Variations of two classical Turán-type extremal results *

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Abstract. We consider a variation of a classical Turán-type extremal problem due to Bollobás [2, p. 398, no. 13] as follows: determine the smallest even integer $\sigma(C^k, n)$ such that every graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with term sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(C^k, n)$ has a realization G containing a cycle with k chords incident to a vertex on the cycle. Moreover, we also consider a variation of a classical Turán-type extremal result due to Faudree and Schelp [7] as follows: determine the smallest even integer $\sigma(P_\ell, n)$ such that every graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $\sigma(\pi) \geq \sigma(P_\ell, n)$ has a realization G containing P_ℓ as a subgraph, where P_ℓ is the path of length ℓ. In this paper, we determine the values of $\sigma(P_\ell, n)$ for $n \geq \ell + 1$ and the values of $\sigma(C^k, n)$ for $n \geq (k+3)(2k+5)$.
Keywords. graph, degree sequence, potentially H-graphic sequence.
Mathematics Subject Classification(2000): 05C35, 05C07

1. Introduction

The set of all non-increasing nonnegative integer sequences $\pi = (d_1, d_2, \ldots, d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic sequences in NS_n is denoted by GS_n. For a nonnegative integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, define $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. A *chord* of a cycle C is an edge not in C whose endpoints lie in C. For a given graph H, a sequence $\pi \in GS_n$ is said to be *potentially* (resp. *forcibly*) *H-graphic* if there exists a realization

*Supported by NNSF of China (Nos. 10861006 and 10401010), the 2009 Scientific Research Foundation of Hainan University (No. hd09xm87) and SRF for ROCS, SEM.
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of π containing H as a subgraph (resp. each realization of π contains H as a subgraph). Moreover, a sequence $\pi \in GS_n$ is said to be *potentially* (resp. *forcibly*) *C^k-graphic* if there exists a realization of π containing a cycle with k chords incident to a vertex on the cycle (resp. each realization of π contains a cycle with k chords incident to a vertex on the cycle).

Let $e(G)$ be the number of edges in the graph G. It is well known (see [2], chapter 6 for example) that one of the classical extremal problems in extremal graph theory is to determine the smallest positive integer $t(H,n)$ such that every graph G on n vertices with $e(G) \geq t(H,n)$ contains H as a subgraph. The number $t(H,n)$ is called the *Turán number* of H. The classical Turán theorem (see [2], chapter 6) determined the Turán number $t(K_r,n)$ for K_r, the complete graph on r vertices. Faudree and Schelp [7] proved that the Turán number $t(P_\ell,n)$ for P_ℓ is $\binom{\ell}{2}\lfloor\frac{n}{\ell}\rfloor + \binom{r}{2} + 1$, where $r \equiv n \bmod (\ell)$. Let $t(C^k,n)$ denote the smallest positive integer such that every graph G on n vertices with $e(G) \geq t(C^k,n)$ contains a cycle with k chords incident to a vertex on the cycle. Since the complete bipartite graph $K_{k+1,n-(k+1)}$ and the graph with at most $k+2$ vertices contain no cycles with k chords incident to a vertex on the cycle, it follows that $t(C^k,n) \geq (k+1)n - (k+1)^2 + 1$ for $n \geq k+3$. In [1], Alon used the function $t(C^k,n)$ to give an upper bound on anti-Ramsey function. Erdös conjectured that $t(C^k,n) = (k+1)n - (k+1)^2 + 1$ for $n \geq 2k+2$. This was disproved by Lewin [2, p. 398, no. 12] for $2k \leq n < \frac{5(k-1)}{2}$. Bollobás [2, p. 398, no. 13] conjectured that there exists a function $n(k)$ such that $t(C^k,n) = (k+1)n - (k+1)^2 + 1$ for all $n \geq n(k)$. Recently, Jiang [9] proved the conjecture, and showed that $n(k) \leq 3k+3$.

In terms of graphic sequences, the number $2t(H,n)$ (resp. $2t(C^k,n)$) is the smallest even integer such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geq 2t(H,n)$ (resp. $2t(C^k,n)$) is forcibly H (resp. C^k)-graphic. In [6], Erdös, Jacobson and Lehel considered the following variation of the classical Turán number $t(K_r,n)$: determine the smallest even integer $\sigma(K_r,n)$ such that every sequence $\pi \in GS_n$ without zero terms and with $\sigma(\pi) \geq \sigma(K_r,n)$ is potentially K_r-graphic. They showed that $\sigma(K_3,n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_r,n) = (r-2)(2n-r+1)+2$ for sufficiently large n. Gould et al. [8] and Li and Song [13] independently proved it for $r = 4$. Recently, Li et al. [14,15] proved that the conjecture is true for $r = 5$ and $n \geq 10$ and for $r \geq 6$ and $n \geq \binom{r-1}{2} + 3$. In [8], Gould et al. generalized the above variation as follows: for a given graph H, determine the smallest even integer $\sigma(H,n)$ such that every sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H,n)$ is potentially H-graphic. The purpose of the paper is to consider the variations of two classical Turán numbers $t(P_\ell,n)$ and $t(C^k,n)$. The paper is organized as follows. In section 2, we will determine the values of $\sigma(P_\ell,n)$ for $n \geq \ell+1$ (see Theorem 2.5). In section 3, we will consider the variation of the classical Turán number $t(C^k, n)$, that is, we will determine the smallest even integer $\sigma(C^k, n)$ such that every sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(C^k, n)$ is potentially C^k-graphic for $n \geq (k+3)(2k+5)$ (see Theorem 3.8).

