On the generalized k-Fibonacci hyperbolic functions

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Abstract

The hyperbolic Fibonacci function, which is the being extension of Binet's formula for the Fibonacci number in continuous domain, transform the Fibonacci number theory into "continuous" theory because every identity for the hyperbolic Fibonacci function has its discrete analogy in the framework of the Fibonacci number. In this new paper, it is defined three important generalizations of the k-Fibonacci sine, cosine and quasi-sine hyperbolic functions and then many number of concepts and techniques that we learned in a standard setting for the k-Fibonacci sine, cosine and quasi-sine hyperbolic functions is carried over to the generalizations of these functions.

Keywords: Hyperbolic functions; Fibonacci hyperbolic functions.

1 Introduction

One of the major achievements of modern science is an understanding that the world of Nature is hyperbolic.The theory of hyperbolic functions has

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developed in ways that, at first sight, does not appear to have any connection to hyperbolic functions. However, a new class of the hyperbolic functions based on the Golden Section could have far going consequences for future progress of mathematics, physics, biology and cosmology. Also, it does relate to the theory of Fibonacci numbers, an actively developing branch of modern mathematics [1-3]. The Fibonacci numbers are given by the sequence $0, 1, 1, 2, 3, 5, \ldots$ where each term is the sum of the previous two. This sequence can be defined via the recursive formulas: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ [2]. It is well known that the ratio of two consecutive classical Fibonacci numbers converges to Golden Mean, or Golden Section, $\tau = \frac{1+\sqrt{5}}{2}$.

In 1993, the Ukrainian mathematicians Alexey Stakhov and Ivan Tkachenko developed a new approach to the theory of hyperbolic functions [4]. Using the so-called *Binet formulas*, they developed a new class of hyperbolic functions called *hyperbolic Fibonacci and Lucas functions* [4-5]. This idea was further developed in Stakhov and Rozin's paper [6] where they defined a class of *symmetric hyperbolic Fibonacci and Lucas functions.* In Stakhov and Rozin's article [7] a new surface of the second degree called the *Golden Shofar* was developed. The hyperbolic Fibonacci and Lucas functions and the Golden Shofar surface are the most important ingredients of the *"golden" mathematical models* applicable to the description of the "hyperbolic worlds" of Nature.

In 2008, k-Fibonacci sequence $\{F_{k,n}\}$ was defined by Falcón and Plaza [8]. The k-Fibonacci sequence generalizes, between others, both the classic Fibonacci sequence and the Pell sequence. In [8], Falcón and Plaza showed the relation between the 4-triangle longest-edge (4TLE) partition and the k-Fibonacci numbers, as another example of the relation between geometry and numbers, and many properties of these numbers are deduced directly from elementary matrix algebra. In [9], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In [10], the 3-dimensional k-Fibonacci spirals are studied from a geometric point of view. These curves appear naturally from studying the k-th Fibonacci numbers and the related hyperbolic k-Fibonacci functions.

We predict that hyperbolic Fibonacci and Lucas functions will have great importance for the future development of Fibonacci and Lucas number theory. They generalize Fibonacci and Lucas numbers to the continuous domain since Fibonacci and Lucas numbers are embedded in them. Each discrete identity for Fibonacci and Lucas numbers has its continuous analogue in the form of a corresponding identity for the hyperbolic Fibonacci and Lucas functions, and conversely. Therefore, the theory of the hyperbolic Fibonacci and Lucas functions is more general than traditional Fibonacci and Lucas number theory.

The purpose of the present article is to develop continuous functions within the Fibonacci number theory which lead to the hyperbolic Fibonacci and Lucas functions [10, 6, 7, 12, 13] and the sinusoidal Fibonacci and k-Fibonacci functions (17) and (18). In this new paper, we will extend the results (3), (4) and (18) for the area of the generalized k-Fibonacci sequence.

2 The generalized k-Fibonacci hyperbolic functions

Definition 1 *For any positive real numbers $k, t > 0$; the nth generalized (k,t)-Fibonacci sequence, say $\{G_{k,t,n}\}_{n\in\mathrm{N}}$ is defined recurrently by*

$$G_{k,t,n+1} = sG_{k,t,n} + tG_{k,t,n-1} \quad \textit{for } n \geq 1, \tag{1}$$

where $G_{k,t,0} = 0$ and $G_{k,t,1} = 1$.

Particular case of the generalized (k,t)-Fibonacci sequence $\{G_{k,t,n}\}$ for

$t = 1$ is the k-Fibonacci sequence $\{F_{k,n}\}$.

