(2.1) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2.1).

(v) The equilibrium point $\overline{x}$ of Eq.(2.1) is unstable if $\overline{x}$ is not locally stable.

The linearized equation associated with Eq.(2.1) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \overline{x}, ..., \overline{x})\, y_{n-i}, \quad n = 0, 1, ... \tag{2.2}$$

The characteristic equation associated with Eq.(2.2) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \overline{x}, ..., \overline{x})\, \lambda^{k-i} = 0. \tag{2.3}$$

Theorem 1. [13]*Assume that f is a C^1 function and let $\overline{x}$ be an equilibrium point of Eq.(2.1). Then the following statements are true.*

(i) If all roots of Eq.(2.3) lie in open disk $|\lambda| < 1$, then $\overline{x}$ is locally asymptotically stable.

(ii) If at least one root of Eq.(2.3) has absolute value greater than one, then $\overline{x}$ is unstable.

3. Dynamics of Eq.(1.1)

In this section, we investigate the dynamics of *Eq.*(1.1) under the assumptions that all parameters are nonnegative with $c \neq 0$, the initial conditions are nonnegative and l, k are nonnegative.

The change of variables $x_n = \sqrt[k+2]{\frac{b}{c}} y_n$ reduces *Eq.*(1.1) to the difference equation

$$y_{n+1} = \frac{\gamma y_{n-2l}}{1 + \prod_{i=0}^{k+1} y_{n-2i}}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where $\gamma = \frac{a}{b}$. We can see that $\overline{y}_1 = 0$ is always an equilibrium point of *Eq.*(3.1). When $\gamma > 1$, *Eq.*(3.1) also possesses the unique positive equilibrium $\overline{y}_2 = \sqrt[k+2]{\gamma - 1}$.

Theorem 2. *The following statements are true:*

(i) If $\gamma < 1$, then the equilibrium point $\overline{y}_1 = 0$ of Eq.(3.1) is locally asymptotically stable,

(ii) If $\gamma > 1$, then the equilibrium points $\overline{y}_1 = 0$ and $\overline{y}_2 = \sqrt[k+2]{\gamma - 1}$ are unstable.

Proof. The linearized equation associated with *Eq.*(3.1) about $\overline{y}$ is

$$z_{n+1} + \frac{\gamma \overline{y}^{k+2}}{\left(1 + \overline{y}^{k+2}\right)^2} \left(\sum_{i=0}^{k+1} z_{n-2i} - z_{n-2l} \right) - \frac{\gamma}{\left(1 + \overline{y}^{k+2}\right)^2} z_{n-2l} = 0, \ n = 0, 1, \ldots.$$

The characteristic equation associated with this equation is

$$\lambda^{2k+3} + \frac{\gamma \overline{y}^{k+2}}{\left(1 + \overline{y}^{k+2}\right)^2} \left(\sum_{i=0}^{k+1} \lambda^{2i} - \lambda^{2k+2-2l} \right) - \frac{\gamma}{\left(1 + \overline{y}^{k+2}\right)^2} \lambda^{2k+2-2l} = 0.$$

Then the linearized equation of *Eq.*(3.1) about the equilibrium point $\overline{y}_1 = 0$ is

$$z_{n+1} - \gamma z_{n-2l} = 0, \quad n = 0, 1, \ldots.$$

The characteristic equation of *Eq.*(3.1) about the equilibrium point $\overline{y}_1 = 0$ is

$$\lambda^{2k+2-2l} \left(\lambda^{2l+1} - \gamma \right) = 0.$$

So

$$\lambda = 0 \text{ and } \lambda = \sqrt[2l+1]{\gamma}.$$

In view of Theorem 1:

If $\gamma < 1$, then $|\lambda| < 1$ for all roots and the equilibrium point $\overline{y}_1 = 0$ is locally asymptotically stable.

If $\gamma > 1$, it follows that the equilibrium point $\overline{y}_1 = 0$ is unstable.

The linearized equation of $Eq.$(3.1) about the equilibrium point $\overline{y}_2 = \sqrt[k+2]{\gamma - 1}$ becomes

$$z_{n+1} + \left(1 - \frac{1}{\gamma}\right)\left(\sum_{i=0}^{k+1} z_{n-2i} - z_{n-2l}\right) - \frac{1}{\gamma} z_{n-2l} = 0, \ n = 0, 1, \ldots.$$

The characteristic equation of $Eq.$(3.1) about the equilibrium point $\overline{y}_2 = \sqrt[k+2]{\gamma - 1}$ is,

$$\lambda^{2k+3} + \left(1 - \frac{1}{\gamma}\right)\left(\sum_{i=0}^{k+1} \lambda^{2i} - \lambda^{2k+2-2l}\right) - \frac{1}{\gamma}\lambda^{2k+2-2l} = 0.$$

It is clear that this equation has a root in the interval $(-\infty, -1)$. Then the equilibrium point $\overline{y}_2 = \sqrt[k+2]{\gamma - 1}$ is unstable.

Theorem 3. *Assume that $\gamma < 1$, then the equilibrium point $\overline{y}_1 = 0$ of $Eq.$(3.1) is globally asymptotically stable.*

□

Proof. Let $\{y_n\}_{n=-(k+1)}^{\infty}$ be a solution of $Eq.$(3.1). From Theorem 2 we know that the equilibrium point $\overline{y}_1 = 0$ of $Eq.$(3.1) is locally asymptotically stable. So it is sufficed to show that

$$\lim_{n\to\infty} y_n = 0.$$

Since

$$y_{n+1} = \frac{\gamma y_{n-2l}}{1 + \prod_{i=0}^{k+1} y_{n-2i}} \leq \gamma y_{n-2l}.$$

We obtain

$$y_{n+1} \leq \gamma y_{n-2l}.$$

Then it can be written for $t = 0, 1, \ldots$ and $p = 1, 2, \ldots, 2l + 1$

$$y_{t(2l+1)+p} \leq \gamma^{t+1} y_{-(2l+1-p)}. \tag{3.2}$$

If $\gamma < 1$, then $\lim_{t\to\infty} \gamma^{(t+1)} = 0$
and

$$\lim_{n\to\infty} y_n = 0.$$

The proof is complete. □

Corollary 1. *Assume that $\gamma = 1$. Then every solution of $Eq.$(3.1) is bounded.*

Proof. Let $\{y_n\}_{n=-(k+1)}^{\infty}$ be a solution of $Eq.(3.1)$.We have from our assumption and inequality (3.2)

$$y_{t(2l+1)+1} \leq y_{-2l},$$

$$y_{t(2l+1)+2} \leq y_{-(2l-1)},$$

$$\ldots$$

$$y_{t(2l+1)+2l+1} \leq y_0.$$

It is obvious that every solution of $Eq.(3.1)$ is bounded from above by $A = \max\left\{y_{-2l}, y_{-(2l-1)}, \ldots, y_0\right\}$. □

Theorem 4. *Assume that at least one of the initial conditions y_{-i} ($i = 0, 1, \ldots, 2l$) of $Eq.(3.1)$ is zero. Then the following statements are true:*

(i) If $\gamma > 1$, then every solution of $Eq.(3.1)$ is unbounded except zero.

(ii) If $\gamma = 1$, then $Eq.(3.1)$ has periodic solutions of period $(2l+1)$.

Proof. Let $\{y_n\}_{n=-(k+1)}^{\infty}$ be a solution of $Eq.(3.1)$.We have from $Eq.(3.1)$

$$y_{t(2l+1)+1} = \frac{\gamma^{t+1} y_{-2l}}{1 + \prod_{i=0}^{k+1} y_{-2i}},$$

$$y_{t(2l+1)+2} = \frac{\gamma^{t+1} y_{-(2l-1)}}{1 + \prod_{i=0}^{k+1} y_{-(2i-1)}},$$

$$\ldots$$

$$y_{t(2l+1)+2l+1} = \frac{\gamma^{t+1} y_0}{1 + \prod_{i=0}^{k+1} y_{-(2i-2l)}}.$$

When at least one of the initial conditions y_{-i} ($i = 0, 1, \ldots, 2l$) is zero and $\gamma > 1$, it follows that every solution of $Eq.(3.1)$ is unbounded except zero from the above equalities. If $\gamma = 1$, then $Eq.(3.1)$ has periodic solutions of period $(2l+1)$.

The proof is complete. □

4. Numerical Results

In this section, we give a few numerical results for some special values of the parameters.

Example 1. *Let* $y_{n+1} = \frac{\gamma y_{n-2l}}{1+\prod_{i=0}^{k+1} y_{n-2i}}$, $n = 0, 1, , ..., 99$ *and* $l = 1, k = 1, \gamma = 0.4, y_{-4} = 2, y_{-3} = 1, y_{-2} = 5, y_{-1} = 6, y_0 = 3$. *Then we have the following results for* $\overline{y}_1 = 0$ *:*

n	y_n	n	y_n
1	0,06451612	55	$4,354798.10^{-9}$
13	0,00162228	67	$1,114828.10^{-10}$
35	0,00006410	85	$4,566337.10^{-13}$
44	0,00000410	100	$4,675929.10^{-15}$

Example 2. *Let* $y_{n+1} = \frac{\gamma y_{n-2l}}{1+\prod_{i=0}^{k+1} y_{n-2i}}$, $n = 0, 1, , ..., 99$ *and* $l = 2, k = 2, \gamma = 1, y_{-6} = 0.2, y_{-5} = 3, y_{-4} = 3, y_{-3} = 4, y_{-2} = 1, y_{-1} = 5, y_0 = 3$. *Then we have the following results for Corollary 1:*

n	y_n	n	y_n
1	1,07142857	60	2,09075356
15	2,63852683	76	0,06252139
33	0,55374906	88	0,52619345
44	0,33493443	100	1,78498919

Example 3. *Let* $y_{n+1} = \frac{\gamma y_{n-2l}}{1+\prod_{i=0}^{k+1} y_{n-2i}}$, $n = 0, 1, , ..., 49$ *and* $l = 1, k = 1, \gamma = 5, y_{-4} = 2, y_{-3} = 1, y_{-2} = 5, y_{-1} = 6, y_0 = 0$. *Then we have the following results for Theorem 4 (i):*

n	y_n	n	y_n
1	25	31	$2,441406.10^{8}$
14	124, 172185	38	$4,850475.10^{7}$
19	$3,90625.10^{5}$	45	0
24	0	50	$3,0315474.10^{10}$

Example 4. *Let* $y_{n+1} = \frac{\gamma y_{n-2l}}{1+\prod_{i=0}^{k+1} y_{n-2i}}$, $n = 0, 1, , ..., 7$ *and* $l = 1, k = 1, \gamma = 1, y_{-4} = 2, y_{-3} = 1, y_{-2} = 5, y_{-1} = 6, y_0 = 0$. *Then we have the following results for Theorem 4 (ii):*

n	y_n	n	y_n
1	5	5	0, 193548387
2	0, 193548387	6	0
3	0	7	5
4	5	8	0, 193548387

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ON QUOTIENT CURVES OF THE SUZUKI CURVE

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ABSTRACT. Inspired by a recent paper by Giulietti, Korchmáros and Torres [3], we provide equations for some quotient curves of the Deligne-Lusztig curve associated to the Suzuki group $Sz(q)$.