2. $\sigma(P_\ell, n)$ for $n \geq \ell + 1$

In order to determine $\sigma(P_\ell, n)$, we need the following known results.

Theorem 2.1 [16] Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $n \geq 3$ and $d_n \geq 1$. Then π is potentially C_3-graphic if and only if $d_3 \geq 2$ except for two cases: $\pi = (2^4)$ and $\pi = (2^5)$, where C_ℓ is a cycle of length ℓ and the symbol x^y in a sequence stands for y consecutive terms, each equal to x.

Theorem 2.2 [16] Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_n \geq 1$. Then π is potentially C_4-graphic if and only if all the following conditions must be satisfied:

(1) $d_4 \geq 2$;

(2) $d_1 = n-1$ implies that $d_2 \geq 3$;

(3) If $n = 5, 6$, then $\pi \neq (2^n)$.

Theorem 2.3 [12,19] Let $\ell \geq 5$ and $n \geq \ell$. Then

$$\sigma(C_\ell, n) = \begin{cases} (m-1)(2n-m)+2 & \text{if } \ell = 2m-1 \text{ and } n \geq \frac{5m-5}{2}, \\ 2n+4m^2-14m+12 & \text{if } \ell = 2m-1 \text{ and } n \leq \frac{5m-5}{2}, \\ (m-1)(2n-m)+4 & \text{if } \ell = 2m \text{ and } n \geq \frac{5m-1}{2}, \\ 2n+4m^2-10m+6 & \text{if } \ell = 2m \text{ and } n \leq \frac{5m-1}{2}. \end{cases}$$

Theorem 2.4 [5] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any t, $1 \leq t \leq n-1$,

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{j=t+1}^{n} \min\{t, d_j\}.$$

We also need the following

Lemma 2.1 If $\ell \geq 3$ and $n \geq \ell + 1$, then

$$\sigma(P_\ell, n) \geq \begin{cases} (m-1)(2n-m)+2 & \text{if } \ell = 2m-1 \text{ and } n \geq \frac{5m-2}{2}, \\ 4m^2-6m+4 & \text{if } \ell = 2m-1 \text{ and } n \leq \frac{5m-2}{2}, \\ (m-1)(2n-m)+4 & \text{if } \ell = 2m \text{ and } n \geq \frac{5m+2}{2}, \\ 4m^2-2m+2 & \text{if } \ell = 2m \text{ and } n \leq \frac{5m+2}{2}. \end{cases}$$

Proof. Let $\pi = ((\ell-1)^\ell, 0^{n-\ell})$. Clearly, the only graph realizing π is $K_\ell \cup \overline{K_{n-\ell}}$, where $\overline{K_{n-\ell}}$ denotes the complement graph of $K_{n-\ell}$. Since $K_\ell \cup \overline{K_{n-\ell}}$ contains no P_ℓ, π is not potentially P_ℓ-graphic. Hence $\sigma(P_\ell, n) \geq \sigma(\pi)+2 = \ell(\ell-1)+2$, that is, $\sigma(P_{2m-1}, n) \geq 4m^2-6m+4$ and $\sigma(P_{2m}, n) \geq$

$4m^2 - 2m + 2$. Now consider $\pi = ((n-1)^{m-1}, (m-1)^{n-m+1})$. It is easy to see that $K_{m-1} + \overline{K_{n-m+1}}$ is the only graph realizing π, and has no path of length $2m-1$, so that π is not potentially P_{2m-1}−graphic, where + denotes 'join'. Hence $\sigma(P_{2m-1}, n) \geq \sigma(\pi) + 2 = (m-1)(2n-m) + 2$. By a similar argument using the degree sequence of the graph obtained from $K_{m-1} + \overline{K_{n-m+1}}$ by adding an extra edge joining two vertices of degree $m-1$, we have $\sigma(P_{2m}, n) \geq \sigma(\pi) + 2 = (m-1)(2n-m) + 4$. Thus

$$\sigma(P_\ell, n) \geq \begin{cases} (m-1)(2n-m)+2 & \text{if } \ell = 2m-1 \text{ and } n \geq \frac{5m-2}{2}, \\ 4m^2 - 6m + 4 & \text{if } \ell = 2m-1 \text{ and } n \leq \frac{5m-2}{2}, \\ (m-1)(2n-m)+4 & \text{if } \ell = 2m \text{ and } n \geq \frac{5m+2}{2}, \\ 4m^2 - 2m + 2 & \text{if } \ell = 2m \text{ and } n \leq \frac{5m+2}{2}. \end{cases}$$

□

We now prove the following Theorem 2.5 which is the main result of this section.

Theorem 2.5 (1) Let $n \geq 2$. Then $\sigma(P_1, n) = 2$.

