A characterization of trees with equal domination and total domination numbers*

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Abstract Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a total dominating set of G if every vertex of V is adjacent to some vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. In this paper, we provide a constructive characterization of those trees with equal domination and total domination numbers.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, the set of vertices adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For $S \subseteq V$, the open neighborhood of S is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of S by $N[S] = N(S) \cup S$. The private neighborhood $PN(v, S)$ of $v \in S$ is defined by $PN(v, S) = N[v] - N[S - \{v\}]$. The subgraph of G induced by the vertices in S is denoted by $< S >$. For $X, Y \subseteq V$, if X dominates Y, we write $X \succ Y$, or $X \succ G$ if $Y = V$, or $X \succ y$ if $Y = \{y\}$.

A set $S \subseteq V$ is a dominating set of G if every vertex not in S is adjacent

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to some vertex in S. (That is $N[S] = V$.) The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G.

Let $G = (V, E)$ be a graph without isolated vertices. A set $S \subseteq V$ is a total dominating set of G if every vertex of V is adjacent to some vertex in S. (That is $N(S) = V$.) The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of G of cardinality $\gamma_t(G)$ is called a γ_t-set of G. A set $S \subseteq V$ is a paired-dominating set of G if S dominates V and $< S >$ contains at least one perfect matching. The paired-domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired-dominating set of G. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [1]. Paired-domination in graphs was introduced by Haynes and Slater [7]. The concept of domination in graphs, with its many variations, is well studied in graph theory (see [4, 5]).

An area of research in domination of graphs that has received considerable attention is the study of classes of graphs with equal domination parameters. For any two graph theoretic parameters λ and μ, G is called a (λ, μ)-graph if $\lambda(G) = \mu(G)$. The class of (γ, i)-trees, that is trees with equal domination and independent domination numbers, was characterized in [2]. In [3], the authors provided a constructive characterization of trees with equal independent domination and restrained domination numbers, and a constructive characterization of trees with equal independent domination and weak domination numbers is also given. In [6], those trees with strong equality of domination parameters were characterized. In [8], the authors characterized those trees with equal domination and paired-domination numbers.

An immediate consequence directly of the definitions of domination, total domination and paired-domination numbers, we have

Proposition 1 *([7]) Let G be a graph without isolated vertices. Then* $\gamma(G) \leq \gamma_t(G) \leq \gamma_p(G)$.

By Proposition 1, those trees with equal domination and paired-domination numbers must be trees with equal domination and total domination numbers. In this paper, we give a constructive characterization of trees with equal domination and total domination numbers. Figure 1(a) gives the tree T of minimum order with $\gamma(T) = \gamma_t(T) = \gamma_p(T)$, (b) gives the tree T of minimum order with $\gamma(T) = \gamma_t(T) < \gamma_p(T)$.

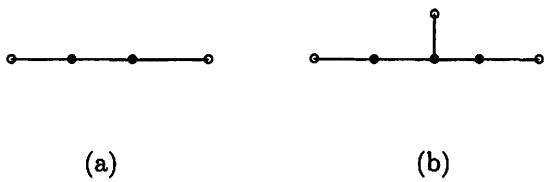

Figure 1: (a) $\gamma(P_4) = \gamma_t(P_4) = \gamma_p(P_4) = 2$ (b) $\gamma(T) = \gamma_t(T) = 3 < \gamma_p(T) = 4$

2 Main result

Let $T = (V, E)$ be a tree with vertex set V and edge set E. A vertex of T is said to be remote if it is adjacent to a leaf. The set of leaves of T is denoted by $L(T)$. In this paper, we use T_v denote the subtree of $T - uv$ containing v for $uv \in E(T)$. P_l represents a path with l vertices. To state the characterization of trees with equal domination and total domination numbers, we introduce five types of operations.

Type-1 operation: Attach a path P_1 to a vertex v of T, where v is in a γ_t-set of T.

Type-2 operation: Attach an end vertex of a path P_2 to a vertex v of a tree T, where v is in a γ_t-set of T and for every γ-set X of T, there is no vertex $u \in X$ such that $PN(u, X) = \{v\}$ in T.

Type-3 operation: Attach an end vertex of a path P_5 to a vertex v of a tree T, where v is in a γ_t-set of T and for every γ-set X of T, there is no vertex $u \in X$ such that $PN(u, X) = \{v\}$ in T.

