Characterizing powers of cycles that are divisor graphs

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Abstract.
It was conjectured in a recently published paper that for any integer $k \geq 8$ and any even integer n with $2k + 3 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$, the k-th power C_n^k of the n-cycle is not a divisor graph. In this paper we prove this conjecture, hence obtaining a complete characterization of those powers of cycles which are divisor graphs.
2000 Mathematics Subject Classification: 05C12, 05C20, 05C38, 05C75
Key words: Divisor graph; Divisor orientation; Power of a cycle.

1 Introduction

We consider only simple graphs, i.e. graphs with no loops or multiple edges. For undefined notions, the reader is referred to [9].

For a positive integer n, the divisor graph G_n is the graph with vertex set $\{1, 2, \cdots, n\}$ in which two distinct vertices i and j are adjacent if and only if either i divides j or j divides i. Erdős et al. [6], Pollington [7] and Pomerance [8] have studied some properties of the length of a longest path in G_n. Chartrand et al. [5] investigated divisor graphs on a set S of positive integers rather than the set $\{1, 2, \cdots, n\}$. The divisor graph $G(S)$ has S as its vertex set and two distinct vertices i and j are adjacent if and only if either i divides j or j divides i.

A graph G is a *divisor graph* if G is isomorphic to $G(S)$ for some set S of positive integers. Chartrand et al. [5] proved that all bipartite graphs are divisor graphs. Block graphs which are divisor graphs were precisely determined in [2].

An *odd hole* in a graph G is an induced odd cycle C_{2k+1} of G for some $k \geq 2$. Divisor graphs do not contain odd holes, see [5]. Chartrand et al. [4] have characterized nontrivial connected divisor graphs in terms of the upper orientable hull number.

For any integer $k \geq 2$, Al-Addasi et al. [1] have determined precisely the values of n for which the k-th power P_n^k of the path P_n is a divisor graph. They studied the same problem for the k-th power C_n^k of the cycle C_n and obtained partial results. Namely, they proved that C_n^k is a divisor graph when $n \leq 2k+2$, but not a divisor graph when $n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3$. For odd n with $2k+2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$, they proved that C_n^k is not a divisor graph. This provided a complete characterization of the values of n for which C_n^k is a divisor graph when $k \in \{2, 3\}$. In the same paper, they settled the unique missing case of even n for each $k \in \{4, 5, 6, 7\}$ by proving that C_{12}^4, C_{14}^5, C_{16}^6 and C_{18}^7 are not divisor graphs. For each $k \geq 8$, there are more than one missing case of even n. It was conjectured in [1] that for each integer $k \geq 8$, the graph C_n^k is also not a divisor graph for every even integer n with $2k+2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$. The aim of the present paper is to prove this conjecture and hence to determine precisely those values of n for which C_n^k is a divisor graph.

2 Preliminaries

In a digraph D, a *transmitter* is a vertex having indegree 0, a *receiver* is a vertex having outdegree 0, while a vertex x is a *transitive vertex* if it has both positive outdegree and positive indegree such that $(u, w) \in E(D)$ whenever $(u, x), (x, w) \in E(D)$. An orientation D of a graph G in which every vertex is a transmitter, a receiver, or a transitive vertex is called a *divisor orientation* of G, see [1]. The following result was shown in [1] and will be used in the sequel.

Lemma 1. *If D is a divisor orientation of a graph G, then the converse of D is also a divisor orientation of G.*

An interesting characterization of divisor graphs in terms of divisor orientation was given by Chartrand et al. in [5]. We include it here since we will use it frequently in the proofs in the last section.

Lemma 2. *A graph G is a divisor graph if and only if G has a divisor orientation.*

For any integer $k \geq 2$, Al-Addasi et al. [1] have determined some values of n for which C_n^k is a divisor graph, as stated in the next result.

Lemma 3. *For any integer $k \geq 2$, if $n \leq 2k+2$, then C_n^k is a divisor graph.*

They also have specified some values of n for which C_n^k is not a divisor graph. For the remaining values of n, we will see in the next section that C_n^k is indeed not a divisor graph.