Binet's formula for the generalized (k,t)-Fibonacci sequence $\{G_{k,t,n}\}$ is given by (see [14])

$$G_{k,t,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{2}$$

where r_1, r_2 are the roots of the characteristic equation (1) defining $G_{k,t,n}$.

Alexey Stakhov, Ivan Tkachenko, Boris Rozin, Sergio Falcón and Ángel Plaza developed recently a theory of the *hyperbolic Fibonacci, Lucas and k-Fibonacci functions* [10, 6, 7, 12, 13], which are extensions of Binet formulas for a continuous domain, have a strategic importance for the development of both mathematics and theoretical physics. The classical Fibonacci hyperbolic functions have been defined as

$$\mathrm{sFh}\,(x) = \frac{\tau^x - \tau^{-x}}{\sqrt{5}}$$

$$\mathrm{cFh}\,(x) = \frac{\tau^x + \tau^{-x}}{\sqrt{5}},$$

where sFh and cFh are called, respectively, the Fibonacci hyperbolic sine and cosine, and $\tau = \frac{1+\sqrt{5}}{2}$. In [10], the above functions have been extended to the k-Fibonacci hyperbolic functions in the following way:

$$\mathrm{sF}_k\mathrm{h}\,(x) = \frac{\tau_k^x - \tau_k^{-x}}{\sqrt{k^2+4}}$$

$$\mathrm{cF}_k\mathrm{h}\,(x) = \frac{\tau_k^x + \tau_k^{-x}}{\sqrt{k^2+4}},$$

where $\tau_k = \frac{k+\sqrt{k^2+4}}{2}$ is the positive root of the characteristic equation associated to the k-Fibonacci sequence. Note that the hyperbolic Fibonacci and kFibonacci functions are connected to the Fibonacci and k-Fibonacci numbers by the following correlations:

$$F_n = \begin{cases} \mathrm{sFh}\,(n), & \text{for } n = 2k \\ \mathrm{cFh}\,(n), & \text{for } n = 2k+1 \end{cases},$$

and

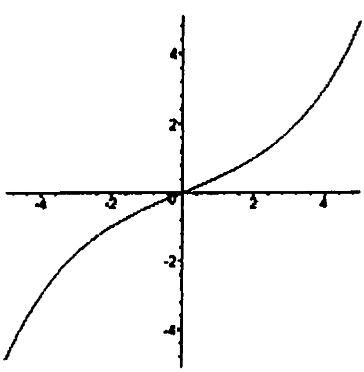
$$F_{k,n} = \begin{cases} \mathrm{sF}_k\mathrm{h}\,(n)\,, & \text{for } n = 2k \\ \mathrm{cF}_k\mathrm{h}\,(n)\,, & \text{for } n = 2k+1 \end{cases}.$$

From now on, in analogous way followed by Stakhov, Rozin [6] and Falcón, Plaza [10] the so-called the generalized k-Fibonacci hyperbolic sine and cosine functions are, respectively, defined as follows:

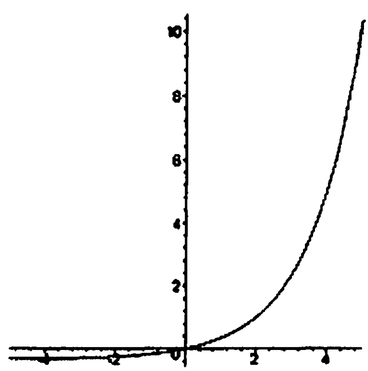
$$\mathrm{sG}_{k,t}\mathrm{h}\,(x) = \frac{\tau_{k,t}^{x} - t^{x}\tau_{k,t}^{-x}}{\delta} \tag{3}$$

$$\mathrm{cG}_{k,t}\mathrm{h}\,(x) = \frac{\tau_{k,t}^{x} + t^{x}\tau_{k,t}^{-x}}{\delta}, \tag{4}$$

where $\delta = \sqrt{k^2+4t}$ and $\tau_{k,t}$ is the positive root of the characteristic equation (1) associated to the generalized k-Fibonacci sequence, that is $\tau_{k,t} = \frac{k+\sqrt{k^2+4t}}{2}$. For some the values k and t, the graphics of these new generalized k-Fibonacci hyperbolic sine and cosine functions are shown in Fig. 1 in below.