Type-4 operation: Attach a remote vertex of a path P_4 to a vertex v of a tree T, where v is a vertex such that for every γ-set X of T, there is no vertex $u \in X$ such that $PN(u, X) = \{v\}$ in T.

Type-5 operation: Attach a vertex u_0 of T_1 to a vertex v of a tree T, where T_1 is a tree with $V(T_1) = \{u_0, u_1, u_2, u_3, u_4\}$ and $E(T_1) = \{u_0u_1, u_1u_2, u_1u_3, u_2u_4\}$.

Let $\mathcal{J}_t$ be the family of trees with equal domination and total domination numbers. Then

$$\mathcal{J}_t = \{T : \gamma(T) = \gamma_t(T)\}.$$

We define the family $\mathcal{F}_t$ as :

$$\mathcal{F}_t = \{T : T \text{ is obtained from } P_4 \text{ by a finite sequence of operations of Type-}i, i = 1, 2, 3, 4, 5\}.$$

We shall prove that

Theorem 2

$$\mathcal{J}_t = \mathcal{F}_t.$$

3 The proof of Theorem 2

We begin with a simple observation.

Lemma 3 *If v is a remote vertex of a tree T, then for each total dominating set S, $v \in S$.*

Lemma 4 *If T is a tree with $\gamma_t(T) = \gamma(T)$, S is a γ_t-set of T, then for each $v \in S$, $PN(v, S) \neq \phi$.*

Proof. Suppose to the contrary that there exists a vertex $v \in S$ such that $PN(v, S) = \phi$, then $S - \{v\}$ is a dominating set of T, contradicts to $\gamma_t(T) = \gamma(T)$. □

From Lemma 3 and Lemma 4, or the Observation 2 in [9], we have

Lemma 5 *Let T be a tree with $\gamma_t(T) = \gamma(T)$, S is a γ_t-set of T. Then $S \cap L(T) = \phi$.*

By Lemma 5, we know that every γ_t-set of a tree T with $\gamma_t(T) = \gamma(T)$ contains no leaves of T.

Lemma 6 *If T is a tree with $\gamma_t(T) = \gamma(T)$, then T has a unique γ_t-set.*

Proof. We proceed by induction on n, the order of the tree T. If $n \leq 4$, then $T \in \mathcal{J}_t = \{P_4\}$ and T has a unique γ_t-set. Let $n \geq 5$ and assume that the result is true for all trees $T' \in \mathcal{J}_t$ of order n', $n' < n$. Let $T \in \mathcal{J}_t$ be a tree of order n and let $v_0 v_1 v_2 \cdots v_l$ be a longest path in T. If $d(v_1) \geq 3$, then there exists a leaf u adjacent to v_1. Let $T' = T - \{u\}$. Then, by Lemma 3 and Lemma 5, T' and T have the same γ_t-set. Clearly, T' has a $\gamma(T')$-set containing v_1, Such a $\gamma(T')$-set is also a dominating set of T. So, $\gamma_t(T') = \gamma_t(T) = \gamma(T) = \gamma(T')$. By inductive hypothesis, T' has a unique γ_t-set. It follows that T has a unique γ_t-set. Hence, we may assume that $d(v_1) = 2$.

Case 1: $d(v_2) \geq 3$.

Case 1.1: There is a remote vertex $u \in N(v_2) \setminus \{v_1, v_3\}$.

Let S be a γ_t-set of T and $T' = T_{v_2}$ be the subtree of $T - v_2u$ containing v_2. Then, by Lemma 3 and Lemma 5, $\{v_1, v_2, u\} \subseteq S$. So, $S - \{u\}$ is a total dominating set of T' and $\gamma_t(T') \leq \gamma_t(T) - 1$. However, let S' be any γ_t-set of T', by Lemmas 3 and 5, $\{v_1, v_2\} \subseteq S'$. Then $S' \cup \{u\}$ is a total dominating set of T. So $\gamma_t(T) \leq \gamma_t(T') + 1$. Consequently, $\gamma_t(T) = \gamma_t(T') + 1$. Since $\gamma(T) \leq \gamma(T') + 1 \leq \gamma_t(T') + 1 = \gamma_t(T) = \gamma(T)$, we have $\gamma(T') = \gamma_t(T')$ and $T' \in \mathcal{J}_t$. By inductive hypothesis, T' has a unique γ_t-set S' and $\{v_1, v_2\} \subseteq S'$. It follows that $S = S' \cup \{u\}$ is the unique γ_t-set of T.