1. INTRODUCTION

The Deligne-Lusztig curve of Suzuki type (shortly DLS-curve) is the (projective geometrically irreducible, non-singular) algebraic curve defined to be the non-singular model over the finite field $\mathbf{F}_q$ of the (absolutely irreducible) plane curve $\mathcal{C}$ of equation $X^{q_0}(X^q + X) = Y^q + Y$, where $q_0 = 2^s, s \geq 1$ and $q = 2q_0^2$. Several authors have studied the DLS-curve also in connection with coding theory, see [1], [2], [6], [7], [8], [9]. Here we only mention that the DLS-curve has genus $g = q_0(q-1)$ and that the number of its $\mathbf{F}_q$-rational points is $q^2 + 1$. Actually, the two latter properties characterize the DLS-curve, see [2]. The automorphism group of the DLS-curve is the Suzuki group $\mathcal{S}z(q)$.

In [3] the quotient curves of the DLS-curve arising from the subgroups of $\mathcal{S}z(q)$ are thoroughly investigated. For tame covering, that is for subgroups of odd order, the authors obtain an exhaustive list of such curves. A similar complete list for non-tame coverings cannot be produced because the Suzuki group contains a huge number of pairwise non-isomorphic subgroups of even order.

Our contribution here is to provide such a complete list for the cases $q = 8$ and $q = 32$. For all curves in the list, a plane equation is given as well.

A motivation for the present work comes from the current interest in curves over finite fields with many rational points, see van der Geer's survey [4]. Indeed, the number of $\mathbf{F}_q$-rational points of a curve of genus g which is $\mathbf{F}_q$-covered by the DLS-curve is $N = 1 + q + 2q_0g$ [3, Proposition 3.1] and this value is in the interval from which the entries of the tables of curves with many rational points are taken for $g \leq 50$, $q \leq 128$ in [5]. It should be

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noted that we provide here (see Theorem 3.9) plane equations for quotient curves of the DLS-curve defined over $\mathbf{F}_{32}$ with genus $g \in \{12, 28, 30\}$. By [5] these curves attain the largest value of $\mathbf{F}_q$-rational points for which an $\mathbf{F}_q$-rational curve of genus g is previously known to exist.

2. Preliminary results on the DLS-curve

Throughout the present chapter $\mathcal{X}$ will stand for the DLS-curve over $\mathbf{F}_q$. As we have mentioned in the Introduction $\mathcal{X}$ has genus $q_0(q-1)$ and contains exactly q^2+1 $\mathbf{F}_q$-rational points. The proposition below will be useful in the sequel.

Proposition 2.1. *For any $b \in \mathbf{F}_q$, $b \neq 0$, there are elements $x, y \in \mathbf{F}_q(\mathcal{X})$ such that*

$$\mathbf{F}_q(\mathcal{X}) = \mathbf{F}_q(x,y), \quad x^{2q_0}(x^q+x) = b(y^q+y).$$

Proof. We have $\mathbf{F}_q(\mathcal{X}) = \mathbf{F}_q(x,t)$ with $x^{q_0}(x^q+x) = t^q+t$. Let $y = b^{-1}(x^{2q_0+1}+t^{2q_0})$, that is $t^q = b^{q_0}y_0^q + x^{q+q_0}$. Then $\mathbf{F}_q(\mathcal{X}) = \mathbf{F}_q(x,y)$. Furthermore, $y^{q_0} = b^{-q_0}(x^{q+q_0}+t^q) = b^{-q_0}(x^{q_0+1}+t)$, and hence $y^q = b^{-1}(x^{q+q_0}+t^{2q_0})$. Now, since $y^q+y = b^{-1}(x^{q+q_0}+t^{2q_0}+x^{2q_0+1}+t^{2q_0}) = b^{-1}x^{2q_0}(x^q+x)$, the claim follows. □

Let C_b be the plane curve of equation $X^{2q_0}(X^q+X) = b(Y^q+Y)$. C_b has only one singular point, namely the infinite point Y_∞ of the Y-axis which point is a q_0-fold point. We know from [8] that $\bar{\mathbf{F}}_q(\mathcal{X})$ has just one place centered at Y_∞.

For $a, c, d \in \mathbf{F}_q$ with $d \neq 0$, we define the following automorphisms of $\mathbf{F}_q(\mathcal{X})$:

$$(2.1) \qquad \psi_{a,c} := \begin{cases} x \mapsto x+a, \\ y \mapsto a^{2q_0}x+y+c; \end{cases} \qquad \gamma_d := \begin{cases} x \mapsto dx, \\ y \mapsto d^{2q_0+1}y; \end{cases}$$

for $h := xy + x^{2q_0+2} + y^{2q_0}$,

$$(2.2) \qquad \varphi := \begin{cases} x \mapsto y/h, \\ y \mapsto x/h. \end{cases}$$

The automorphism group of $\bar{\mathbf{F}}_q(\mathcal{X})$ generated by $\psi_{a,c}$, γ_d and φ is the full automorphism group of $\bar{\mathbf{F}}_q(\mathcal{X})$, it is isomorphic to $Sz(q)$ and it acts on the set of places of $\mathcal{X}(\mathbf{F}_q)$ as $Sz(q)$ in its unique 2-transitive permutation representation [3, Proposition 3.5].

By Proposition 2.1 $\bar{\mathbf{F}}_q(\mathcal{X}) = \bar{\mathbf{F}}_q(x,y)$ with $x^{2q_0}(x^q+x) = y^q+y$. The extension $\bar{\mathbf{F}}_q(\mathcal{X})|\bar{\mathbf{F}}_q(x)$ is Galois of degree q, and x has a unique pole in $\mathbf{F}_q(\mathcal{X})$ that we denote by $\mathcal{P}_\infty$. Such a place is totally ramified in $\bar{\mathbf{F}}_q(\mathcal{X})$, while all the other rational places of $\bar{\mathbf{F}}_q(x)$ split completely in $\bar{\mathbf{F}}_q(\mathcal{X})|\bar{\mathbf{F}}_q(x)$. The Galois group of $\bar{\mathbf{F}}_q(\mathcal{X})|\bar{\mathbf{F}}_q(x)$ is $\bar{\mathbf{T}}_0 := \{\psi_{0,c} \mid c \in \mathbf{F}_q\}$. Note that $\bar{\mathbf{T}}_0$ comprises the identity and the elements of order 2 of the Sylow 2-subgroup $\bar{\mathbf{T}} = \{\psi_{a,c} \mid a, c \in \mathbf{F}_q\}$ of $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$.

The stabilizer of $\mathcal{P}_\infty$ in $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$ is the group $\bar{\mathbf{T}}\bar{\mathbf{N}}$, where $\bar{\mathbf{N}} := \{\gamma_d \mid d \in \mathbf{F}_q, d \neq 0\}$, and the normalizer $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{N}})$ is the dihedral group generated by $\bar{\mathbf{N}}$ together with φ. Moreover, $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$ contains two conjugacy classes of subgroups of Singer type, one consisting of cyclic subgroups $\bar{\mathbf{D}}^+$ of order $q+2q_0+1$ and the other of cyclic subgroups $\bar{\mathbf{D}}^-$ of order $q-2q_0+1$. The normalizer $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{D}}^+)$ has order $4(q+2q_0+1)$ and is the semidirect product of $\bar{\mathbf{D}}^+$ by a cyclic group of order 4. All these results hold true for $\bar{\mathbf{D}}^-$.

In some cases, $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$ contains subgroups isomorphic to the Suzuki group over a subfield $\mathbf{F}_{\bar{q}}$ of $\mathbf{F}_q$. This occurs if and only if $\bar{q} = 2^{2\bar{s}+1}$ with a divisor $\bar{s}$ of s such that $2\bar{s}+1$ divides $2s+1$.

Proposition 2.2. *Any subgroup of* $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$ *is conjugate to either a subgroup isomorphic to* $Sz(\bar{q})$, *or to a subgroup of one of the following groups:* $\bar{\mathbf{T}}\bar{\mathbf{N}}$, $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{N}})$, $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{D}}^+)$, $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{D}}^-)$.

For tame covering, that is for subgroups of odd order, an exhaustive list of quotient curves of the DLS-curve of Suzuki type has been recently obtained in [3].