$\mathrm{SG}_{1,1}\mathrm{h}(x)$



$\mathrm{sG}_{1,2}\mathrm{h}(x)$

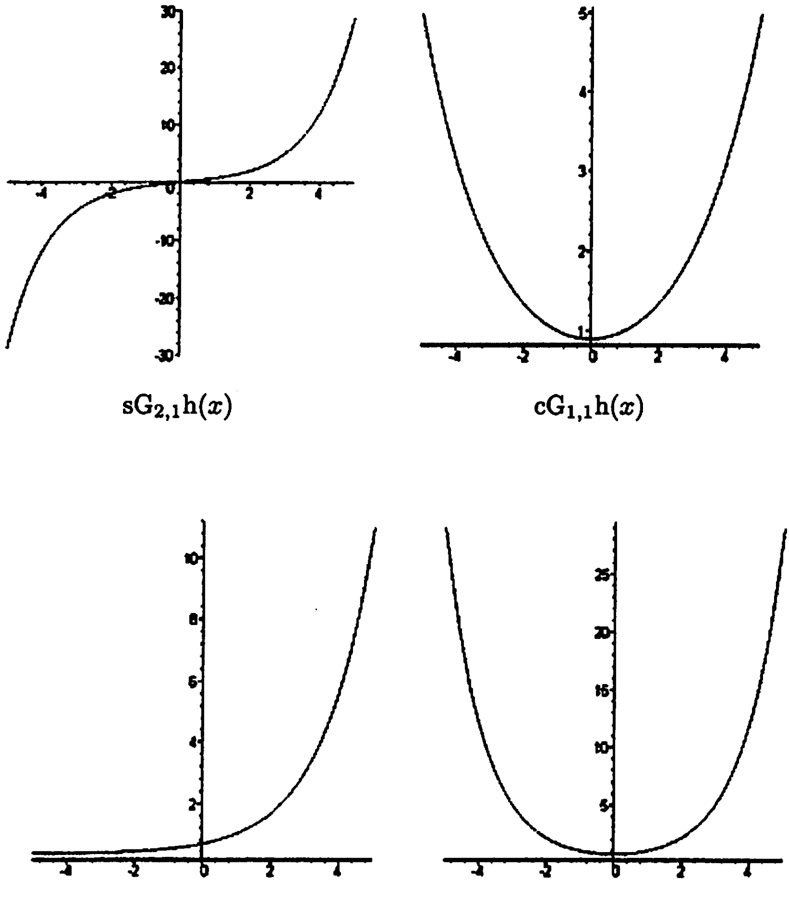


Fig. 1. For some the values k, t, the generalized k-Fibonacci hyperbolic sine and cosine functions.

From (2) Binet's formula for the generalized k-Fibonacci sequence establishes that the general term of this sequence can be written as:

$$G_{k,t,n} = \frac{\tau_{k,t}^{n} - (-t)^{n}\,\tau_{k,t}^{-n}}{\delta}. \tag{5}$$

Notice that considering $(3-5)$, these functions verify the property that, if

x is an even number, $x = 2n$ then $\mathrm{sG}_{k,t}\mathrm{h}(x) = G_{k,t,2n}$, while if x is an odd number, $x = 2n+1$ then $\mathrm{sG}_{k,t}\mathrm{h}(x) = G_{k,t,2n+1}$.

From now on for simplicity we will write τ instead of $\tau_{k,t}$.

We now present several results associated to the generalized k-Fibonacci hyperbolic functions.

Theorem 2

$$[cG_{k,t}h\,(x)]^2 - [sG_{k,t}h\,(x)]^2 = \frac{4t^x}{k^2+4t}.$$

Proof. Let $\delta = \sqrt{k^2+4t}$. By the definitions of the generalized k-Fibonacci hyperbolic functions, we get

$$\begin{aligned}[\mathrm{cG}_{k,t}\mathrm{h}\,(x)]^2 - [\mathrm{sG}_{k,t}\mathrm{h}\,(x)]^2 &= \left[\frac{\tau^x + t^x\tau^{-x}}{\delta}\right]^2 - \left[\frac{\tau^x - t^x\tau^{-x}}{\delta}\right]^2 \\ &= \frac{4t^x}{\delta^2},\ \text{after some algebra}\end{aligned}$$

which proves the theorem. ■

This theorem given for the generalized k-Fibonacci hyperbolic functions can be seen as a version of famous the Pythagorean Theorem given for the classical hyperbolic functions.