Case 1.2: All of $u \in N(v_2) \setminus \{v_1, v_3\}$ are leaves. Let S be a γ_t-set of T, then, by Lemma 3, $\{v_1, v_2\} \subseteq S$.

If $v_3 \in S$, then $v_3 \notin PN(v_2, S)$. Let $T' = T_{v_2}$ be the subtree of $T - v_1v_2$ containing v_2, then $S - \{v_1\}$ is a total dominating set of T'. So $\gamma_t(T') \leq \gamma_t(T) - 1$. On the other hand, let S' be any γ_t-set of T', then, by Lemma 3, $v_2 \in S'$. So S' can be extended to a total dominating set of T by adding to it the vertex v_1. Hence $\gamma_t(T) \leq \gamma_t(T') + 1$. Consequently, $\gamma_t(T) = \gamma_t(T') + 1$. Since $\gamma(T) \leq \gamma(T') + 1 \leq \gamma_t(T') + 1 = \gamma_t(T) = \gamma(T)$, we have $\gamma(T') = \gamma_t(T')$ and $T' \in \mathcal{J}_t$. By inductive hypothesis, T' has a unique γ_t-set S' with $v_2 \in S'$. It follows that $S = S' \cup \{v_1\}$ is the unique γ_t-set of T.

If $v_3 \notin S$ and $v_3 \notin PN(v_2, S)$, let $T' = T_{v_3}$ denote the subtree of $T - v_2v_3$ containing v_3, then $S - \{v_1, v_2\}$ is a total dominating set of T'. So, $\gamma_t(T') \leq \gamma_t(T) - 2$. However, any γ_t-set of T' can be extended to a total dominating set by adding to it the vertices v_1 and v_2. So, $\gamma_t(T) \leq \gamma_t(T') + 2$. Consequently, $\gamma_t(T) = \gamma_t(T') + 2$. Since $\gamma(T) \leq \gamma(T') + 2 \leq \gamma_t(T') + 2 = \gamma_t(T) = \gamma(T)$, we have $\gamma(T') = \gamma_t(T')$ and $T' \in \mathcal{J}_t$. By inductive hypothesis, T' has a unique γ_t-set S'. It follows that $S = S' \cup \{v_1, v_2\}$ is the unique γ_t-set of T.

If $v_3 \in PN(v_2, S)$, then $v_3 \notin S$ and for any vertex $u \in N[v_3] \setminus \{v_2\}$, $u \notin S$. By Lemma 3 and Lemma 5, v_3 is not a remote vertex, and neither

v_3 is adjacent to a remote vertex nor v_3 is adjacent to a vertex which is adjacent to a remote vertex. Then, $d(v_3) = 1$ or $d(v_3) = 2$. If $d(v_3) = 1$, then T has a unique γ_t-set $S = \{v_1, v_2\}$. If $d(v_3) = 2$, let $T' = T_{v_4}$ be the subtree of $T - v_3v_4$ containing v_4, then for any γ_t-set S of T, $S - \{v_1, v_2\}$ is a total dominating set of T'. So, $\gamma_t(T') \leq \gamma_t(T) - 2$. However, any γ_t-set of T' can be extended to a total dominating set of T by adding to it the vertices v_1 and v_2. So $\gamma_t(T) \leq \gamma_t(T') + 2$. Consequently, $\gamma_t(T) = \gamma_t(T') + 2$. Since $\gamma(T) \leq \gamma(T') + 2 \leq \gamma_t(T') + 2 = \gamma_t(T) = \gamma(T)$, we have $\gamma_t(T') = \gamma(T')$ and $T' \in \mathcal{J}_t$. By the inductive hypothesis, T' has a unique γ_t-set S'. It follows that $S = S' \cup \{v_1, v_2\}$ is the unique γ_t-set of T.

Case 2: $d(v_2) = 2$.