(2) Let $n \geq 3$. Then $\sigma(P_2, n) = \begin{cases} n+1 & \text{if } n \text{ is odd}, \\ n+2 & \text{if } n \text{ is even}. \end{cases}$

(3) Let $\ell \geq 3$ and $n \geq \ell + 1$. Then

$$\sigma(P_\ell, n) = \begin{cases} (m-1)(2n-m)+2 & \text{if } \ell = 2m-1 \text{ and } n \geq \frac{5m-2}{2}, \\ 4m^2 - 6m + 4 & \text{if } \ell = 2m-1 \text{ and } n \leq \frac{5m-2}{2}, \\ (m-1)(2n-m)+4 & \text{if } \ell = 2m \text{ and } n \geq \frac{5m+2}{2}, \\ 4m^2 - 2m + 2 & \text{if } \ell = 2m \text{ and } n \leq \frac{5m+2}{2}. \end{cases}$$

Proof. (1) and (2) are trivial. In order to prove (3), by Lemma 2.1, it is enough to prove that if $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (m-1)(2n-m)+2 & \text{if } \ell = 2m-1 \text{ and } n \geq \frac{5m-2}{2}, \\ 4m^2 - 6m + 4 & \text{if } \ell = 2m-1 \text{ and } n \leq \frac{5m-2}{2}, \\ (m-1)(2n-m)+4 & \text{if } \ell = 2m \text{ and } n \geq \frac{5m+2}{2}, \\ 4m^2 - 2m + 2 & \text{if } \ell = 2m \text{ and } n \leq \frac{5m+2}{2}, \end{cases}$$

then π is potentially P_ℓ-graphic. We consider the following cases.

Case 1. $\ell = 3$. Since $\sigma(\pi) \geq 2n$ for $n \geq 4$, we have $d_2 \geq 2$. If $d_1 = 2$, then $\pi = (2^n)$ and C_n is a realization of π. Thus π is potentially P_3-graphic. So we may assume that $d_1 \geq 3$. Let G be a realization of π and $x, y \in V(G)$ with $d_G(x) = d_1$ and $d_G(y) = d_2$. Suppose $xy \in E(G)$. Since $d_G(x) \geq 3$ and $d_G(y) \geq 2$, let $z \in N_G(y) - \{x\}$ and $w \in N_G(x) - \{y, z\}$. Then w, x, y, z form a path of length three in G. Thus π is potentially P_3-graphic. Suppose $xy \notin E(G)$. If x and y have no common neighbor in G, let $z \in N_G(y)$ and $w \in N_G(x)$. If $zw \in E(G)$, then x, w, z, y form a path of length three in G. So we may assume that $zw \notin E(G)$. Let G' be obtained from G by deleting edges xw, yz but adding edges xy, zw. Clearly, G' is a realization of π with $xy \in E(G')$, the rest of the proof is similar to the case when $xy \in E(G)$. If x and y have a common neighbor, say u, let $v \neq u$ be another neighbor of x. Then v, x, u, y form a path of length three in G. In either case, π is potentially P_3-graphic.

Case 2. $\ell = 4$. Then $\sigma(\pi) \geq 2n+2$ for $n \geq 6$ and $\sigma(\pi) \geq 14$ for $n = 5$. By Case 1, π is potentially P_3-graphic, and hence we may assume that G is a realization of π and contains $P_3 = u_1u_2u_3u_4$ as a subgraph. By Theorem 3.2, we may further assume that $\{d_G(u_1), d_G(u_2), d_G(u_3), d_G(u_4)\} = \{d_1, d_2, d_3, d_4\}$. If $d_4 = 1$, then by Theorem 2.4, we have $\sigma(\pi) \leq d_1 + d_2 + d_3 + (n-3) \leq 6 + (n-3) + (n-3) = 2n$, a contradiction. Hence $d_4 \geq 2$. In other words, $d_G(u_1), d_G(u_2), d_G(u_3), d_G(u_4) \geq 2$. If there exists a vertex $x \in V(G) - \{u_1, u_2, u_3, u_4\}$ such that $xu_1 \in E(G)$ or $xu_4 \in E(G)$, then x, u_1, u_2, u_3, u_4 or u_1, u_2, u_3, u_4, x form a path of length four in G. Thus π is potentially P_4-graphic. Suppose $xu_1 \notin E(G)$ for any $x \in V(G) - \{u_1, u_2, u_3, u_4\}$ and $xu_4 \notin E(G)$ for any $x \in V(G) - \{u_1, u_2, u_3, u_4\}$. We will prove that G contains C_4 as a subgraph. If $u_1u_4 \in E(G)$, then $u_1u_2u_3u_4u_1$ is a cycle of length four in G. If $u_1u_4 \notin E(G)$, then by $d_G(u_1), d_G(u_4) \geq 2$, we have $u_1u_3, u_2u_4 \in E(G)$, and hence $u_1u_3u_4u_2u_1$ is a cycle of length four in G. Hence G contains C_4 as a subgraph. Denote $G_1 = G \setminus C_4$. If there exist $x \in G_1$ and $y \in C_4$ such that $xy \in E(G)$, then G contains a path of length four. Assume that there is no edge between $V(G_1)$ and $V(C_4)$. Take $xy \in E(C_4)$ and $x'y' \in E(G_1)$, and let $G' = G - \{xy + x'y'\} + \{xx' + yy'\}$. Clearly, G' is also a realization of π, and contains a path of length four. Thus π is potentially P_4-graphic.