Particular case is:

- If $t = 1$, the Pythagorean Theorem for the k-Fibonacci hyperbolic functions appears [10, Proposition 1]:

$$[\mathrm{cF}_k\mathrm{h}\,(x)]^2 - [\mathrm{sF}_k\mathrm{h}\,(x)]^2 = \frac{4}{k^2+4}.$$

Theorem 3

$$\frac{2cG_{k,t}h\,(x+y)}{\sqrt{k^2+4t}} = [cG_{k,t}h\,(x)\,.cG_{k,t}h\,(y) + sG_{k,t}h\,(x)\,.cG_{k,t}h\,(y)]\,, \qquad (6)$$

$$\frac{2t^y cG_{k,t}h\,(x-y)}{\sqrt{k^2+4t}} = [cG_{k,t}h\,(x)\,.cG_{k,t}h\,(y) - sG_{k,t}h\,(x)\,.cG_{k,t}h\,(y)]\,,$$

$$\frac{2sG_{k,t}h(x+y)}{\sqrt{k^2+4t}} = [sG_{k,t}h(x)\, cG_{k,t}h(y) + cG_{k,t}h(x)\, cG_{k,t}h(y)], \quad (7)$$

$$\frac{2t^y sG_{k,t}h(x-y)}{\sqrt{k^2+4t}} = [sG_{k,t}h(x)\, .cG_{k,t}h(y) - cG_{k,t}h(x)\, .cG_{k,t}h(y)].$$

Proof. Let $\delta = \sqrt{k^2+4t}$ and

$$LHS = cG_{k,t}h(x)\, .cG_{k,t}h(y) + sG_{k,t}h(x)\, .cG_{k,t}h(y).$$

Let us prove the identity (6):

$$\begin{aligned} LHS &= \left[\left(\frac{\tau^x + t^x\tau^{-x}}{\delta}\right)\left(\frac{\tau^y + t^y\tau^{-y}}{\delta}\right) + \right. \\ &\quad \left. + \left(\frac{\tau^x - t^x\tau^{-x}}{\delta}\right)\left(\frac{\tau^y - t^y\tau^{-y}}{\delta}\right)\right] \\ &= \frac{2}{\delta^2}\left(\tau^{x+y} + t^{x+y}\tau^{-(x+y)}\right), \text{ after some algebra} \end{aligned}$$

from where the identity (6) is obtained. ∎

By doing $y = x$ in the Eq. (6) and (7), the following corollary is obtained.

Corollary 4

$$cG_{k,t}h(2x) = \frac{\sqrt{k^2+4t}}{2}\left[(cG_{k,t}h(x))^2 + (sG_{k,t}h(x))^2\right], \quad (8)$$

$$sG_{k,t}h(2x) = \sqrt{k^2+4t}.sG_{k,t}h(x)\, .cG_{k,t}h(y).$$

Corollary 5

$$[cG_{k,t}h(x)]^2 = \frac{1}{\sqrt{k^2+4t}}\left[cG_{k,t}h(2x) + \frac{2t^x}{\sqrt{k^2+4t}}\right], \quad (9)$$

$$[sG_{k,t}h(x)]^2 = \frac{1}{\sqrt{k^2+4t}}\left[cG_{k,t}h(2x) - \frac{2t^x}{\sqrt{k^2+4t}}\right].$$

Proof. Now, let us prove the identity (9). Let $\delta = \sqrt{k^2+4t}$. From Eq. (8) we have

$$\begin{aligned} [\mathrm{cG}_{k,t}\mathrm{h}(x)]^2 &= \frac{2}{\delta}\mathrm{cG}_{k,t}\mathrm{h}(2x) - [\mathrm{sG}_{k,t}\mathrm{h}(x)]^2 \\ &= \frac{2}{\delta}\mathrm{cG}_{k,t}\mathrm{h}(2x) - \frac{\tau^{2x} + t^{2x}\tau^{-2x} - 2t^x}{\delta^2} \\ &= \frac{1}{\delta}\mathrm{cG}_{k,t}\mathrm{h}(2x) + \frac{2t^x}{\delta^2}, \end{aligned}$$

which completes the proof. ■

The generalized k-Fibonacci hyperbolic functions have recurrent properties that are similar to the generalized k-Fibonacci numbers as shown in below.