Theorem 2.3. *Let* $\mathcal{X}$ *be a tame quotient curve of the DLS-curve. Then one of the following holds.*

I) *r is any divisor of $q-1$, $\mathcal{X}$ has genus $g = \frac{q-1}{r}q_0$ and is a non-singular model over $\mathbf{F}_q$ of the plane curve of equation*

$$Y^{(q-1)/r}\Big(1+\sum_{i=0}^{s-1} X^{2^i(2q_0+1)-(q_0+1)}(1+X)^{2^i}\Big) = (X^{q_0}+1)(Y^{2(q-1)/r}+X^{q-1}),$$

II) *r is any divisor of $q+2q_0+1$, $\mathcal{X}$ has genus $g = \frac{q_0(q-1)-1}{r}+1$ and is a non-singular model over $\mathbf{F}_{q^4}$ of the plane curve of equation*

$$Y^{(q+2q_0+1)/r}\Big(1+\sum_{i=0}^{s-1} X^{2^i q_0}(1+X)^{2^i(q_0+1)-q_0} + X^{q/2}\Big) =$$
$$X^{q+2q_0+1} + Y^{2(q+2q_0+1)/r},$$

III) *r is any divisor of $q-2q_0+1$, $\mathcal{X}$ has genus $g = \frac{q_0(q-1)+1}{r}-1$ and is a non-singular model over $\mathbf{F}_{q^4}$ of the plane curve of equation*

$$bY^{(q-2q_0+1)/r}\Big(1+\sum_{i=0}^{s-1} X^{2^i(2q_0+1)-(q_0+1)}(1+X)^{2^i}\Big) =$$
$$(X^{q-2q_0+1}+Y^{2(q-2q_0+1)/r})(X^{q_0-1}+X^{2q_0-1}),$$

where $b = \lambda^{q_0}+\lambda^{q_0-1}+\lambda^{-q_0}+\lambda^{-q_0+1}$ and $\lambda \in \mathbf{F}_{q^4}$ is an element of order $q-2q_0+1$.

A similar complete list for non-tame coverings cannot be produced because the Suzuki group contains a huge number of pairwise non-isomorphic subgroups of even order. However, the existence of non-tame quotient curves of the DLS-curve of genus g has been given in Theorem 2.4 (see [3]). For some of these curves also a plane equation has been provided, see Theorem 2.5 ([3]).

Theorem 2.4. *Let v, u, r be positive integers. For the following values of g the DLS-curve has a quotient curve $\mathcal{X}$ of genus g.*

i) $g = 2^{s-u+v}(2^{2s+1-v} - 1)$, $v \le 2s+1$, $v \le u \le v + \log_2(v+1)$,

ii) $g = \frac{1}{r}2^s(2^{2s+1-v} - 1)$, $v \le 2s+1$, $r|(q-1)$, $r|(2^{2s+1-v} - 1)$,

iii) $g = \frac{q_0(q-r-1)}{2r}$, $r|(q-1)$,

iv) $g = \frac{1}{2}\left[\frac{q_0(q-1)-1}{r} - (q_0 - 1)\right]$, $r|(q + 2q_0 + 1)$,

v) $g = \frac{1}{2}\left[\frac{q_0(q-1)+1}{r} - (q_0 + 1)\right]$, $r|(q - 2q_0 + 1)$,

vi) $g = \frac{1}{4}\left[\frac{q_0(q-1)-1}{r} - (q_0 - 1)\right]$, $r|(q + 2q_0 + 1)$,

vii) $g = \frac{1}{4}\left[\frac{q_0(q-1)+1}{r} - (q_0 + 1)\right]$, $r|(q - 2q_0 + 1)$,

viii) $g = \frac{q_0(q-1)-1+(\bar{q}^2+1)\bar{q}^2(\bar{q}-1)+\Delta}{(\bar{q}^2+1)\bar{q}^2(\bar{q}-1)}$, $\bar{q} = 2^{2\bar{s}+1}$, $\bar{s}|s$, $(2\bar{s}+1)|(2s+1)$, $\Delta := (\bar{q}^2+1)[(2q_0+2)(\bar{q}-1) + 2\bar{q}(\bar{q}-1)] + \bar{q}^2(\bar{q}^2+1)(\bar{q}-2) + \bar{q}^2(\bar{q}+2\bar{q}_0+1)(\bar{q}-1)(\bar{q}-2\bar{q}_0)$,

ix) $g = 2^4(2^{9-v} - 1)$, $3 \le v \le 2s+1$, *for* $q = 512$.

Theorem 2.5. i') *For $u = v$, $v|(2s+1)$ a non-singular model over $\mathbf{F}_q$ of the plane curve of equation*

$$X^{2q_0}(X^q + X) = b \sum_{i=0}^{(2s+1/v)-1} Y^{(2^v)^i}$$

is a quotient curve of the DLS-curve of genus g as in i).

ii') *For $u = 2$, $v = 1$ a non-singular model over $\mathbf{F}_q$ of the plane curve of equation*

$$\sum_{i=0}^{2s} X^{2^i} + \sum_{i=0}^{s} X^{2^i}\Big(\sum_{j=i}^{s} X^{2^j}\Big) + \sum_{i=s+1}^{2s} X^{2^i}\Big(\sum_{j=0}^{i-s-2} X^{2^j}\Big)^{2q_0} = \sum_{i=0}^{2s} Y^{2^i}$$

is a quotient curve of the DLS-curve of genus g as in i).

iii') *A non-singular model over $\mathbf{F}_q$ of the plane curve of equation*

$$1 + \sum_{i=0}^{s-1} X^{2^i(2q_0+1)-(q_0+1)}(1+X)^{2^i} =$$

$$\sum (-1)^{i+j}\frac{(i+j-1)!k}{i!j!} Y^i(X^{rj}(X^{q_0}+1)$$

where the summation is extended over all pairs (i,j) of non-negative integers with $i+2j=(q+2q_0+1)/r$, is a quotient curve of the DLS-curve of genus g as in iii).

$$1+\sum_{i=0}^{s-1} X^{2^i q_0}(1+X)^{2^i(q_0+1)-q_0}+X^{q/2}=\sum(-1)^{i+j}\frac{(i+j-1)!}{i!j!}X^{ri}Y^j$$

is a quotient curve of the DLS-curve of genus g as in iv).

v') *Let b be as in III). A non-singular model over $\mathbf{F}_{q^4}$ of the plane curve of equation*

$$b\Big(1+\sum_{i=0}^{s-1} X^{2^i(2q_0+1)-(q_0+1)}(1+X)^{2^i}\Big)=$$

$$(X^{q_0-1}+X^{2q_0-1})\sum(-1)^{i+j}\frac{(i+j-1)!}{i!j!}X^{ri}Y^j$$

where the summation is extended over all pairs (i,j) of non-negative integers with $i+2j=(q+2q_0+1)/r$, is a quotient curve of the DLS-curve of genus g as in v).

In the following sections we are going to investigate the quotient curves of the DLS-curve arising from its automorphism groups of even order. In particular, we provide an equation for the cases not covered by the above theorems, for $q=8$ and $q=32$.

3. Quotient curves arising from 2-subgroups

Throughout this section the following notation will be used:

- $\mathcal{U}$ is a subgroup of $\bar{\mathbf{T}}$;
- $\mathcal{U}_2$ is the subgroup of $\mathcal{U}$ consisting of all elements of order 2 together with the identity;
- $\Phi:\bar{\mathbf{T}}\to\mathbf{F}_q$ is the map given by $\Phi(\psi_{a,c})=a$;
- $\mathcal{X}_{\mathcal{U}}$ is the quotient curve of $\mathcal{X}$ arising from $\mathcal{U}$;
- $g_{\mathcal{U}}$ is the genus of $\mathcal{X}_{\mathcal{U}}$.

Proposition 3.1. [3, Proposition 7.1] *Let $\mathcal{U}$ have order 2^u. If $\mathcal{U}_2$ has order 2^v, then*

$$g_{\mathcal{U}}=2^{s-u+v}(2^{2s+1-v}-1).$$

The map Φ is a homomorphism from $\bar{\mathbf{T}}$ onto the additive subgroup of $\mathbf{F}_q$. The restriction of Φ to $\mathcal{U}$ is the homomorphism $\Phi_{|\mathcal{U}}$ with kernel $\mathrm{Ker}(\Phi_{|\mathcal{U}})=\{\psi_{0,c}|c\in\mathbf{F}_q\}$ isomorphic to $\mathcal{U}_2$.

Lemma 3.2. *If $\mathcal{U}=\mathcal{U}_2$, then the fixed field of $\mathcal{U}$ is generated by x and $F(y)$, where $F(T)=\prod_{\psi_{0,c}\in\mathcal{U}_2}(T+c)$.*

Proof. It is straightforward to check that both x and $F(y)$ are fixed by $\mathcal{U}$. On the other hand, $\bar{\mathbf{F}}_q(x, F(y))$ cannot be a proper subfield of the fixed field of $\mathcal{U}$. In fact, the degree of the extension $\bar{\mathbf{F}}_q(x,y) \mid \bar{\mathbf{F}}_q(x, F(y))$ is less than or equal to $\#\mathcal{U}$, as $\deg F = \#\mathcal{U}$. $\square$

Lemma 3.3. *Let $\mathcal{U}$ have order $2\#\mathcal{U}_2$, and assume that $\psi_{a,0} \in \mathcal{U}$, $a \in \mathbf{F}_q \setminus \{0\}$. Then the fixed field of $\mathcal{U}$ is generated by $x(x+a)$ and $F(y) + a^{-1}xF(a^{2q_0}x)$, where $F(T) = \prod_{\psi_{0,c} \in \mathcal{U}_2}(T+c)$.*

Proof. The group $\mathcal{U}$ is generated by $\mathcal{U}_2$ and $\psi_{a,0}$. As the set $\{c \in \mathbf{F}_q \mid \psi_{0,c} \in \mathcal{U}_2\}$ is a linear subspace of $\mathbf{F}_q \mid \mathbf{F}_2$, the polynomial $F(T)$ is such that $F(T_1 + T_2) = F(T_1) + F(T_2)$ ([11]). Also, $\psi^2_{a,0} = \psi_{0,a^{2q_0+1}} \in \mathcal{U}_2$ implies $F(a^{2q_0+1}) = 0$. Then it is straightforward to check that $x(x+a)$ and $F(y) + a^{-1}xF(a^{2q_0}x)$ are fixed by $\mathcal{U}_2$ and $\psi_{a,0}$. On the other hand, $\bar{\mathbf{F}}_q(x(x+a), F(y) + a^{-1}xF(a^{2q_0}x))$ cannot be a proper subfield of the fixed field of $\mathcal{U}$. In fact, by the proof of Lemma 3.2 the degree of the extension $\bar{\mathbf{F}}_q(\mathcal{X}) \mid \bar{\mathbf{F}}_q(x, F(y))$ is equal to $\#\mathcal{U}_2$. As the degree of $\bar{\mathbf{F}}_q(x, F(y)) \mid \bar{\mathbf{F}}_q(x(x+a), F(y) + a^{-1}xF(a^{2q_0}x))$ is at most 2, the claim follows. $\square$