By Lemmas 3 and 4, for any γ_t-set S of T, $v_1, v_2 \in S$, $v_3 \in PN(v_2, S)$, $v_4 \notin S$. As discussed in the case $v_3 \in PN(v_2, S)$ of case 1.2, we can infer $d(v_3) = 2$. Furthermore, we claim that $d(v_4) = 2$ and $v_5 \in S$. Otherwise, there is a vertex $u \in N(v_4) \setminus \{v_3, v_5\}$. By Lemma 3, u is not a leaf. Let T_u denote the subtree of $T - uv_4$ containing u. From the above discussions, T_u must be a path P_l with $l \leq 4$. By Lemma 5, $T_u \neq P_2$. So, $T_u = P_3$ or P_4. Let T_{v_4} denote the subtree of $T - v_4v_5$ containing v_4 and T_{v_5} denote the subtree of $T - v_4v_5$ containing v_5. Then, $\gamma(T_{v_4}) \leq d(v_4) < 2(d(v_4) - 1) = |S \cap V(T_{v_4})|$. Since $v_4 \notin S$, $S \cap V(T_{v_5})$ is a total dominating set of T_{v_5}, so, $\gamma_t(T_{v_5}) \leq \gamma_t(T) - 2(d(v_{v_4}) - 1)$ and $\gamma(T) \leq \gamma(T_{v_4}) + \gamma(T_{v_5}) \leq d(v_4) + \gamma(T_{v_5})$. Consequently, $\gamma(T) < \gamma_t(T)$, a contradiction. So, $d(v_4) = 2$ and $v_5 \in S$. Let $T' = T - \{v_0, v_1, v_2, v_3\}$, then $S - \{v_1, v_2\}$ is a total dominating set of T'. So $\gamma_t(T') \leq \gamma_t(T) - 2$. However, any γ_t-set of T' can be extended to a total dominating set of T by adding to it the vertices v_1 and v_2. So $\gamma_t(T) \leq \gamma_t(T') + 2$. Consequently, $\gamma_t(T) = \gamma_t(T') + 2$. Since $\gamma(T) \leq 2 + \gamma(T') \leq 2 + \gamma_t(T') = \gamma_t(T) = \gamma(T)$, we have $\gamma_t(T') = \gamma(T')$ and $T' \in \mathcal{J}_t$. By inductive hypothesis, T' has a unique γ_t-set S'. It follows that $S = S' \cup \{v_1, v_2\}$ is the unique γ_t-set of T. □

Lemma 7 *If $T' \in \mathcal{J}_t$ and T is obtained from T' by a Type-i operation, $i = 1, 2, 3, 4, 5$. Then $T \in \mathcal{J}_t$.*

Proof. Let $T' \in \mathcal{J}_t$. Then, by Lemma 6, T' has a unique γ_t-set S'. Suppose that T is obtained from T' by attaching a graph G_i to the vertex v of T' satisfying the conditions required by the operation of Type-i, $i = 1, 2, 3, 4$ or 5.

If $i = 1$, then $G_1 = P_1$. Let $P_1 = u$, then u is a leaf of T adjacent to $v \in S'$. Thus S' is also a total dominating set of T, consequently, S' is a γ_t-set of T. Since $\gamma(T) \geq \gamma(T') = \gamma_t(T') = \gamma_t(T) \geq \gamma(T)$, we have, $\gamma(T) = \gamma_t(T)$ and $T \in \mathcal{J}_t$.

If $i = 2$, then $G_2 = P_2 = u_0u_1$, assume that u_1 is adjacent to $v \in S'$. Then $S = S' \cup \{u_1\}$ is a total dominating set of T, so $\gamma_t(T) \leq \gamma_t(T') + 1$. Let X be any γ-set of T with $u_1 \in X$. If $v \notin PN(u_1, X)$, then $X \setminus \{u_1\}$ is a dominating set of T', so $\gamma(T') \leq \gamma(T) - 1$. If $v \in PN(u_1, X)$, then $X \cap V(T') \succ T' - \{v\}$. We claim that $\gamma(T') \leq \gamma(T) - 1$. Otherwise, suppose that $\gamma(T') = \gamma(T)$, then $X' = (X \cap V(T')) \cup \{v\}$ is a γ-set of T' and $PN(v, X') = \{v\}$ in T', contradicts to the choice of v required by Type-2 operation. So, $\gamma(T') \leq \gamma(T) - 1$. By $\gamma(T) \geq \gamma(T') + 1 = \gamma_t(T') + 1 \geq \gamma_t(T) \geq \gamma(T)$, we have $\gamma(T) = \gamma_t(T)$ and $T \in \mathcal{J}_t$.