3 Main results

Let $k \geq 2$ be an integer. In this section, we will determine precisely those values of n for which the graph C_n^k is a divisor graph. We start with the following lemma.

Lemma 4. *Let n, k be two integers with $k \geq 2$ and $n > 2k+2$. If C_n^k is a divisor graph and D is a divisor orientation of C_n^k where $xy \in E(C_n)$ and $(x, y) \in E(D)$, then y is a receiver in D.*

Proof. Let C_n be the cycle $12 \cdots n1$. Suppose that C_n^k is a divisor graph and D is a divisor orientation of C_n^k where $xy \in E(C_n)$ and $(x, y) \in E(D)$. By Lemma 1 and the symmetry in C_n^k, we can assume that $x = 1$ and $y = 2$. Since $n > 2k+2$, we have $1(k+2) \notin E(C_n^k)$. Then $(k+2, 2) \in E(D)$ because $(1, 2) \in E(D)$ and D is a divisor orientation of C_n^k. But also $(k+2)n \notin E(C_n^k)$ and $k \geq 2$, which implies that $(n, 2) \in E(D)$. Now since $(k+1)n \notin E(C_n^k)$, we must have $(k+1, 2) \in E(D)$. This proves the result when $k = 2$. If $k > 2$, we must have $(n-1, 2) \in E(D)$ because $(k+1)(n-1) \notin E(C_n^k)$. Then, since $k(n-1) \notin E(C_n^k)$, we have $(k, 2) \in E(D)$. Now if $k > 3$, then we repeat applying a similar argument to obtain that $(n-2, 2), (k-1, 2), (n-3, 2), (k-2, 2), \cdots, (n-(k-2), 2), (k+1-(k-2), 2)$ all belong to $E(D)$. Therefore 2 is a receiver in D. □

Combining the previous lemma and Lemma 1, we get the following result.

Corollary 1. *Let n, k be two integers with $k \geq 2$ and $n > 2k+2$. If C_n^k is a divisor graph and D is a divisor orientation of C_n^k where $xy \in E(C_n)$ and $(x, y) \in E(D)$, then x is a transmitter and y is a receiver in D.*

Proof. By Lemma 4, the vertex y is a receiver in D. According to Lemma 1, the converse of D is also a divisor orientation of C_n^k containing the arc (y, x). Thus, again by Lemma 4, the vertex x is a receiver in the converse of D. This means that x is a transmitter in D. □

For $k \geq 2$, the following result assures that C_n^k is not a divisor graph for any $n > 2k + 2$.

Theorem 1. *Let n, k be two integers with $k \geq 2$ and $n > 2k+2$. Then the graph C_n^k is not a divisor graph.*

Proof. Let C_n be the cycle $12 \cdots n1$. Assume to the contrary that C_n^k is a divisor graph and let D be a divisor orientation of C_n^k. In view of Lemma 1, we can assume that $(1, 2) \in E(D)$. Then, by Corollary 1, the vertex 1 is a transmitter while the vertex 2 is a receiver in D. Thus $(3, 2) \in E(D)$. Again by Corollary 1, the vertex 3 is a transmitter in D. But $k \geq 2$ and hence $13 \in E(C_n^k)$, which implies that either $(1, 3) \in E(D)$ or $(3, 1) \in E(D)$. This leads to a contradiction in any case because both 1 and 3 are transmitters in D. □

It was proved in Al-Addasi et al. [1] that C_n^k is not a divisor graph for $n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3$ and for odd n with $2k+2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$. The proof of the previous theorem provides a simpler argument for these two cases and completes the remaining case when n is even and $2k+2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$.

Now we are in a position to give a complete characterization of those powers of cycles that are divisor graphs.

Theorem 2. *For any integer $k \geq 2$, the graph C_n^k is a divisor graph if and only if $n \leq 2k + 2$.*

Proof. The result follows from Theorem 1 and Lemma 3. □

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