Lemma 3.4. *Let $\mathcal{U}$ have order $4\#\mathcal{U}_2$, and assume that $\{\psi_{a,\bar{a}}, \psi_{d,\bar{d}}\} \subset \mathcal{U}$, $a, \bar{a}, d, \bar{d} \in \mathbf{F}_q$, $ad \neq 0$, $a \neq d$. Then the fixed field of $\mathcal{U}$ is generated by $x(x+a)(x+d)(x+a+d)$ and $ad(a+d)F(y) + G(x)$, where $F(T) = \prod_{\psi_{0,c} \in \mathcal{U}_2}(T+c)$, and*

$$\begin{aligned} G(T) = \ & (T+a)^3F(a^{2q_0}T + \bar{a}) + (T+d)^3F(d^{2q_0}T + \bar{d}) + \\ & (T+a+d)^3F((a+d)^{2q_0}T + d^{2q_0}a + \bar{a} + \bar{d}). \end{aligned}$$

Proof. The group $\mathcal{U}$ is generated by $\mathcal{U}_2$, $\psi_{a,\bar{a}}$ and $\psi_{d,\bar{d}}$. As the set $\{c \in \mathbf{F}_q \mid \psi_{0,c} \in \mathcal{U}_2\}$ is a linear subspace of $\mathbf{F}_q \mid \mathbf{F}_2$, the polynomial $F(T)$ is such that $F(T_1 + T_2) = F(T_1) + F(T_2)$. Also, $\psi^2_{a,\bar{a}} = \psi_{0,a^{2q_0+1}}$, $\psi^2_{d,\bar{d}} = \psi_{0,d^{2q_0+1}}$ and $(\psi_{a,\bar{a}}\psi_{d,\bar{d}})^2 = \psi_{0,a^{2q_0+1}+a^{2q_0}d+ad^{2q_0}+d^{2q_0+1}}$ imply $F(a^{2q_0+1}) = F(d^{2q_0+1}) = F(a^{2q_0}d + ad^{2q_0}) = 0$. Notice that $ad(a+d)F(y) + G(x)$ can be written as

$$x^3\Big(\prod_{\psi_{0,c} \in \mathcal{U}_2}(y+c)\Big) + (x+a)^3\Big(\prod_{\psi_{0,c} \in \mathcal{U}_2}(a^{2q_0}x + y + c + \bar{a})\Big) +$$

$$(x+d)^3\Big(\prod_{\psi_{0,c} \in \mathcal{U}_2}(d^{2q_0}x + y + c + \bar{d})\Big) +$$

$$(x+a+d)^3\Big(\prod_{\psi_{0,c} \in \mathcal{U}_2}((a+d)^{2q_0}x + y + c + d^{2q_0}a + \bar{a} + \bar{d})\Big).$$

Then it is straightforward to check that $\xi := x(x+a)(x+d)(x+a+d)$ and $\eta := ad(a+d)F(y) + G(x)$ are fixed by $\mathcal{U}_2$, $\psi_{a,\bar{a}}$ and $\psi_{d,\bar{d}}$. On the other hand, $\bar{\mathbf{F}}_q(\xi, \eta)$ cannot be a proper subfield of the fixed field of $\mathcal{U}$. In fact, by the proof of Lemma 3.2 the degree of the extension $\bar{\mathbf{F}}_q(\mathcal{X}) \mid \bar{\mathbf{F}}_q(x, F(y))$

is equal to $\#\mathcal{U}_2$. By $\bar{\mathbf{F}}_q(x, F(y)) = \bar{\mathbf{F}}_q(\xi, \eta, x)$ it follows that the degree of $\bar{\mathbf{F}}_q(x, F(y)) \mid \bar{\mathbf{F}}_q(\xi, \eta)$ is at most 4, whence the claim. □

3.1. **The case** $q = 8$. Let w be a primitive element of $\mathbf{F}_8$ satisfying $w^3 = w + 1$. As a result of a computer search, a set of representatives of the conjugacy classes of 2-subgroups of $Sz(8)$ is the following:

- $\mathcal{V}_1 :=< \{\psi_{0,1}\} >$;
- $\mathcal{V}_2 :=< \{\psi_{1,0}\} >$;
- $\mathcal{V}_3 :=< \{\psi_{0,1}, \psi_{0,w}\} >$;
- $\mathcal{V}_4 := \bar{\mathbf{T}}_0$;
- $\mathcal{V}_5 :=< \{\psi_{1,0}, \psi_{0,w}\} >$;
- $\mathcal{V}_6 :=< \{\psi_{1,0}, \psi_{0,w^2}\} >$;
- $\mathcal{V}_7 :=< \{\psi_{1,0}, \psi_{0,w^4}\} >$;
- $\mathcal{V}_8 :=< \{\psi_{1,0}, \psi_{0,c} \mid c \in \mathbf{F}_8\} >$;
- $\mathcal{V}_9 :=< \{\psi_{1,0}, \psi_{w,0}, \psi_{0,c} \mid c \in \mathbf{F}_8\} >$;
- $\mathcal{V}_{10} := \bar{\mathbf{T}}$.

By Proposition 3.1, for $\mathcal{U} \in \{\mathcal{V}_4, \mathcal{V}_8, \mathcal{V}_9, \mathcal{V}_{10}\}$ the curve $\mathcal{X}_\mathcal{U}$ is rational. For $\mathcal{U} \in \{\mathcal{V}_1, \mathcal{V}_2\}$ Theorems 7.8 and 7.9 in [3] provide an equation for a plane model of $\mathcal{X}_\mathcal{U}$. Therefore only the equation of $\mathcal{X}_{\mathcal{V}_i}$ for $i \in \{3, 5, 6, 7\}$ has to be computed.

Theorem 3.5. *The curve $\mathcal{X}_{\mathcal{V}_3}$ has genus 2 and it is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

$$X^{12} + X^5 = Y^2 + (\omega + 1)Y.$$

Proof. The curve $\mathcal{X}_{\mathcal{V}_3}$ has genus 2 by Proposition 3.1. By Lemma 3.2 we have to prove that $x^{12} + x^5 + F(y)^2 + (\omega + 1)F(y) = 0$ in $\bar{\mathbf{F}}_q(\mathcal{X})$, where $F(y) = y(y+1)(y+\omega)(y+\omega+1) = y^4 + (\omega^2+\omega+1)y^2 + (\omega^2+\omega)y$. This follows from $x^{12} + x^5 = y^8 + y$, $\omega^2 + \omega + 1 = \omega^5$ and $\omega^2 + \omega = \omega^4$. □

Theorem 3.6. *The curve $\mathcal{X}_{\mathcal{V}_5}$ has genus 1 and it is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

$$X^6 + X^4 + \omega X^3 + X^2 = Y^2 + (\omega + 1)Y.$$

Proof. The curve $\mathcal{X}_{\mathcal{V}_3}$ has genus 1 by Proposition 3.1. By Lemma 3.3 we have to prove that

$$(x^2+x)^6 + (x^2+x)^4 + \omega(x^2+x)^3 + (x^2+x)^2 =$$

$$(F(y) + xF(x))^2 + (\omega+1)(F(y) + xF(x))$$

in $\bar{\mathbf{F}}_q(\mathcal{X})$. By straightforward computation,

$$(x^2+x)^6 + (x^2+x)^4 + \omega(x^2+x)^3 + (x^2+x)^2 =$$

$$x^{12} + x^{10} + (1+\omega)x^6 + \omega x^5 + \omega x^4 + \omega x^3 + x^2$$

and

$$(F(y) + xF(x))^2 + (\omega + 1)(F(y) + xF(x)) =$$
$$y^8 + y + x^{10} + (1 + \omega)x^6 + (\omega + 1)x^5 + \omega x^4 + \omega x^3 + x^2.$$

Then the claim follows from $y^8 + y = x^{12} + x^5$. □

Theorem 3.7. *The curve $\mathcal{X}_{\mathcal{V}_6}$ has genus* 1 *and it is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

$$X^6 + X^4 + \omega^2 X^3 + X^2 = Y^2 + \omega^6 Y.$$

Proof. The proof is similar to that of Theorem 3.6. □

Theorem 3.8. *The curve $\mathcal{X}_{\mathcal{V}_7}$ has genus* 1 *and it is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

$$X^6 + X^4 + \omega X^4 + X^2 = Y^2 + \omega^5 Y.$$

Proof. The proof is similar to that of Theorem 3.6. □

3.2. **The case $q = 32$.** Let w be a primitive element of $\mathbf{F}_{32}$ satisfying $w^5 = w^2 + 1$. As a result of a computer search, a set of representatives of the conjugacy classes of 2-subgroups of $\mathcal{S}z(32)$ is the following:

- $\mathcal{V}_1 :=< \{\psi_{0,1}\} >;$
- $\mathcal{V}_2 :=< \{\psi_{0,w^{28}}, \psi_{0,w^{23}}\} >;$
- $\mathcal{V}_3 :=< \{\psi_{0,w^{28}}, \psi_{0,w^{7}}\} >;$
- $\mathcal{V}_4 :=< \{\psi_{0,w^{28}}, \psi_{0,w^{5}}\} >;$
- $\mathcal{V}_5 :=< \{\psi_{0,w^{28}}, \psi_{0,w^{10}}\} >;$
- $\mathcal{V}_6 :=< \{\psi_{0,w^{28}}, \psi_{0,w^{6}}\} >;$
- $\mathcal{V}_7 :=< \{\psi_{1,0}\} >;$
- $\mathcal{V}_8 :=< \{\psi_{0,w^{28}}, \psi_{0,w^{7}}, \psi_{0,w^{10}}\} >;$
- $\mathcal{V}_9 :=< \{\psi_{0,w^{28}}, \psi_{0,w^{7}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{10} :=< \{\psi_{0,w^{28}}, \psi_{0,w^{7}}, \psi_{0,w^{4}}\} >;$
- $\mathcal{V}_{11} :=< \{\psi_{0,w^{28}}, \psi_{0,w^{7}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{12} :=< \{\psi_{0,w^{28}}, \psi_{0,w^{7}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{13} :=< \{\psi_{w,0}, \psi_{0,w^{26}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{14} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{15} :=< \{\psi_{w,0}, \psi_{0,w^{28}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{16} :=< \{\psi_{w,0}, \psi_{0,w^{1}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{17} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{18} :=< \{\psi_{w,0}, \psi_{0,w^{15}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{19} :=< \{\psi_{w,0}, \psi_{0,w^{29}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{20} :=< \{\psi_{w,0}, \psi_{0,w^{14}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{21} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{22} :=< \{\psi_{w,0}, \psi_{0,w^{24}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{23} :=< \{\psi_{w,0}, \psi_{0,w^{3}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{24} :=< \{\psi_{w,0}, \psi_{0,w^{6}}, \psi_{0,w^{4}}\} >;$