Case 3. $\ell \geq 5$. Then, by Theorem 2.3, it is easy to see that $\sigma(\pi) \geq \sigma(C_\ell, n)$. Hence π has a realization G containing C_ℓ as a subgraph. Denote $G_1 = G \setminus C_\ell$. If there exist $x \in G_1$ and $y \in C_\ell$ such that $xy \in E(G)$, then G contains a path of length ℓ. Now we assume that there is no edge between $V(G_1)$ and $V(C_\ell)$. Take $xy \in E(C_\ell)$ and $x'y' \in E(G_1)$, and let $G' = G - \{xy + x'y'\} + \{xx' + yy'\}$. Clearly, G' is also a realization of π, and contains a path of length ℓ. □

3. $\sigma(C^k, n)$ for $n \geq (k+3)(2k+5)$

In order to determine $\sigma(C^k, n)$, we need the following known results. Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $1 \leq k \leq n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) \\ \text{if } d_k \geq k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n) \\ \text{if } d_k < k. \end{cases}$$

Let $\pi'_k = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ is a rearrangement of the $n-1$ terms of π''_k. π'_k is called the *residual sequence* obtained by laying off d_k from π.

Theorem 3.1 [10] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $1 \leq k \leq n$. Then $\pi \in GS_n$ if and only if $\pi'_k \in GS_{n-1}$.

Theorem 3.2 [8] If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Let $K_4 - e$ denote the graph obtained from K_4 by deleting one edge. Lai [11] proved the following

Theorem 3.3 [11] For $n = 4, 5$ and $n \geq 7$,

$$\sigma(K_4 - e, n) = \begin{cases} 3n - 1 & \text{if } n \text{ is odd,} \\ 3n - 2 & \text{if } n \text{ is even.} \end{cases}$$

For $n = 6$, if π is a 6-term graphic sequence with $\sigma(\pi) \geq 16$, then either π is potentially $K_4 - e$-graphic or $\pi = (3^6)$.

Theorem 3.4 [18] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $d_1 = m$ and $\sigma(\pi)$ is even. If there exists an integer $n_1 \leq n$ such that $d_{n_1} \geq h \geq 1$ and $n_1 \geq \frac{1}{h}\left[\frac{(m+h+1)^2}{4}\right]$, then $\pi \in GS_n$.

Theorem 3.5 [17] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{r+1} \geq r$. If $d_{2r+2} \geq r - 1$, then π is potentially K_{r+1}-graphic.

Theorem 3.6 [4] Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$. Then π has a nonseparable realization if and only if $d_n \geq 1$ and $\sigma(\pi) \geq 2(n-1)$.

Theorem 3.7 [3] Let G be a simple nonseparable graph of minimum degree δ on n vertices, where $n \geq 3$. Then G contains either a cycle of length at least 2δ or a Hamilton cycle.

We also need the following lemmas.

Lemma 3.1 Let $k \geq 2$, $n \geq k+3$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > (k+1)n$. If $d_{k+3} \leq \frac{k+1}{2}$, then π is potentially C^k-graphic.

Proof. Since $\sigma(\pi) > (k+1)n$, we have $d_1 \geq k+2$. Furthermore, by Theorem 2.3, $\sigma(\pi) \geq \sigma(C_{k+3}, n)$. If $n = k+3$, then π is potentially C^k-graphic. Assume that $n \geq k+4$. Let $\rho_0 = \pi$, and for $i = 1, \ldots, n-k-3$ in turn, ρ_i be the residual sequence obtained by laying off the last term from ρ_{i-1}. Then ρ_{n-k-3} is a $k+3$-term graphic sequence with $\sigma(\rho_{n-k-3}) \geq \sigma(\pi) - 2(d_n + d_{n-1} + \cdots + d_{k+4}) > (k+1)n - (n-k-3)(k+1) = (k+1)(k+3)$. Hence, ρ_{n-k-3} is potentially C^k-graphic. By Theorem 3.1, for $i = n-k-4, \ldots, 0$ in turn, ρ_i is also potentially C^k-graphic. □

Lemma 3.2 Let $k \geq 2$, $n \geq k+3$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > (k+1)n$. Then

(1) $d_i \geq (k+3) - i$ for $i = 1, 2, \ldots, [\frac{k}{2}] + 2$.

(2) If $d_i = (k+3) - i$ for some integer i, where $2 \le i \le [\frac{k}{2}] + 2$, then $d_{i-1} \ge k+2$ and $d_{k+3} = (k+3) - i$.

Proof. (1) Assume that there is an r, $1 \le r \le [\frac{k}{2}] + 2$ such that $d_r \le (k+2) - r$. Then $\sigma(\pi) \le (r-1)(n-1) + (k+2-r)(n-r+1) = r^2 - (k+4)r + (k+1)n + k + 3 \le 1 - (k+4) \times 1 + (k+1)n + k + 3 = (k+1)n$, a contradiction.

(2) If $d_{i-1} \le k+1$, then $\sigma(\pi) \le (i-2)(n-1) + k + 1 + (n-i+1)(k+3-i) = (k+1)n + i^2 - (k+5)i + 2k + 6 \le (k+1)n$, a contradiction. Hence $d_{i-1} \ge k+2$. If $d_{k+3} \le (k+2) - i$, then $\sigma(\pi) \le (i-1)(n-1) + (k+3-i)(k+2-(i-1)) + (k+2-i)(n-(k+2)) = (k+1)n + i^2 - (k+5)i + 2k + 6 \le (k+1)n$, a contradiction. Hence $d_{k+3} = (k+3) - i$. □