Theorem 6

$$sG_{k,t}h(x+1) = kcG_{k,t}h(x) + tsG_{k,t}h(x-1) - \frac{t^x \tau^{-x+1}(1-t)}{\sqrt{k^2+4t}}. \quad (10)$$

Proof. Let $\delta = \sqrt{k^2+4t}$ and $LHS = k\mathrm{cG}_{k,t}\mathrm{h}(x) + t\mathrm{sG}_{k,t}\mathrm{h}(x-1)$. Thus, by definitions (3) and (4), we obtain

$$\begin{aligned} LHS &= k\left(\frac{\tau^x + t^x\tau^{-x}}{\delta}\right) + t\left(\frac{\tau^{x-1} - t^{x-1}\tau^{-(x-1)}}{\delta}\right) \\ &= \frac{k\tau^x + t\tau^{x-1} + t^x\left(k\tau^{-x} - t\tau^{-(x-1)}\right)}{\delta} \\ &= \frac{\tau^{x-1}(k\tau+t) + t^x\left(\frac{k-t\tau}{\tau^x}\right)}{\delta} \\ &= \frac{\tau^{x+1} + t^x\left(\frac{k\tau-t\tau^2}{\tau^{x+1}}\right)}{\delta}, \text{ since } \tau^2 = k\tau + t \\ &= \frac{\tau^{x+1} + t^x\left(\frac{\tau^2-t-t\tau^2}{\tau^{x+1}}\right)}{\delta} \\ &= \frac{\tau^{x+1} - t^{x+1}\tau^{-(x+1)}}{\delta} + \frac{t^x\tau^{-x+1}(1-t)}{\delta}. \end{aligned}$$

Thus, the result is obtained. ■

Particular case is:

- If $t = 1$ in Eq. (1), the k-Fibonacci numbers verify that

$$F_{k,0} = 0, F_{k,1} = 1, F_{k,n+1} = F_{k,n} + F_{k,n-1}.$$

In Eq. (10), an analogous equation for the k-Fibonacci hyperbolic functions is the following [10, Proposition 1]

$$\mathrm{sF}_k\mathrm{h}(x+1) = k\mathrm{cF}_k\mathrm{h}(x) + \mathrm{sF}_k\mathrm{h}(x-1).$$

Now, we present several versions of Catalan's identity for the generalized k-Fibonacci hyperbolic functions via the following theorem.

Theorem 7

$$cG_{k,t}h(x-r).cG_{k,t}h(x+r)-[cG_{k,t}h(x)]^2=t^{x-r}[sG_{k,t}h(r)]^2,$$

$$cG_{k,t}h(x-r).cG_{k,t}h(x+r)-[sG_{k,t}h(x)]^2=t^{x-r}[cG_{k,t}h(r)]^2, \quad (11)$$

$$sG_{k,t}h(x-r).sG_{k,t}h(x+r)-[sG_{k,t}h(x)]^2=-t^{x-r}[sG_{k,t}h(r)]^2, \quad (12)$$

$$sG_{k,t}h(x-r).sG_{k,t}h(x+r)-[cG_{k,t}h(x)]^2=-t^{x-r}[cG_{k,t}h(r)]^2. \quad (13)$$

Proof. Let $\delta=\sqrt{k^2+4t}$ and

$$LHS=sG_{k,t}h(x-r).sG_{k,t}h(x+r)-[\mathrm{sG}_{k,t}\mathrm{h}(x)]^2.$$

Let us prove the identity (12). By definitions (3) and (4), we have

$$\begin{aligned} LHS &= \frac{1}{\delta^2}\Big[\left(\tau^{x-r}-t^{x-r}\tau^{-(x-r)}\right)\left(\tau^{x+r}-t^{x+r}\tau^{-(x+r)}\right) \\ &\quad -\left(\tau^{x}-t^{x}\tau^{-x}\right)^2\Big] \\ &= -\frac{t^{x-r}}{\delta^2}\left(\tau^{2r}+t^{2r}\tau^{-2r}-2t^r\right) \\ &= -t^{x-r}\left(\frac{\tau^r-t^r\tau^{-r}}{\delta}\right)^2, \end{aligned}$$

which proves the identity. ■

By doing $r=1$ in Eq. (11) and (13), the following identities (14) and (15) are a generalization of the "Cassini formula" known as

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n,$$

for the generalized k-Fibonacci hyperbolic functions.

Theorem 8

$$[sG_{k,t}h(x)]^2-cG_{k,t}h(x-1).cG_{k,t}h(x+1)=-t^{x-1}, \quad (14)$$

$$[cG_{k,t}h(x)]^2-sG_{k,t}h(x-1).sG_{k,t}h(x+1)=t^{x-1}. \quad (15)$$

Theorem 9 *If*

$$LHS1 = cG_{k,t}h\left(x\right).cG_{k,t}h\left(y+r\right) - sG_{k,t}h\left(x+r\right).sG_{k,t}h\left(y\right),$$

then

$$LHS1 = t^{y}cG_{k,t}h\left(r\right).cG_{k,t}h\left(x-y\right), \tag{16}$$

and, similarly, if

$$LHS2 = cG_{k,t}h\left(x\right).sG_{k,t}h\left(y+r\right) - cG_{k,t}h\left(x+r\right).sG_{k,t}h\left(y\right),$$