- $\mathcal{V}_{25} :=< \{\psi_{w,0}, \psi_{0,w^{13}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{26} :=< \{\psi_{w,0}, \psi_{0,w^{22}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{27} :=< \{\psi_{w,0}, \psi_{0,w^{18}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{28} :=< \{\psi_{0,w^{6}}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{29} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{7}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{30} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{28}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{31} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{9}}, \psi_{0,w^{14}}\} >;$
- $\mathcal{V}_{32} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{33} :=< \{\psi_{w,0}, \psi_{0,w}, \psi_{0,w^{10}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{34} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{14}}, \psi_{0,w^{15}}\} >;$
- $\mathcal{V}_{35} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{13}}, \psi_{0,w^{24}}\} >;$
- $\mathcal{V}_{36} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{7}}\} >;$
- $\mathcal{V}_{37} :=< \{\psi_{w,0}, \psi_{0,w}, \psi_{0,w^{4}}, \psi_{0,w^{6}}\} >;$
- $\mathcal{V}_{38} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{9}}, \psi_{0,w^{13}}\} >;$
- $\mathcal{V}_{39} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{40} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{22}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{41} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{22}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{42} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{9}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{43} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{24}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{44} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{3}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{45} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{24}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{46} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{15}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{47} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{15}}, \psi_{0,w^{13}}\} >;$
- $\mathcal{V}_{48} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{18}}\} >;$
- $\mathcal{V}_{49} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{22}}\} >;$
- $\mathcal{V}_{50} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{13}}, \psi_{0,w^{22}}\} >;$
- $\mathcal{V}_{51} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{7}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{52} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{53} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{13}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{54} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{55} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{6}}, \psi_{0,w^{10}}\} >;$
- $\mathcal{V}_{56} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{18}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{57} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{14}}, \psi_{0,w^{22}}\} >;$
- $\mathcal{V}_{58} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{59} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{14}}, \psi_{0,w^{24}}\} >;$
- $\mathcal{V}_{60} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{61} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{15}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{62} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{63} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{64} := \bar{\mathbf{T}}_0;$
- $\mathcal{V}_{65} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{13}}, \psi_{0,w^{16}}, \psi_{0,w^{24}}\} >;$
- $\mathcal{V}_{66} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{67} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{6}}, \psi_{0,w^{10}}\} >;$

- $\mathcal{V}_{68} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^7}, \psi_{0,w^{13}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{69} :=< \{\psi_{w,0}, \psi_{0,w^4}, \psi_{0,w^6}, \psi_{0,w^7}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{70} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^6}, \psi_{0,w^{10}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{71} :=< \{\psi_{w,0}, \psi_{0,w^4}, \psi_{0,w^6}, \psi_{0,w^{22}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{72} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{16}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{73} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^7}, \psi_{0,w^{16}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{74} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{75} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{76} :=< \{\psi_{w,0}, \psi_{0,w^4}, \psi_{0,w^6}, \psi_{0,w^7}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{77} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^6}, \psi_{0,w^7}, \psi_{0,w^{10}}\} >;$
- $\mathcal{V}_{78} :=< \{\psi_{w,0}, \psi_{0,w^4}, \psi_{0,w^6}, \psi_{0,w^{22}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{79} :=< \{\psi_{w,0}, \psi_{0,w^7}, \psi_{0,w^{10}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{80} :=< \{\psi_{w,w^{13}}, \psi_{w^{17},w^{10}}, \psi_{0,w^9}, \psi_{0,w^{15}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{81} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,w^9}, \psi_{0,w^{15}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{82} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{0,w^9}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{83} :=< \{\psi_{w,w^3}, \psi_{w^{24},w^8}, \psi_{0,w^9}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{84} :=< \{\psi_{w,w^{24}}, \psi_{w^2,w^{22}}, \psi_{0,w^7}, \psi_{0,w^9}, \psi_{0,w^{14}}\} >;$
- $\mathcal{V}_{85} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{86} :=< \{\psi_{w,w^{18}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{87} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{88} :=< \{\psi_{w,w^{14}}, \psi_{w^{27},w^{19}}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{89} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^2}, \psi_{0,w^7}, \psi_{0,w^9}, \psi_{0,w^{14}}\} >;$
- $\mathcal{V}_{90} :=< \{\psi_{w,0}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{91} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{92} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,1}, \psi_{0,w^{16}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{93} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,w^4}, \psi_{0,w^6}, \psi_{0,w^{22}}, \psi_{0,w^{29}}\} >;$
- $\mathcal{V}_{94} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,w^7}, \psi_{0,w^{10}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{95} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{0,w^4}, \psi_{0,w^6}, \psi_{0,w^7}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{96} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w^7}, \psi_{0,w^{16}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{97} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w^{13}}, \psi_{0,w^{16}}, \psi_{0,w^{24}}\} >;$
- $\mathcal{V}_{98} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^6}, \psi_{0,w^{10}}\} >;$
- $\mathcal{V}_{99} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,1}, \psi_{0,w^6}, \psi_{0,w^{10}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{100} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,w^7}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{101} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{102} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^2}, \psi_{0,w^7}, \psi_{0,w^{10}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{103} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^2}, \psi_{0,1}, \psi_{0,w^7}, \psi_{0,w^{13}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{104} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^2}, \psi_{0,w^4}, \psi_{0,w^6}, \psi_{0,w^7}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{105} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{0,1}, \psi_{0,w^{16}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{106} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{107} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^2}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{108} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{109} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{110} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$

- $\mathcal{V}_{111} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{w^{21},w^{16}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{112} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{w^{18},w^{2}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{113} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{w^{27},w^{19}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{114} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{w^{3},w^{21}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{115} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{w^{17},w^{10}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{116} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{w^{21},w^{16}}, \psi_{w^{27},w^{19}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{117} := \bar{\mathbf{T}}$.

By Proposition 3.1, the genus $g_{\mathcal{V}_i}$ of the curve $\mathcal{X}_{\mathcal{V}_i}$ is equal to

$$g_{\mathcal{V}_i} = \begin{cases} 60 & \text{for } i = 1, \\ 28 & \text{for } 2 \le i \le 6, \\ 30 & \text{for } i = 7, \\ 12 & \text{for } 8 \le i \le 12, \\ 14 & \text{for } 13 \le i \le 27, \\ 4 & \text{for } i = 28, \\ 6 & \text{for } 29 \le i \le 63, \\ 2 & \text{for } 65 \le i \le 79, \\ 3 & \text{for } 80 \le i \le 89, \\ 1 & \text{for } 91 \le i \le 105, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for $\mathcal{U} = \mathcal{V}_i$, $i = 64, 90$ and $106 \le i \le 117$ the curve $\mathcal{X}_{\mathcal{U}}$ is rational. For $\mathcal{U} \in \{\mathcal{V}_1, \mathcal{V}_7\}$ Theorems 7.8 and 7.9 in [3] provide an equation for a plane model of $\mathcal{X}_{\mathcal{U}}$. An equation for a plane model of $\mathcal{X}_{\mathcal{U}}$ for the remaining cases is given in the following theorem.

Theorem 3.9. *The curve $\mathcal{X}_{\mathcal{V}_i}$ is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

- $X^{40} + X^9 + Y^8 + w^{17}Y^4 + w^{26}Y^2 + w^{17}Y = 0$, *for $i = 2$;*
- $X^{40} + X^9 + Y^8 + w^{25}Y^4 + w^{23}Y^2 + w^{26}Y = 0$, *for $i = 3$;*
- $X^{40} + X^9 + Y^8 + w^{5}Y^4 + w^{25}Y^2 + w^{12}Y = 0$, *for $i = 4$;*
- $X^{40} + X^9 + Y^8 + Y^4 + w^{8}Y^2 + w^{13}Y = 0$, *for $i = 5$;*
- $X^{40} + X^9 + Y^8 + w^{27}Y^4 + w^{29}Y^2 + w^{15}Y = 0$, *for $i = 6$;*
- $X^{40} + X^9 + Y^4 + w^{6}Y^2 + w^{14}Y = 0$, *for $i = 8$;*
- $X^{40} + X^9 + Y^4 + w^{18}Y^2 + w^{16}Y = 0$, *for $i = 9$;*
- $X^{40} + X^9 + Y^4 + w^{10}Y^2 + w^{2}Y = 0$, *for $i = 10$;*
- $X^{40} + X^9 + Y^4 + w^{21}Y^2 + w^{26}Y = 0$, *for $i = 11$;*
- $X^{40} + X^9 + Y^4 + w^{27}Y^2 + w^{3}Y = 0$, *for $i = 12$;*
- $X^{12} + w^{29}X^8 + w^{14}X^6 + w^4X^5 + w^6X^4 + wX^3 + w^{10}X^2 + w^5Y^8 + w^4Y^4 + w^{27}Y^2 + w^{24}Y = 0$, *for $i = 13$;*
- $X^{12} + w^{15}X^8 + w^4X^6 + w^{21}X^5 + w^{23}X^4 + w^{30}X^3 + w^{27}X^2 + w^{22}Y^8 + w^4Y^4 + w^{30}Y^2 + w^{28}Y = 0$, *for $i = 14$;*
- $X^{12} + w^{27}X^8 + w^8X^6 + w^2X^5 + w^4X^4 + w^{24}X^3 + w^8X^2 + w^3Y^8 + w^4Y^4 + w^{22}Y^2 + w^8Y = 0$, *for $i = 15$;*