We now define a new graph $H(k)$ on $k+3$ vertices as follows: If k is odd and $V(K_{[\frac{k}{2}]+2}) = \{v_1, v_2, \ldots, v_{[\frac{k}{2}]+2}\}$, let $H(k)$ be the graph obtained from $K_{[\frac{k}{2}]+2}$ by adding new vertices $x_1, x_2, \ldots, x_{[\frac{k}{2}]+2}$, and joining x_1 to v_1 and v_2, x_i to $v_1, v_2, \ldots, v_{i+1}$ for $2 \le i \le [\frac{k}{2}] + 1$ and $x_{[\frac{k}{2}]+2}$ to $v_1, v_2, \ldots, v_{[\frac{k}{2}]+2}$. If k is even and $V(K_{[\frac{k}{2}]+2}) = \{v_1, v_2, \ldots, v_{[\frac{k}{2}]+2}\}$, let $H(k)$ be the graph obtained from $K_{[\frac{k}{2}]+2}$ by adding new vertices $x_1, x_2, \ldots, x_{[\frac{k}{2}]+1}$, and joining x_1 to v_1 and v_2, x_i to $v_1, v_2, \ldots, v_{i+1}$ for $2 \le i \le [\frac{k}{2}]$ and $x_{[\frac{k}{2}]+1}$ to $v_1, v_2, \ldots, v_{[\frac{k}{2}]+2}$.

Lemma 3.3 $H(k)$ contains a Hamilton cycle with k chords incident to v_1 on the cycle.

Proof. Let

$$C = \begin{cases} x_{[\frac{k}{2}]+2} v_{[\frac{k}{2}]+2} x_{[\frac{k}{2}]+1} v_{[\frac{k}{2}]+1} \cdots x_2 v_2 x_1 v_1 x_{[\frac{k}{2}]+2} & \text{if } k \text{ is odd,} \\ v_{[\frac{k}{2}]+2} x_{[\frac{k}{2}]+1} v_{[\frac{k}{2}]+1} x_{[\frac{k}{2}]} v_{[\frac{k}{2}]} \cdots x_2 v_2 x_1 v_1 v_{[\frac{k}{2}]+2} & \text{if } k \text{ is even.} \end{cases}$$

Then C is a Hamilton cycle of $H(k)$ with v_1 adjacent to each vertex of C. □

Let $k \ge 2$, $n \ge k+3$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > (k+1)n$. We now define sequences $\pi_0, \ldots, \pi_{k+3}$ as follows. Let $\pi_0 = \pi$. We define

$$\pi_1 = (d_2^{(1)}, \ldots, d_{k+3}^{(1)}, d_{k+4}^{(1)}, \ldots, d_n^{(1)}) \quad \text{by}$$

(1) deleting d_1 from π_0,
(2) subtracting one from the first d_1 remaining nonzero terms to get the resulting sequence,
(3) re-ordering the last $n-k-3$ terms of the resulting sequence to make them non-increasing.

For $2 \le i \le k+3$, given $\pi_{i-1} = (d_i^{(i-1)}, \ldots, d_{k+3}^{(i-1)}, d_{k+4}^{(i-1)}, \ldots, d_n^{(i-1)})$, we define

$$\pi_i = (d_{i+1}^{(i)}, \ldots, d_{k+3}^{(i)}, d_{k+4}^{(i)}, \ldots, d_n^{(i)}) \quad \text{by}$$

(1) deleting $d_i^{(i-1)}$ from π_{i-1},
(2) subtracting one from the first $d_i^{(i-1)}$ remaining nonzero terms to get the resulting sequence,
(3) re-ordering the last $n-k-3$ terms of the resulting sequence to make them non-increasing.

By the definition of π_{k+3}, the following lemma is obvious.

Lemma 3.4 Let $k \geq 2$, $n \geq k+3$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ satisfy

(1) $d_1 \geq k+2$ and $d_i \geq (k+4) - i$ for $2 \leq i \leq [\frac{k}{2}] + 2$;

(2) $d_{k+3} \geq [\frac{k}{2}] + 2$ if k is odd, and $d_{[\frac{k}{2}]+3} \geq [\frac{k}{2}] + 2$ and $d_{k+3} \geq [\frac{k}{2}] + 1$ if k is even;

(3) π_{k+3} is graphic.

Then π is potentially $H(k)$-graphic.

Lemma 3.5 Let $k \geq 2$ be even, $n \geq k+3$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq f(k,n)$, where

$$f(k,n) = \begin{cases} (k+2)n - \frac{(k+2)(k+4)}{4} + 2 & \text{if } n \geq \frac{(k+2)(k+4)}{4} - 1, \\ (k+1)n + 2 & \text{if } n \leq \frac{(k+2)(k+4)}{4} - 1 \text{ and } n \text{ is even}, \\ (k+1)n + 1 & \text{if } n \leq \frac{(k+2)(k+4)}{4} - 1 \text{ and } n \text{ is odd}. \end{cases}$$

Then $d_{[\frac{k}{2}]+3} \geq [\frac{k}{2}] + 2$.