then

$$LHS2 = t^{y}sG_{k,t}h\left(r\right).cG_{k,t}h\left(x-y\right).$$

Proof. We only will prove Eq. (16). Let $\delta = \sqrt{k^2+4t}$ and $LHS1 = \mathrm{cG}_{k,t}\mathrm{h}(x).\mathrm{cG}_{k,t}\mathrm{h}(y+r) - \mathrm{sG}_{k,t}\mathrm{h}(x+r).\mathrm{sG}_{k,t}\mathrm{h}(y)$. Hence, by definitions (3) and (4), we have

$$\begin{aligned}
LHS1 &= \frac{1}{\delta^2}\Big[\left(\tau^{x}+t^{x}\tau^{-x}\right)\left(\tau^{y+r}+t^{y+r}\tau^{-(y+r)}\right) \\
&\quad -\left(\tau^{x+r}-t^{x+r}\tau^{-(x+r)}\right)\left(\tau^{y}-t^{y}\tau^{-y}\right)\Big] \\
&= \frac{1}{\delta^2}\left[t^{y+r}\tau^{x-y-r}+t^{x}\tau^{-x+y+r}+\right. \\
&\quad \left.+t^{y}\tau^{x-y-r}+t^{x+r}\tau^{-x+y+r}\right] \\
&= \frac{1}{\delta^2}\Big[t^{y+r}\tau^{-r}\left(\tau^{x-y}+t^{x-y}\tau^{-(x-y)}\right)+ \\
&\quad t^{y}\tau^{r}\left(\tau^{x-y}+t^{x-y}\tau^{-(x-y)}\right)\Big] \\
&= t^{y}\mathrm{cG}_{k,t}\mathrm{h}\left(r\right).\mathrm{cG}_{k,t}\mathrm{h}\left(x-y\right).
\end{aligned}$$

■

3 The generalized quasi-sine k-Fibonacci function

The sinusoidal Fibonacci function [7] is a further development of a continuous approach to the Fibonacci numbers theory begun in a series of papers

[12,13]. The definition of the function is

$$\mathrm{F}(x) = \frac{\tau^{x} - \cos(\pi x)\,\tau^{-x}}{\sqrt{5}}, \tag{17}$$

where $\tau = \frac{1+\sqrt{5}}{2}$. On the other hand, Falcón and Plaza [10] have defined the quasi-sine k-Fibonacci function, which is a generalization of the sinusoidal Fibonacci function, by

$$\mathrm{FF}_k(x) = \frac{\tau_k^{x} - \cos(\pi x)\,\tau_k^{-x}}{\sqrt{k^2+4}}, \tag{18}$$

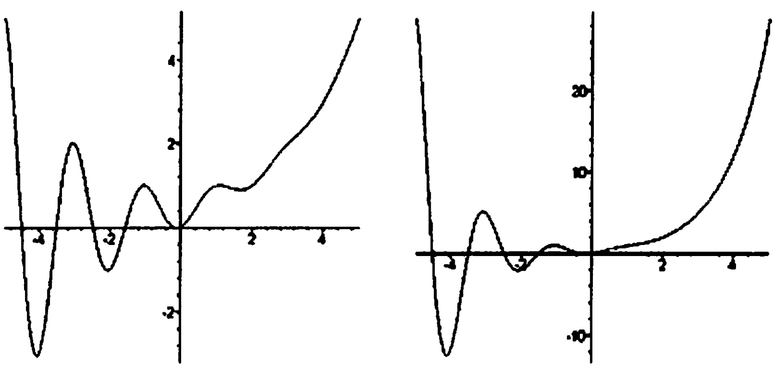
where $\tau = \frac{k+\sqrt{k^2+4}}{2}$. Considering Eq. (5) given for the generalized k-Fibonacci numbers and taking into account that $\cos(n\pi) = (-1)^n$, naturally we introduce the following definition.

Definition 10 *The generalized quasi-sine k-Fibonacci function is defined by*

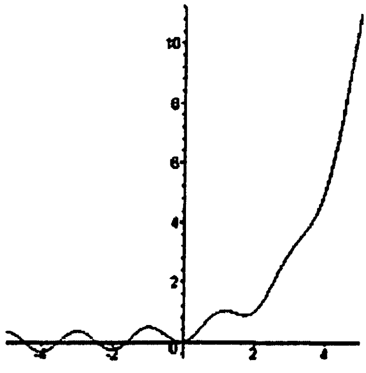
$$FG_{k,t}(x) = \frac{\tau^{x} - \cos(\pi x)\,t^{x}\tau^{-x}}{\sqrt{k^2+4t}}, \tag{19}$$

where $\tau = \tau_{k,t}$ is the positive root of the characteristic equation (1).