- $X^{12}+w^{21}X^8+w^{28}X^6+w^{27}X^5+w^{29}X^4+w^6X^3+w^2X^2+w^{28}Y^8+w^4Y^4+w^{25}Y^2+w^{28}Y=0$, *for* $i=16$;
- $X^{12}+w^{19}X^8+w^{16}X^6+w^{25}X^5+w^{27}X^4+w^{15}X^3+X^2+w^{26}Y^8+w^4Y^4+w^9Y^2+w^{29}Y=0$, *for* $i=17$;
- $X^{12}+w^{30}X^8+w^{20}X^6+w^5X^5+w^7X^4+w^{12}X^3+w^{11}X^2+w^6Y^8+w^4Y^4+w^4Y^2+w^8Y=0$, *for* $i=18$;
- $X^{12}+w^{17}X^8+X^6+w^{23}X^5+w^{25}X^4+w^{25}X^3+w^{29}X^2+w^{24}Y^8+w^4Y^4+w^{27}Y^2+Y=0$, *for* $i=19$;
- $X^{12}+w^{18}X^8+w^9X^6+w^{24}X^5+w^{26}X^4+w^{12}X^3+w^{30}X^2+w^{25}Y^8+w^4Y^4+w^{25}Y^2+w^{22}Y=0$, *for* $i=20$;
- $X^{12}+w^{13}X^8+w^{26}X^6+w^{19}X^5+w^{21}X^4+w^{15}X^3+w^{25}X^2+w^{20}Y^8+w^4Y^4+w^4Y^2+w^5Y=0$, *for* $i=21$;
- $X^{12}+w^3X^8+w^{13}X^6+w^9X^5+w^{11}X^4+w^{25}X^3+w^{15}X^2+w^{10}Y^8+w^4Y^4+w^5Y^2+w^6Y=0$, *for* $i=22$;
- $X^{12}+w^2X^8+w^{10}X^6+w^8X^5+w^{10}X^4+w^{21}X^3+w^{14}X^2+w^9Y^8+w^4Y^4+w^{18}Y^2+w^{29}Y=0$, *for* $i=23$;
- $X^{12}+w^5X^8+w^{11}X^6+w^{11}X^5+w^{13}X^4+w^4X^3+w^{17}X^2+w^{12}Y^8+w^4Y^4+w^{16}Y^2+w^{24}Y=0$, *for* $i=24$;
- $X^{12}+w^{28}X^8+w^6X^6+w^3X^5+w^5X^4+w^6X^3+w^9X^2+w^4Y^8+w^4Y^4+w^5Y^2+w^{25}Y=0$, *for* $i=25$;
- $X^{12}+w^{22}X^8+w^{27}X^6+w^{28}X^5+w^{30}X^4+w^{11}X^3+w^3X^2+w^{29}Y^8+w^4Y^4+w^{15}Y^2+w^6Y=0$, *for* $i=26$;
- $X^{12}+w^{26}X^8+w^{19}X^6+wX^5+w^3X^4+wX^3+w^7X^2+w^2Y^8+w^4Y^4+w^9Y^2+w^{12}Y=0$, *for* $i=27$;
- $X^{40}+X^9+Y^2+w^{20}Y=0$, *for* $i=28$;
- $X^{20}+w^{16}X^{12}+w^{10}X^{10}+w^{24}X^8+w^8X^6+wX^5+wX^4+w^{14}X^3+w^5X^2+Y^4+w^{10}Y^2+w^4Y=0$, *for* $i=29$;
- $X^{20}+w^{16}X^{12}+w^{15}X^{10}+w^{24}X^8+w^{18}X^6+X^5+wX^4+w^{24}X^3+w^5X^2+Y^4+w^{15}Y^2+w^{17}Y=0$, *for* $i=30$;
- $X^{20}+w^{16}X^{12}+w^{15}X^{10}+w^{24}X^8+w^{17}X^6+w^{27}X^5+wX^4+w^8X^3+w^5X^2+Y^4+w^{15}Y^2+w^{25}Y=0$, *for* $i=31$;
- $X^{20}+w^{16}X^{12}+w^{14}X^{10}+w^{24}X^8+w^{23}X^6+w^{15}X^5+wX^4+w^{10}X^3+w^5X^2+Y^4+w^{14}Y^2+w^8Y=0$, *for* $i=32$;
- $X^{20}+w^{16}X^{12}+X^{10}+w^{24}X^8+w^{15}X^6+w^{18}X^5+wX^4+w^{23}X^3+w^5X^2+Y^4+Y^2+w^{10}Y=0$, *for* $i=33$;
- $X^{20}+w^{16}X^{12}+w^8X^{10}+w^{24}X^8+w^{18}X^6+X^5+wX^4+w^{17}X^3+w^5X^2+Y^4+w^8Y^2+w^{17}Y=0$, *for* $i=34$;
- $X^{20}+w^{16}X^{12}+w^{27}X^{10}+w^{24}X^8+w^8X^6+wX^5+wX^4+X^3+w^5X^2+Y^4+w^{27}Y^2+w^4Y=0$, *for* $i=35$;
- $X^{20}+w^{16}X^{12}+w^{28}X^{10}+w^{24}X^8+wX^6+w^{16}X^5+wX^4+w^{13}X^3+w^5X^2+Y^4+w^{28}Y^2+w^{29}Y=0$, *for* $i=36$;
- $X^{20}+w^{16}X^{12}+w^{11}X^{10}+w^{24}X^8+w^{10}X^6+w^7X^5+wX^4+w^{16}X^3+w^5X^2+Y^4+w^{11}Y^2+w^{19}Y=0$, *for* $i=37$;

- $X^{20}+w^{16}X^{12}+w^{7}X^{10}+w^{24}X^{8}+w^{24}X^{6}+w^{17}X^{5}+wX^{4}+w^{19}X^{3}+w^{5}X^{2}+Y^{4}+w^{7}Y^{2}+Y=0$, *for* $i=38$;
- $X^{20}+w^{16}X^{12}+w^{30}X^{10}+w^{24}X^{8}+w^{12}X^{6}+w^{9}X^{5}+wX^{4}+w^{5}X^{3}+w^{5}X^{2}+Y^{4}+w^{30}Y^{2}+w^{3}Y=0$, *for* $i=39$;
- $X^{20}+w^{16}X^{12}+w^{5}X^{10}+w^{24}X^{8}+w^{15}X^{6}+w^{18}X^{5}+wX^{4}+w^{28}X^{3}+w^{5}X^{2}+Y^{4}+w^{5}Y^{2}+w^{10}Y=0$, *for* $i=40$;
- $X^{20}+w^{16}X^{12}+w^{24}X^{10}+w^{24}X^{8}+w^{30}X^{6}+w^{23}X^{5}+wX^{4}+w^{8}X^{3}+w^{5}X^{2}+Y^{4}+w^{24}Y^{2}+w^{14}Y=0$, *for* $i=41$;
- $X^{20}+w^{16}X^{12}+w^{17}X^{10}+w^{24}X^{8}+w^{10}X^{6}+w^{7}X^{5}+wX^{4}+w^{22}X^{3}+w^{5}X^{2}+Y^{4}+w^{17}Y^{2}+w^{19}Y=0$, *for* $i=42$;
- $X^{20}+w^{16}X^{12}+w^{3}X^{10}+w^{24}X^{8}+w^{19}X^{6}+w^{3}X^{5}+wX^{4}+w^{28}X^{3}+w^{5}X^{2}+Y^{4}+w^{3}Y^{2}+w^{9}Y=0$, *for* $i=43$;
- $X^{20}+w^{16}X^{12}+w^{28}X^{10}+w^{24}X^{8}+w^{6}X^{6}+w^{24}X^{5}+wX^{4}+X^{3}+w^{5}X^{2}+Y^{4}+w^{28}Y^{2}+w^{20}Y=0$, *for* $i=44$;
- $X^{20}+w^{16}X^{12}+w^{22}X^{10}+w^{24}X^{8}+w^{29}X^{6}+w^{11}X^{5}+wX^{4}+w^{21}X^{3}+w^{5}X^{2}+Y^{4}+w^{22}Y^{2}+w^{22}Y=0$, *for* $i=45$;
- $X^{20}+w^{16}X^{12}+w^{5}X^{10}+w^{24}X^{8}+w^{5}X^{6}+w^{2}X^{5}+wX^{4}+w^{23}X^{3}+w^{5}X^{2}+Y^{4}+w^{5}Y^{2}+w^{28}Y=0$, *for* $i=46$;
- $X^{20}+w^{16}X^{12}+w^{10}X^{10}+w^{24}X^{8}+w^{6}X^{6}+w^{24}X^{5}+wX^{4}+w^{13}X^{3}+w^{5}X^{2}+Y^{4}+w^{10}Y^{2}+w^{20}Y=0$, *for* $i=47$;
- $X^{20}+w^{16}X^{12}+w^{4}X^{10}+w^{24}X^{8}+w^{22}X^{6}+w^{29}X^{5}+wX^{4}+w^{15}X^{3}+w^{5}X^{2}+Y^{4}+w^{4}Y^{2}+w^{16}Y=0$, *for* $i=48$;
- $X^{20}+w^{16}X^{12}+X^{10}+w^{24}X^{8}+w^{19}X^{6}+w^{3}X^{5}+wX^{4}+w^{25}X^{3}+w^{5}X^{2}+Y^{4}+Y^{2}+w^{9}Y=0$, *for* $i=49$;
- $X^{20}+w^{16}X^{12}+w^{12}X^{10}+w^{24}X^{8}+w^{25}X^{6}+w^{14}X^{5}+wX^{4}+w^{9}X^{3}+w^{5}X^{2}+Y^{4}+w^{12}Y^{2}+w^{23}Y=0$, *for* $i=50$;
- $X^{20}+w^{16}X^{12}+w^{21}X^{10}+w^{24}X^{8}+w^{13}X^{6}+w^{5}X^{5}+wX^{4}+w^{12}X^{3}+w^{5}X^{2}+Y^{4}+w^{21}Y^{2}+w^{26}Y=0$, *for* $i=51$;
- $X^{20}+w^{16}X^{12}+w^{29}X^{10}+w^{24}X^{8}+w^{26}X^{6}+w^{8}X^{5}+wX^{4}+w^{11}X^{3}+w^{5}X^{2}+Y^{4}+w^{29}Y^{2}+w^{15}Y=0$, *for* $i=52$;
- $X^{20}+w^{16}X^{12}+w^{16}X^{10}+w^{24}X^{8}+w^{11}X^{6}+w^{22}X^{5}+wX^{4}+w^{6}X^{3}+w^{5}X^{2}+Y^{4}+w^{16}Y^{2}+w^{11}Y=0$, *for* $i=53$;
- $X^{20}+w^{16}X^{12}+w^{2}X^{10}+w^{24}X^{8}+w^{7}X^{6}+w^{13}X^{5}+wX^{4}+w^{21}X^{3}+w^{5}X^{2}+Y^{4}+w^{2}Y^{2}+w^{12}Y=0$, *for* $i=54$;
- $X^{20}+w^{16}X^{12}+w^{21}X^{10}+w^{24}X^{8}+w^{2}X^{6}+w^{6}X^{5}+wX^{4}+w^{22}X^{3}+w^{5}X^{2}+Y^{4}+w^{21}Y^{2}+w^{21}Y=0$, *for* $i=55$;
- $X^{20}+w^{16}X^{12}+w^{19}X^{10}+w^{24}X^{8}+w^{16}X^{6}+w^{28}X^{5}+wX^{4}+w^{27}X^{3}+w^{5}X^{2}+Y^{4}+w^{19}Y^{2}+w^{2}Y=0$, *for* $i=56$;
- $X^{20}+w^{16}X^{12}+w^{22}X^{10}+w^{24}X^{8}+w^{7}X^{6}+w^{13}X^{5}+wX^{4}+w^{10}X^{3}+w^{5}X^{2}+Y^{4}+w^{22}Y^{2}+w^{12}Y=0$, *for* $i=57$;
- $X^{20}+w^{16}X^{12}+w^{9}X^{10}+w^{24}X^{8}+w^{27}X^{6}+w^{19}X^{5}+wX^{4}+w^{7}X^{3}+w^{5}X^{2}+Y^{4}+w^{9}Y^{2}+w^{7}Y=0$, *for* $i=58$;