Proof. It is easy to check that $f(k,n) \geq (k+2)n - \frac{(k+2)(k+4)}{4} + 2$ for $n \geq k+3$. If $d_{\frac{k}{2}+3} \leq \frac{k}{2} + 1$, then by Theorem 2.4, $\sigma(\pi) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{\frac{k}{2}+2} d_i + \sum_{i=\frac{k}{2}+3}^{n} d_i \leq ((\frac{k}{2}+1)(\frac{k}{2}+2) + \sum_{i=\frac{k}{2}+3}^{n} \min\{\frac{k}{2}+2, d_i\}) + \sum_{i=\frac{k}{2}+3}^{n} d_i = (\frac{k}{2}+1)(\frac{k}{2}+2) + 2\sum_{i=\frac{k}{2}+3}^{n} d_i \leq (\frac{k}{2}+1)(\frac{k}{2}+2) + 2(\frac{k}{2}+1)(n - \frac{k}{2} - 2) = (k+2)n - \frac{(k+2)(k+4)}{4} < f(k,n)$, a contradiction. Hence $d_{[\frac{k}{2}]+3} \geq [\frac{k}{2}] + 2$. □

Lemma 3.6 Let $n \geq (k+3)(2k+5)$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{2k+6} \leq k$ and $\sigma(\pi) > (k+1)n$. Then $d_1 \geq 2k+4$.

Proof. If $d_1 \leq 2k+3$, then $\sigma(\pi) \leq (2k+3)(2k+5) + k(n-2k-5) = (k+1)n + (k+3)(2k+5) - n \leq (k+1)n$, a contradiction. □

Lemma 3.7 Let $k \geq 2$, $n \geq 2k+6$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{2k+6} \geq k+1$ and $\sigma(\pi) > (k+1)n$. Then π is potentially C^k-graphic.

Proof. If $d_{k+3} \geq k+2$, then by Theorem 3.5, π is potentially K_{k+3}-graphic, and hence π is potentially C^k-graphic. Assume that $d_{k+3} = \cdots = d_{2k+6} = k+1$. Clearly, $d_1 \geq k+2$. We now consider the following two cases.

Case 1. $d_2 \geq k+2$. Then $d_i \geq (k+4) - i$ for $2 \leq i \leq [\frac{k}{2}] + 2$ and $d_{k+3} \geq [\frac{k}{2}] + 2$. For $i = 0, 1, \ldots, k+3$, the values of $d_{k+4}^{(i)}, \ldots, d_{2k+6}^{(i)}$ differ by at most one. Hence $\pi_{k+3} = (d_{k+4}^{(k+3)}, \ldots, d_{2k+6}^{(k+3)}, \ldots, d_n^{(k+3)})$ satisfies

$$k+1 \geq m = d_{k+4}^{(k+3)} \geq \cdots \geq d_{2k+6}^{(k+3)} \geq m-1$$

for some $m \geq 1$. If $m = 1$, then π_{k+3} must be graphic as $\sigma(\pi_{k+3})$ is even. If $m \geq 2$, then

$$\frac{1}{m-1}\left[\frac{(m+(m-1)+1)^2}{4}\right] \leq m+2 \leq k+3.$$

By Theorem 3.4, π_{k+3} is graphic. Thus, π is potentially C^k-graphic by Lemmas 3.4 and 3.3.

Case 2. $d_2 = \cdots = d_{2k+6} = k+1$. Then $\pi_1 = (k^{k+2}, d_{k+4}^{(1)}, \ldots, d_{2k+6}^{(1)}, \ldots, d_n^{(1)})$. Let

$$\rho = \begin{cases} (k^{k+2}) & \text{if } k \text{ is even,} \\ (k^{k+1}, k-1) & \text{if } k \text{ is odd,} \end{cases}$$

and

$$\rho' = \begin{cases} (d_{k+4}^{(1)}, d_{k+5}^{(1)}, \ldots, d_{2k+6}^{(1)}, \ldots, d_n^{(1)}) & \text{if } k \text{ is even,} \\ (d_{k+4}^{(1)} - 1, d_{k+5}^{(1)}, \ldots, d_{2k+6}^{(1)}, \ldots, d_n^{(1)}) & \text{if } k \text{ is odd.} \end{cases}$$

Clearly, $\sigma(\rho)$ and $\sigma(\rho')$ are even. Similarly, it is easy to follow from Theorem 3.4 that both ρ and ρ' are graphic. It is easy to see that ρ has a realization containing a Hamilton cycle for $k = 2$ and 3. If $k \geq 4$, then by Theorems 3.6 and 3.7, ρ also has a realization containing a Hamilton cycle. Let G_1 be a realization of ρ containing a Hamilton cycle, and let G_2 be a realization of ρ'. Let $G' = G_1 \cup G_2$ if k is even and G' be the graph obtained from $G_1 \cup G_2$ by joining the vertex of G_1 with degree $k-1$ to the vertex of G_2 with degree $d_{k+4}^{(1)} - 1$ if k is odd. Clearly, G' is a realization of π_1. Let G be the graph obtained from G' by adding a new vertex of degree d_1 and joining it to the vertices whose degrees are reduced by one in going from π to π_1. Then G is a realization of π and contains a cycle with k chords incident to the vertex of degree d_1 on the cycle. □

Lemma 3.8 (1) Let $k = 1$ and $n \geq 4$. Then

$$\sigma(C^k, n) \geq \begin{cases} 3n-1 & \text{if } n \text{ is odd,} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

(2) Let k (≥ 2) be even and $n \geq k+3$. Then

$$\sigma(C^k, n) \geq \begin{cases} (k+2)n - \frac{(k+2)(k+4)}{4} + 2 & \text{if } n \geq \frac{(k+2)(k+4)}{4} - 1, \\ (k+1)n + 2 & \text{if } n \leq \frac{(k+2)(k+4)}{4} - 1 \text{ and } n \text{ is even,} \\ (k+1)n + 1 & \text{if } n \leq \frac{(k+2)(k+4)}{4} - 1 \text{ and } n \text{ is odd.} \end{cases}$$

(3) Let k (≥ 3) be odd and $n \geq k+3$. Then $\sigma(C^k, n) \geq (k+1)n+2$.