For some the values k and t, the graphics of this new generalized quasi-sine k-Fibonacci hyperbolic functions are shown in Fig. 2 in below.



$\mathrm{FG}_{1,1}(x)$ $\qquad$ $\mathrm{FG}_{2,1}(x)$



$\mathrm{FG}_{1,2}(x)$

Fig. 2. For some the values k, t, the generalized quasi-sine k-Fibonacci hyperbolic functions.

In [10], Falcón and Plaza have given several identites for these quasi-sine k-Fibonacci functions. We now extend these identities for the generalized quasi-sine k-Fibonacci.

Theorem 11

$$FG_{k,t}(x+1) = kFG_{k,t}(x) + tFG_{k,t}(x-1) - \frac{t^x \tau^{-x+1}(1-t)}{\sqrt{k^2+4t}} \qquad (20)$$

Proof. Let $\delta = \sqrt{k^2+4t}$. Thus, by definitions (3) and (4), we obtain

$$\begin{aligned}
k\mathrm{FG}_{k,t}(x) + t\mathrm{FG}_{k,t}(x-1) &= k\left(\frac{\tau^x - \cos(\pi x)\, t^x \tau^{-x}}{\delta}\right) + \\
&\quad + t\left(\frac{\tau^{x-1} - \cos(\pi(x-1))\, t^{x-1}\tau^{-(x-1)}}{\delta}\right) \\
&= \frac{1}{\delta}\left\{k\tau^x + t\tau^{x-1} - \cos(\pi x)\, t^x \times \right. \\
&\quad \left. \times \left(k\tau^{-x} - t\tau^{-(x-1)}\right)\right\} \\
&= \frac{\tau^{x-1}(k\tau+t) - \cos(\pi x)\, t^x \left(\frac{k-t\tau}{\tau^x}\right)}{\delta} \\
&= \frac{\tau^{x+1} - \cos(\pi x)\, t^x \left(\frac{k\tau - t\tau^2}{\tau^{x+1}}\right)}{\delta} \\
&= \frac{\tau^{x+1} - \cos(\pi x)\, t^x \left(\frac{\tau^2 - t - t\tau^2}{\tau^{x+1}}\right)}{\delta} \\
&= \frac{\tau^{x+1} - \cos(\pi(x+1))\, t^{x+1}\tau^{-(x+1)}}{\delta} + \\
&\quad + \frac{t^x \tau^{-x+1}(1-t)}{\delta} \\
&= \mathrm{FG}_{k,t}(x+1) + \frac{t^x\tau^{-x+1}(1-t)}{\delta}.
\end{aligned}$$

Thus, the result is obtained. ■

Particular case is:

- If $t = 1$ in Eq. (1), the k-Fibonacci numbers verify that $F_{k,0} = 0$, $F_{k,1} = 1$, $F_{k,n+1} = F_{k,n} + F_{k,n-1}$. In Eq. (20), an analogous equation for the k-Fibonacci hyperbolic functions is the following [10, Theorem 13]

$$\mathrm{FF}_k\mathrm{h}(x+1) = k\mathrm{FF}_k\mathrm{h}(x) + \mathrm{FF}_k\mathrm{h}(x-1).$$

Similarly, Catalan's identity for the generalized quasi-sine kFibonacci functions is given by:

Theorem 12 *Let* $LHS = FG_{k,t}(x-r).FG_{k,t}(x+r) - [FG_{k,t}(x)]^2$. *Then,* *for* $r \in \mathbb{Z}$,