- $X^{20}+w^{16}X^{12}+w^{11}X^{10}+w^{24}X^{8}+w^{2}X^{6}+w^{6}X^{5}+wX^{4}+w^{12}X^{3}+w^{5}X^{2}+Y^{4}+w^{11}Y^{2}+w^{21}Y=0$, *for* $i=59$;
- $X^{20}+w^{16}X^{12}+w^{18}X^{10}+w^{24}X^{8}+w^{4}X^{6}+w^{26}X^{5}+wX^{4}+w^{20}X^{3}+w^{5}X^{2}+Y^{4}+w^{18}Y^{2}+w^{5}Y=0$, *for* $i=60$;
- $X^{20}+w^{16}X^{12}+w^{25}X^{10}+w^{24}X^{8}+w^{20}X^{6}+w^{4}X^{5}+wX^{4}+w^{4}X^{3}+w^{5}X^{2}+Y^{4}+w^{25}Y^{2}+wY=0$, *for* $i=61$;
- $X^{20}+w^{16}X^{12}+w^{24}X^{10}+w^{24}X^{8}+w^{17}X^{6}+w^{27}X^{5}+wX^{4}+w^{17}X^{3}+w^{5}X^{2}+Y^{4}+w^{24}Y^{2}+w^{25}Y=0$, *for* $i=62$;
- $X^{20}+w^{16}X^{12}+w^{2}X^{10}+w^{24}X^{8}+w^{23}X^{6}+w^{15}X^{5}+wX^{4}+w^{29}X^{3}+w^{5}X^{2}+Y^{4}+w^{2}Y^{2}+w^{8}Y=0$, *for* $i=63$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{26}X^{9}+w^{24}X^{8}+w^{15}X^{5}+wX^{4}+w^{26}X^{3}+w^{5}X^{2}+Y^{2}+w^{24}Y=0$, *for* $i=65$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{23}X^{9}+w^{24}X^{8}+w^{3}X^{5}+wX^{4}+wX^{3}+w^{5}X^{2}+Y^{2}+w^{21}Y=0$, *for* $i=66$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{3}X^{9}+w^{24}X^{8}+w^{19}X^{5}+wX^{4}+w^{10}X^{3}+w^{5}X^{2}+Y^{2}+wY=0$, *for* $i=67$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{25}X^{9}+w^{24}X^{8}+w^{4}X^{5}+wX^{4}+w^{28}X^{3}+w^{5}X^{2}+Y^{2}+w^{23}Y=0$, *for* $i=68$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{20}X^{9}+w^{24}X^{8}+w^{23}X^{5}+wX^{4}+w^{7}X^{3}+w^{5}X^{2}+Y^{2}+w^{18}Y=0$, *for* $i=69$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{8}X^{9}+w^{24}X^{8}+w^{17}X^{5}+wX^{4}+X^{3}+w^{5}X^{2}+Y^{2}+w^{6}Y=0$, *for* $i=70$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{28}X^{9}+w^{24}X^{8}+w^{9}X^{5}+wX^{4}+w^{22}X^{3}+w^{5}X^{2}+Y^{2}+w^{26}Y=0$, *for* $i=71$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{18}X^{9}+w^{24}X^{8}+w^{16}X^{5}+wX^{4}+w^{11}X^{3}+w^{5}X^{2}+Y^{2}+w^{16}Y=0$, *for* $i=72$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{14}X^{9}+w^{24}X^{8}+w^{2}X^{5}+wX^{4}+w^{19}X^{3}+w^{5}X^{2}+Y^{2}+w^{12}Y=0$, *for* $i=73$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{15}X^{9}+w^{24}X^{8}+w^{8}X^{5}+wX^{4}+w^{17}X^{3}+w^{5}X^{2}+Y^{2}+w^{13}Y=0$, *for* $i=74$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{30}X^{9}+w^{24}X^{8}+X^{5}+wX^{4}+w^{18}X^{3}+w^{5}X^{2}+Y^{2}+w^{28}Y=0$, *for* $i=75$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{7}X^{9}+w^{24}X^{8}+wX^{5}+wX^{4}+w^{2}X^{3}+w^{5}X^{2}+Y^{2}+w^{5}Y=0$, *for* $i=76$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{6}X^{9}+w^{24}X^{8}+w^{5}X^{5}+wX^{4}+w^{4}X^{3}+w^{5}X^{2}+Y^{2}+w^{4}Y=0$, *for* $i=77$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{21}X^{9}+w^{24}X^{8}+w^{11}X^{5}+wX^{4}+w^{5}X^{3}+w^{5}X^{2}+Y^{2}+w^{19}Y=0$, *for* $i=78$;
- $X^{20}+w^{4}X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{24}X^{9}+w^{24}X^{8}+w^{7}X^{5}+wX^{4}+w^{30}X^{3}+w^{5}X^{2}+Y^{2}+w^{22}Y=0$, *for* $i=79$;
- $X^{8}+w^{13}X^{4}+w^{4}X^{3}+w^{16}X^{2}+w^{11}X+w^{13}Y^{4}+wY^{2}+w^{5}Y+w^{21}=0$, *for* $i=80$;

- $X^8+w^{13}X^4+w^4X^3+w^{22}X^2+w^7X+w^{13}Y^4+wY^2+w^5Y+w^{30}=0$, *for* $i=81$;
- $X^8+w^{26}X^4+w^7X^3+w^{15}X^2+w^{17}X+w^{17}Y^4+w^2Y^2+w^{15}Y+w^{11}=0$, *for* $i=82$;
- $X^8+w^{26}X^4+w^7X^3+w^{25}X^2+w^{17}Y^4+w^2Y^2+w^{15}Y+w^{11}=0$, *for* $i=83$;
- $X^8+w^{18}X^4+w^2X^3+w^{25}X^2+w^{13}X+w^{10}Y^4+w^{30}Y^2+w^{27}Y+w^{27}=0$, *for* $i=84$;
- $X^8+w^{10}X^4+w^{19}X^3+w^8X^2+w^{25}X+w^{15}Y^4+w^{27}Y^2+w^{22}Y+w^{10}=0$, *for* $i=85$;
- $X^8+w^{10}X^4+w^{19}X^3+w^{11}X^2+w^{18}X+w^{15}Y^4+w^{27}Y^2+w^{22}Y+w^{11}=0$, *for* $i=86$;
- $X^8+w^{17}X^4+w^{11}X^3+w^{20}X^2+w^{21}X+w^{26}Y^4+w^{18}Y^2+w^{23}Y+w^{19}=0$, *for* $i=87$;
- $X^8+w^{17}X^4+w^{11}X^3+w^{29}X^2+w^{15}X+w^{26}Y^4+w^{18}Y^2+w^{23}Y+w^{19}=0$, *for* $i=88$;
- $X^8+w^{18}X^4+w^2X^3+w^{28}X^2+w^{16}X+w^{10}Y^4+w^{30}Y^2+w^{27}Y=0$, *for* $i=89$;
- $X^{10}+w^{17}X^8+w^3X^6+w^{13}X^5+w^{25}X^4+w^{20}X^3+w^5X^2+w^{26}X+w^{17}Y^2+w^6Y=0$, *for* $i=91$;
- $X^{10}+w^{30}X^8+w^{29}X^6+w^{16}X^5+w^7X^4+w^{26}X^3+w^{25}X^2+w^{25}X+w^6Y^2+w^{19}Y+w^5=0$, *for* $i=92$;
- $X^{10}+w^{30}X^8+w^{29}X^6+w^{26}X^5+w^7X^4+w^{10}X^3+w^{18}X^2+X+w^6Y^2+w^{29}Y+w^{23}=0$, *for* $i=93$;
- $X^{10}+w^{30}X^8+w^{29}X^6+w^{22}X^5+w^7X^4+w^4X^3+w^{16}X^2+w^5X+w^6Y^2+w^{25}Y+w^{20}=0$, *for* $i=94$;
- $X^{10}+w^{17}X^8+w^3X^6+w^{18}X^5+w^{25}X^4+w^{12}X^3+w^{17}X^2+w^2X+w^{17}Y^2+w^{11}Y+w^{24}=0$, *for* $i=95$;
- $X^{10}+w^{21}X^8+w^{11}X^6+w^{12}X^5+w^{21}X^4+w^{24}X^3+w^{18}X^2+wX+w^{18}Y^2+w^{21}Y=0$, *for* $i=96$;
- $X^{10}+w^{21}X^8+w^{11}X^6+w^{24}X^5+w^{21}X^4+w^{11}X^3+X^2+w^3X+w^{18}Y^2+w^2Y+w^{21}=0$, *for* $i=97$;
- $X^{10}+w^{21}X^8+w^{11}X^6+wX^5+w^{21}X^4+w^{23}X^3+w^4X^2+w^{26}X+w^{18}Y^2+w^{10}Y+w^6=0$, *for* $i=98$;
- $X^{10}+w^{25}X^8+w^{19}X^6+w^6X^5+w^{22}X^4+w^2X^3+w^2X^2+w^{22}X+w^{10}Y^2+w^{11}Y+w^{20}=0$, *for* $i=99$;
- $X^{10}+w^{25}X^8+w^{19}X^6+w^{21}X^5+w^{22}X^4+w^9X^3+w^{25}X^2+w^{11}X+w^{10}Y^2+w^{26}Y+w^8=0$, *for* $i=100$;
- $X^{10}+w^{25}X^8+w^{19}X^6+w^{28}X^5+w^{22}X^4+w^4X^3+w^{30}X^2+w^{29}X+w^{10}Y^2+w^2Y+w^{10}=0$, *for* $i=101$;
- $X^{10}+w^{11}X^8+w^{22}X^6+w^{22}X^5+w^{20}X^4+w^{21}X^3+w^{18}X^2+w^9X+w^{26}Y^2+w^4Y+w^{29}=0$, *for* $i=102$;