Proof. (1) Take $\pi = \begin{cases} (n-1, 2^{n-1}) & \text{if } n \text{ is odd,} \\ (n-1, 2^{n-2}, 1) & \text{if } n \text{ is even,} \end{cases}$
and let

$$G = \begin{cases} K_1 + \frac{n-1}{2} K_2 & \text{if } n \text{ is odd,} \\ K_1 + (\frac{n-2}{2} K_2 \cup K_1) & \text{if } n \text{ is even.} \end{cases}$$

Then, it is easy to see that G is the unique realization of π and contains no cycles of length at least 4. Hence π is not potentially C^1-graphic. Thus,

$$\sigma(C^1, n) \geq \sigma(\pi) + 2 = \begin{cases} 3n-1 & \text{if } n \text{ is odd,} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

(2) Firstly, we consider $\pi = ((n-1)^{\frac{k}{2}+1}, (\frac{k}{2}+1)^{n-\frac{k}{2}-1})$. It is easy to see that $K_{\frac{k}{2}+1} + \overline{K_{n-\frac{k}{2}-1}}$ is the only graph realizing π, and has no cycles of length at least $k+3$, so that π is not potentially C^k-graphic. Hence $\sigma(C^k, n) \geq \sigma(\pi) + 2 = (k+2)n - \frac{(k+2)(k+4)}{4} + 2$. Now we consider $\pi = ((k+1)^n)$ if n is even and $\pi = ((k+1)^{n-1}, k)$ if n is odd. Clearly, π is not potentially C^k-graphic. Hence

$$\sigma(C^k, n) \geq \sigma(\pi) + 2 = \begin{cases} (k+1)n+2 & \text{if } n \text{ is even,} \\ (k+1)n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Thus, we have

$$\sigma(C^k, n) \geq \begin{cases} (k+2)n - \frac{(k+2)(k+4)}{4} + 2 & \text{if } n \geq \frac{(k+2)(k+4)}{4} - 1, \\ (k+1)n+2 & \text{if } n \leq \frac{(k+2)(k+4)}{4} - 1 \text{ and } n \text{ is even,} \\ (k+1)n+1 & \text{if } n \leq \frac{(k+2)(k+4)}{4} - 1 \text{ and } n \text{ is odd.} \end{cases}$$

(3) Take $\pi = ((k+1)^n)$. Clearly, π is not potentially C^k-graphic. Hence $\sigma(C^k, n) \geq \sigma(\pi) + 2 = (k+1)n+2$. □

The following Theorem 3.8 is the main result of this section.

Theorem 3.8 (1) Let $k = 1$ and $n \geq 4$. Then

$$\sigma(C^k, n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd,} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

(2) Let $k \geq 2$ and $n \geq (k+3)(2k+5)$. Then

$$\sigma(C^k, n) = \begin{cases} (k+2)n - \frac{(k+2)(k+4)}{4} + 2 & \text{if } k \text{ is even,} \\ (k+1)n+2 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. (1) By Lemma 3.8, we only need to prove that if $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} 3n-1 & \text{if } n \text{ is odd,} \\ 3n-2 & \text{if } n \text{ is even,} \end{cases}$$

then π is potentially C^1-graphic. If $\pi \neq (3^6)$, then by Theorem 3.3, π is potentially $K_4 - e$-graphic, and hence π is potentially C^1-graphic. If $\pi = (3^6)$, then it is easy to check that π is also potentially C^1-graphic.

(2) Assume that $k \geq 2$, $n \geq (k+3)(2k+5)$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (k+2)n - \frac{(k+2)(k+4)}{4} + 2 & \text{if } k \text{ is even,} \\ (k+1)n + 2 & \text{if } k \text{ is odd.} \end{cases}$$

By Lemma 3.8, it is enough to prove that π is potentially C^k-graphic. Clearly, $\sigma(\pi) \geq (k+1)n+2$, and so $d_1 \geq k+2$. If $d_1 = n-1$ or there is an integer t, $k+3 \leq t \leq d_1+1$ such that $d_t > d_{t+1}$, then by Theorem 2.5, the residual sequence $\pi_1' = (d_1', d_2', \ldots, d_{n-1}')$ obtained by laying off d_1 from π satisfies $d_1' = d_2 - 1, \ldots, d_{k+2}' = d_{k+3} - 1$ and

$$\sigma(\pi_1') = \sigma(\pi) - 2d_1 \geq \begin{cases} (k+2)n - \frac{(k+2)(k+4)}{4} + 2 - 2(n-1) \\ = \sigma(P_{k+1}, n-1) \quad \text{if } k \text{ is even,} \\ (k+1)n + 2 - 2(n-1) \geq \sigma(P_{k+1}, n-1) \\ \text{if } k \text{ is odd.} \end{cases}$$

By Theorem 3.2, there is a realization G' of π_1' containing P_{k+1} as a subgraph so that the vertices of P_{k+1} have degrees $d_1', \ldots, d_{k+2}'$. This implies that π is potentially C^k-graphic. We now assume that

$$n-2 \geq d_1 \geq \cdots \geq d_{k+2} \geq d_{k+3} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n.$$