$$LHS = (-1)^{r+1} t^{x-r} \cos(\pi x) [FG_{k,t}(r)]^2 \tag{21}$$

Proof. Let $\delta = \sqrt{k^2+4t}$. By definition (19), we get

$$\begin{aligned}
LHS &= \frac{1}{\delta^2}\left[\left(\tau^{x-r} - \cos(\pi(x-r))\, t^{x-r}\tau^{-(x-r)}\right) \times \right.\\
&\times \left(\tau^{x+r} - \cos(\pi(x+r))\, t^{x+r}\tau^{-(x+r)}\right) - \\
&\left. - \left(\tau^{x} - \cos(\pi x)\, t^{x}\tau^{-x}\right)^2\right] \\
&= \frac{1}{\delta^2}\left[\left(\tau^{x-r} + (-1)^{r+1}\cos(\pi x)\, t^{x-r}\tau^{-x+r}\right) \times \right.\\
&\times \left(\tau^{x+r} + (-1)^{r+1}\cos(\pi x)\, t^{x+r}\tau^{-x-r}\right) \\
&\left. - \left(\tau^{2x} + \cos^2(\pi x)\, t^{2x}\tau^{-2x} - 2\cos(\pi x)\, t^{x}\right)\right] \\
&= \frac{1}{\delta^2}\left[(-1)^{r+1}\cos(\pi x)\, t^{x-r}\tau^{2r} + \right.\\
&\left. + (-1)^{r+1}\cos(\pi x)\, t^{x+r}\tau^{-2r} + 2\cos(\pi x)\, t^{x}\right] \\
&= (-1)^{r+1} t^{x-r}\cos(\pi x)\left(\frac{\tau^{r} - \cos(\pi r)\, t^{r}\tau^{-r}}{\delta}\right)^2 \\
&= (-1)^{r+1} t^{x-r}\cos(\pi x)\,[\mathrm{FG}_{k,t}(r)]^2,
\end{aligned}$$

which proves the identity (21). ■

Particular case is:

- If $t = 1$ in Eq. (1), for the quasi-sine k-Fibonacci functions we have [10, Theorem 14]:

$$\mathrm{FF}_k(x-r).\mathrm{FF}_k(x+r) - [\mathrm{FF}_k(x)]^2 = (-1)^{r+1}\cos(\pi x)\,[\mathrm{FF}_k(r)]^2 \ .$$

Theorem 13 *For* $r \in \mathbb{Z}$ *and* $t < 1 + s$,

$$\lim_{x\to\infty}\frac{FG_{k,t}(x+r)}{FG_{k,t}(x)} = \tau^{r}.$$

Proof. Since $\tau > 1$ and, for $t < 1+s$, $\frac{t}{\tau} < 1$, we obtain

$$\begin{aligned}
\lim_{x\to\infty}\frac{\mathrm{FG}_{k,t}(x+r)}{\mathrm{FG}_{k,t}(x)} &= \lim_{x\to\infty}\frac{\tau^{x+r}-\cos(\pi(x+r))\,t^{x+r}\tau^{-(x+r)}}{\tau^{x}-\cos(\pi x)\,t^{x}\tau^{-x}} \\
&= \lim_{x\to\infty}\frac{\tau^{r}+(-1)^{r+1}\cos(\pi x)\,t^{x+r}\frac{1}{\tau^{2x+r}}}{1-\cos(\pi x)\,t^{x}\frac{1}{\tau^{2x}}} \\
&= \tau^{r}.
\end{aligned}$$

Thus, the proof is completed. ■

4 Conclusion

A new class of the hyperbolic functions based on the Golden Section could have far going consequences for future progress of mathematics, physics, biology and cosmology. In the first place, the hyperbolic Fibonacci and Lucas functions which are the being extension of Binet's formulas for the Fibonacci and Lucas numbers in continuous domain transform the Fibonacci numbers theory into "continuous" theory because every identity for the hyperbolic Fibonacci and Lucas functions has its discrete analogy in the framework of the Fibonacci and Lucas number theory. In the other words, the theory of Fibonacci and Lucas numbers are being the "discrete" case of the theory of the hyperbolic Fibonacci and Lucas functions. Considering the fundamental role of the classical hyperbolic functions in the mathematical tools of the modern science, it is possible to suppose that the new theory of the hyperbolic functions will bring the new results and interpretations in various spheres of natural science.

The creator of non-Euclidean geometry was the Russian mathematician Nikolay Lobachevsky who derived a new geometric system based on hyperbolic functions in 1827. The need for new geometrical ideas became apparent in physics at the beginning of the 20$^{\text{th}}$ century as the result of Einstein's Special Theory of Relativity (1905). In 1908, three years after the publication of this great work, the German mathematician Herman Minkowsky gave a geometrical interpretation of the Special Theory of Relativity based on hyperbolic ideas.

A consequence of the hyperbolic Fibonacci functions' introduction has been the realization that the classical hyperbolic functions, which are useful in mathematics and theoretical physics, are not the only tools for creating mathematical models of the "hyperbolic world". In addition to models based on classical hyperbolic functions (Lobachevsky's hyperbolic geometry, Minkowsky's geometry, etc.), there is a golden hyperbolic world based on hyperbolic Fibonacci and Lucas functions [4-7]

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