- $X^{10}+w^{11}X^8+w^{22}X^6+w^{23}X^5+w^{20}X^4+w^7X^3+w^3X^2+w^{20}X+w^{26}Y^2+w^5Y+w^9=0$, *for* $i=103$;
- $X^{10}+w^{11}X^8+w^{22}X^6+w^5X^5+w^{20}X^4+w^{11}X^3+w^{28}X^2+w^{22}X+w^{26}Y^2+w^{18}Y+w^{11}=0$, *for* $i=104$;
- $X^{10}+w^{17}X^8+w^3X^6+w^{16}X^5+w^{25}X^4+w^9X^3+w^{16}X^2+w^{27}X+w^{17}Y^2+w^9Y+w^{20}=0$, *for* $i=105$.

Proof. The proof is a straightforward computation based on Lemmas 3.2, 3.3, and 3.4. □

4. Quotient curves arising from subgroups of even order of $\bar{\mathbf{T}}\bar{\mathbf{N}}$

Thanks to a computer research, for both $q=8$ and $q=32$ there are only two conjugacy classes of subgroups of even order of $\bar{\mathbf{T}}\bar{\mathbf{N}}$ which are not 2-subgroups, and they have order $q(q-1)$ and $q^2(q-1)$. By [3, Theorem 8.1] the quotient curves of the DLS-curve associated to those subgroups are rational.

5. Quotient curves arising from subgroups of even order of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{N}}$

Theorem 9.2 in [3] gives an equation for the quotient curve of the DLS-curve associated to subgroups of even order of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{N}}$.

6. Quotient curves arising from subgroups of even order of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^+$

Any subgroup of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^+$ has order 2^ir, for some $i=0,1,2$ and for a certain divisor r of $q+2q_0+1$. Theorem 10.1 in [3] gives an equation of the quotient curve associated to a subgroup of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^+$ for the case $i=1$. For both $q=8$ and $q=32$ the integer $q+2q_0+1$ is a prime number, therefore by [3, Proposition 11.1] the case $i=2$ gives rise to rational curves.

7. Quotient curves arising from subgroups of even order of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^-$

Any subgroup of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^-$ has order 2^ir, for some $i=0,1,2$ and for some divisor r of $q-2q_0+1$. Theorem 10.2 in [3] gives an equation of the quotient curve associated to a subgroup of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^+$ for the case $i=1$. For $q=8$ the integer $q-2q_0+1$ is a prime number, therefore by [3, Proposition 11.1] the case $i=2$ gives rise to rational curves. For $q=32$ we have $q-2q_0+1=25$. Therefore for $i=2$ we have the cases $r=25$ giving rise to a rational curve, and $r=5$ giving rise to a curve of genus 5.

8. Classification of elliptic and hyperelliptic curves of genus 2 covered by the DLS-curve of Suzuki type for $q = 32$

In this section we will provide canonical equations for both elliptic and hyperelliptic curves of genus 2 which are quotient curves of the Suzuki curve defined over the finite field with 32 elements.

A computer based investigation, together with Theorem 8.1 below, has proved that the hyperelliptic curves of genus 2 are pairwise non-isomorphic.

Theorem 8.1. [10] *Assume K to be of even characteristic. Let Γ and Δ be two hyperelliptic curves of genus g given in their canonical form, that is*

$$\Gamma = \mathbf{v}(Y^2 + h(X)Y + g(X)), \quad \deg g(X) = 2g+1, \quad \deg h(X) \le g;$$

$$\Delta = \mathbf{v}(Y^2 + h_1(X)Y + g_1(X)), \quad \deg g_1(X) = 2g+1, \quad \deg h_1(X) \le g.$$

Then Γ and Δ are birationally equivalent if and only if each of the following two conditions are satisfied.

i) *$n = m$, and there is rational function $w(X) = (aX+b)/(cX+d)$ with $ad - bc \neq 0$ which maps the set $\{\alpha_0, \dots, \alpha_n, \infty\}$ onto the set $\{\beta_0, \dots, \beta_n, \infty\}$.*

ii) *There is a rational function $v(X) \in K(X)$ such that*

$$v(X)^2 + v(X) = \frac{g(X)}{h(X)^2} + \frac{g_1(w(X))}{h_1(w(X))^2}.$$

8.1. Canonical equations for elliptic curves $\mathcal{X}_{\nu_i}$, $i = 91, \dots, 105$.

- $w^{29}X^3 + w^6X + Y^2 + Y + w^{30} = 0$, for $i = 91$;
- $w^2X^3 + w^{11}X + Y^2 + Y + w^{13} = 0$, for $i = 92$;
- $w^{13}X^3 + w^{16}X + Y^2 + Y + w^{10} = 0$, for $i = 93$;
- $w^{21}X^3 + w^{20}X + Y^2 + Y + w^{24} = 0$, for $i = 94$;
- $w^{19}X^3 + w^{30}X + Y^2 + Y + w^{28} = 0$, for $i = 95$;
- $w^{13}X^3 + wX + Y^2 + Y + w^{27} = 0$, for $i = 96$;
- $w^{23}X^3 + w^{29}X + Y^2 + Y + 1 = 0$, for $i = 97$;
- $w^7X^3 + w^5X + Y^2 + Y + w^{20} = 0$, for $i = 98$;
- $w^{24}X^3 + w^3X + Y^2 + Y + w^9 = 0$, for $i = 99$;
- $w^{25}X^3 + w^{19}X + Y^2 + Y + w^3 = 0$, for $i = 100$;
- $w^{11}X^3 + w^{14}X + Y^2 + Y + w^{21} = 0$, for $i = 101$;
- $X^3 + w^{13}X + Y^2 + Y + 1 = 0$, for $i = 102$;
- $w^{29}X^3 + w^6X + Y^2 + Y + w^{14} = 0$, for $i = 103$;
- $w^3X^3 + wX + Y^2 + Y + w^{11} = 0$, for $i = 104$;
- $w^{23}X^3 + w^9X + Y^2 + Y + w^{12} = 0$, for $i = 105$.

8.2. Canonical equations for hyperelliptic curves $\mathcal{X}_{\nu_i}$, $i = 65, \dots, 79$.

- $w^{30}X^5 + w^{11}X^4 + w^8X^3 + w^5X^2 + Y^2 + w^{24}Y = 0$, for $i = 65$;
- $w^{30}X^5 + w^{19}X^4 + w^{12}X^3 + w^5X^2 + Y^2 + w^{21}Y = 0$, for $i = 66$;

- $w^{30}X^5 + w^{24}X^4 + w^{29}X^3 + w^5X^2 + Y^2 + wY = 0$, for $i = 67$;
- $w^{30}X^5 + w^{30}X^4 + w^{25}X^3 + w^5X^2 + Y^2 + w^{23}Y = 0$, for $i = 68$;
- $w^{30}X^5 + w^4X^4 + w^{22}X^3 + w^5X^2 + Y^2 + w^{18}Y = 0$, for $i = 69$;
- $w^{30}X^5 + X^4 + w^{14}X^3 + w^5X^2 + Y^2 + w^6Y = 0$, for $i = 70$;
- $w^{30}X^5 + w^{28}X^4 + w^{16}X^3 + w^5X^2 + Y^2 + w^{26}Y = 0$, for $i = 71$;
- $w^{30}X^5 + w^7X^4 + w^{10}X^3 + w^5X^2 + Y^2 + w^{16}Y = 0$, for $i = 72$;
- $w^{30}X^5 + w^{13}X^4 + w^{17}X^3 + w^5X^2 + Y^2 + w^{12}Y = 0$, for $i = 73$;
- $w^{30}X^5 + w^{16}X^4 + w^{24}X^3 + w^5X^2 + Y^2 + w^{13}Y = 0$, for $i = 74$;
- $w^{30}X^5 + w^{21}X^4 + w^{13}X^3 + w^5X^2 + Y^2 + w^{28}Y = 0$, for $i = 75$;
- $w^{30}X^5 + w^{10}X^4 + w^{23}X^3 + w^5X^2 + Y^2 + w^5Y = 0$, for $i = 76$;
- $w^{30}X^5 + w^{25}X^4 + X^3 + w^5X^2 + Y^2 + w^4Y = 0$, for $i = 77$;
- $w^{30}X^5 + w^{18}X^4 + w^{28}X^3 + w^5X^2 + Y^2 + w^{19}Y = 0$, for $i = 78$;
- $w^{30}X^5 + w^6X^4 + w^{21}X^3 + w^5X^2 + Y^2 + w^{22}Y = 0$, for $i = 79$.

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