If $i = 3$ then $G_3 = P_5 = u_0u_1u_2u_3u_4$, assume that u_4 is adjacent to $v \in S'$. Then $S = S' \cup \{u_1, u_2\}$ is a total dominating set of T, so $\gamma_t(T) \leq \gamma_t(T') + 2$. Let X be any γ-set of T with $u_4 \in X$. If $v \notin PN(u_4, X)$, then $X \cap V(T')$ is a dominating set of T', so $\gamma(T') \leq \gamma(T) - \gamma(P_5) = \gamma(T) - 2$. If $v \in PN(u_4, X)$, then $X \cap V(T') \succ T' - \{v\}$. So $\gamma(T') \leq \gamma(T' - \{v\}) + 1 \leq \gamma(T) - 1$. We claim that $\gamma(T') \leq \gamma(T) - 2$. Otherwise, suppose that $\gamma(T') = \gamma(T) - 1$, then $X' = (X \cap V(T')) \cup \{v\}$ is a γ-set of T' and $PN(v, X') = \{v\}$ in T', contradicts to the choice of v required by Type-3 operation. So, $\gamma(T') \leq \gamma(T) - 2$. Since $\gamma_t(T) \leq \gamma_t(T') + 2 = \gamma(T') + 2 \leq \gamma(T)$, we have $\gamma(T) = \gamma_t(T)$ and $T \in \mathcal{J}_t$.

If $i = 4$, we can prove $T \in \mathcal{J}_t$ similar to the proof of $i = 2$ and 3.

If $i = 5$, then $G_5 = T_1$, where T_1 is the tree defined in Type-5 operation. Then $S = S' \cup \{u_1, u_2\}$ is a total dominating set of T, so $\gamma_t(T) \leq \gamma_t(T') + 2$. Let X be any γ-set of T. If $u_0 \notin X$, then $X \cap V(T') \succ T'$ and $X \cap V(T_1) \succ T_1$. Since $|X \cap V(T_1)| \geq |\gamma(T_1)| = 2$, $\gamma(T') \leq |X \cap V(T')| \leq \gamma(T) - 2$. If $u_0 \in X$, then $X \cap V(T') \succ T' - \{v\}$ and $|X \cap V(T_1)| \geq 3$. So, $\gamma(T') \leq \gamma(T' - \{v\}) + 1 \leq |X \cap V(T')| + 1 \leq \gamma(T) - 3 + 1 = \gamma(T) - 2$. By $\gamma_t(T) \leq \gamma_t(T') + 2 = \gamma(T') + 2 \leq \gamma(T)$, we have $\gamma(T) = \gamma_t(T)$ and $T \in \mathcal{J}_t$. □

Lemma 8

$$\mathcal{F}_t \subseteq \mathcal{J}_t.$$

Proof. Note that $P_4 \in \mathcal{J}_t$. Let $T \in \mathcal{F}_t$ be a tree obtained from P_4 by a number of operations of Type-1, Type-2, Type-3, Type-4 or Type-5. By Lemma 7, we can easily prove that $T \in \mathcal{J}_t$ by induction on the number of operations required to construct the tree T. □

Lemma 9

$$\mathcal{J}_t \subseteq \mathcal{F}_t.$$

Proof. Let T be any tree in $\mathcal{J}_t$, we prove that $T \in \mathcal{F}_t$ by induction on n, the order of the tree T. If $n \leq 4$, then $T \in \mathcal{J}_t = \{P_4\}$ and clearly $T \in \mathcal{F}_t$. Assume that the result is true for all trees $T' \in \mathcal{J}_t$ of order $n' < n$, where $n \geq 5$. Let $T \in \mathcal{J}_t$ be a tree of order n and let $v_0v_1v_2\cdots v_l$ be a longest path in T. By Lemmas 6 and 3, 5, T has a unique γ_t-set S and $\{v_1, v_2\} \subseteq S$. In the following, we will follow the notation from the proof of Lemma 6 and omit the proof of some results which have been proved in Lemma 6.

If $d(v_1) \geq 3$, then there exists a leaf u adjacent to v_1 in T. Let $T' = T - \{u\}$. As shown in Lemma 6, $\gamma_t(T') = \gamma(T')$. Hence, $T' \in \mathcal{J}_t$. By the inductive hypothesis, $T' \in \mathcal{F}_t$. Hence T is obtained from T' by a type-1 operation. Therefore, $T \in \mathcal{F}_t$. Now assume that $d(v_1) = 2$.