If $d_{k+3} \leq \frac{k+1}{2}$ or $d_{2k+6} \geq k+1$, then by Lemma 3.1 or Lemma 3.7, π is potentially C^k-graphic. Hence, we may further assume that $d_{k+3} \geq [\frac{k}{2}]+2$ if k is odd, $d_{k+3} \geq [\frac{k}{2}]+1$ if k is even and $d_{2k+6} \leq k$. Lemma 3.5 implies that $d_{[\frac{k}{2}]+3} \geq [\frac{k}{2}]+2$ for even k. By Lemma 3.6, we have $d_1 \geq 2k+4$, and hence $n-2 \geq d_1 \geq \cdots \geq d_{k+3} = \cdots = d_{2k+6} = \cdots = d_{d_1+2}$. By Lemma 3.2(1), we only consider the following two cases.

Case 1. $d_i \geq (k+4) - i$ for $2 \leq i \leq [\frac{k}{2}]+2$. By Lemmas 3.4 and 3.3, it is enough to check that π_{k+3} is graphic. Clearly, $\pi_{k+3} = (d_{k+4}^{(k+3)}, \ldots, d_{2k+6}^{(k+3)}, \ldots, d_n^{(k+3)})$ satisfies

$$k \geq m = d_{k+4}^{(k+3)} \geq \cdots \geq d_{2k+6}^{(k+3)} \geq m-1$$

for some $m \geq 1$. If $m = 1$, then π_{k+3} must be graphic as $\sigma(\pi_{k+3})$ is even. If $m \geq 2$, then

$$\frac{1}{m-1}\left[\frac{(m+(m-1)+1)^2}{4}\right] \leq m+2 \leq k+3.$$

By Theorem 3.4, π_{k+3} is graphic.

Case 2. There is an integer i, $2 \le i \le [\frac{k}{2}]+2$ such that $d_i = (k+3)-i$. By Lemma 3.2(2), $d_{i-1} \ge k+2$ and $d_{k+3} = (k+3)-i$. We now consider $\pi_{i-1} = ((k+4-2i)^{k+4-i}, d_{k+4}^{(i-1)}, \dots, d_n^{(i-1)})$.

Subcase 2.1. k is odd. Let

$$\rho = \begin{cases} ((k+4-2i)^{k+4-i}) & \text{if } (k+4-2i)(k+4-i) \text{ is even,} \\ ((k+4-2i)^{k+3-i}, k+3-2i) & \text{if } (k+4-2i)(k+4-i) \text{ is odd,} \end{cases}$$

and

$$\rho' = \begin{cases} (d_{k+4}^{(i-1)}, d_{k+5}^{(i-1)}, \dots, d_{2k+6}^{(i-1)}, \dots, d_n^{(i-1)}) \\ \text{if } (k+4-2i)(k+4-i) \text{ is even,} \\ (d_{k+4}^{(i-1)} - 1, d_{k+5}^{(i-1)}, \dots, d_{2k+6}^{(i-1)}, \dots, d_n^{(i-1)}) \\ \text{if } (k+4-2i)(k+4-i) \text{ is odd.} \end{cases}$$

Clearly, $\sigma(\rho)$ and $\sigma(\rho')$ are even, and it is easy to follow from Theorem 3.4 that both ρ and ρ' are graphic. If $i = [\frac{k}{2}]+2 = \frac{k+3}{2}$, then

$$\rho = \begin{cases} (1^{\frac{k+5}{2}}) & \text{if } \frac{k+5}{2} \text{ is even,} \\ (1^{\frac{k+3}{2}}, 0) & \text{if } \frac{k+5}{2} \text{ is odd.} \end{cases}$$

Since any realization of ρ has at least two edges, it is easy to get that π has a realization G containing a cycle with k chords incident to the vertex of degree d_1 on the cycle. If $i = [\frac{k}{2}]+1 = \frac{k+1}{2}$, then

$$\rho = \begin{cases} (3^{\frac{k+7}{2}}) & \text{if } \frac{3(k+7)}{2} \text{ is even,} \\ (3^{\frac{k+5}{2}}, 2) & \text{if } \frac{3(k+7)}{2} \text{ is odd.} \end{cases}$$

It is easy to see that ρ has a realization containing a Hamilton cycle, and so π is potentially C^k-graphic. If $i \le [\frac{k}{2}] = \frac{k-1}{2}$, then by Theorems 3.6 and 3.7, ρ has a realization containing a cycle of length at least $2k+6-4i$ or a Hamilton cycle. Hence π is also potentially C^k-graphic.

Subcase 2.2. k is even. Let $\rho = ((k+4-2i)^{k+4-i})$ and $\rho' = (d_{k+4}^{(i-1)}, \dots, d_{2k+6}^{(i-1)}, \dots, d_n^{(i-1)})$. By Theorem 3.4, both ρ and ρ' are graphic. If $i = \frac{k}{2}+2$, then $d_{\frac{k}{2}+2} = (k+3)-i = \frac{k}{2}+1$, which is impossible by Lemma 3.5. Hence $2 \le i \le \frac{k}{2}+1$. By Theorems 3.6 and 3.7, ρ has a realization containing a cycle of length at least $2k+8-4i$ or a Hamilton cycle. Therefore, π is potentially C^k-graphic. □

Acknowledgements The authors are grateful to the referee for his valuable comments and suggestions.

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