Case 1: $d(v_2) \geq 3$.

Case 1.1: There is a remote vertex $u \in N(v_2) \setminus \{v_1, v_3\}$.

Let $T' = T_{v_2}$ be the subtree of $T - v_2u$ containing v_2. As shown in Lemma 6, $T' \in \mathcal{J}_t$ and $\gamma_t(T) = \gamma_t(T') + 1$. By inductive hypothesis, $T' \in \mathcal{F}_t$. Now, we prove that v_2 satisfies the conditions required by the Type-2 operation. Let X' be any γ-set of T', we claim that there is no vertex $w \in X'$ such that $PN(w, X') = \{v_2\}$. Otherwise, $X = (X' \setminus \{w\}) \cup \{u\}$ is a dominating set of T, then $\gamma(T) \leq \gamma(T')$, contradicts to the fact that $\gamma(T) = \gamma_t(T) = \gamma_t(T') + 1 = \gamma(T') + 1$. Therefore, T is obtained from T' by a Type-2 operation. Thus $T \in \mathcal{F}_t$.

Case 1.2: All of $u \in N(v_2) \setminus \{v_1, v_3\}$ are leaves of T.

If $v_3 \in S$, then $v_3 \notin PN(v_2, S)$. Let $T' = T_{v_2}$ be the subtree of $T - v_1v_2$ containing v_2. As shown in Lemma 6, $T' \in \mathcal{J}_t$ and $\gamma_t(T) = \gamma_t(T') + 1$. By inductive hypothesis, $T' \in \mathcal{F}_t$. We prove that v_2 satisfies the conditions required by Type-2 operation. For every γ-set X' of T', we claim that there is no vertex $w \in X'$ such that $PN(w, X') = \{v_2\}$. Otherwise, $X = (X' \setminus \{w\}) \cup \{v_1\}$ is a dominating set of T. Then $\gamma(T) \leq \gamma(T')$, contradicts to the fact that $\gamma(T) = \gamma_t(T) = \gamma_t(T') + 1 = \gamma(T') + 1$. Therefore, T is obtained from T' by a Type-2 operation. Thus $T \in \mathcal{F}_t$.

If $v_3 \notin S$ and $v_3 \notin PN(v_2, S)$, let $T' = T_{v_3}$ denote the subtree of $T - v_2v_3$ containing v_3, as shown in Lemma 6, $T' \in \mathcal{J}_t$ and $\gamma_t(T) = \gamma_t(T') + 2$. By inductive hypothesis, $T' \in \mathcal{F}_t$. We prove that v_3 satisfies the conditions required by Type-4 operation. For every γ-set X' of T', we claim that there is no vertex $w \in X'$ such that $PN(w, X') = \{v_3\}$. Otherwise, $X = (X' \setminus \{w\}) \cup \{v_1, v_2\}$ is a dominating set of T. Then $\gamma(T) \leq \gamma(T') + 1$, contradicts to the fact that $\gamma(T) = \gamma_t(T) = \gamma_t(T') + 2 = \gamma(T') + 2$. Therefore, T is obtained from T' by a Type-4 operation and a finite sequence of operations of Type-1. Thus $T \in \mathcal{F}_t$.

If $v_3 \in PN(v_2, S)$, as shown in Lemma 6, $d(v_3) = 1$ or $d(v_3) = 2$. If $d(v_3) = 1$, then T is obtained from P_4 by a finite sequence of operations of Type-1. Then $T \in \mathcal{F}_t$. If $d(v_3) = 2$, let T_{v_3} be the subtree of $T - v_3v_4$ containing v_3 and $T' = T_{v_4}$ be the subtree of $T - v_3v_4$ containing v_4, then T_{v_3} is obtained from T_1 (T_1 is the tree defined in Type-5 operation) by a finite sequence of operations of Type-1. As discussed in Lemma 6, $T' \in \mathcal{J}_t$. By inductive hypothesis, $T' \in \mathcal{F}_t$. So, T is obtained from T' by a Type-5 operation and a finite sequence of operations of Type-1. Thus $T \in \mathcal{F}_t$.

Case 2: $d(v_2) = 2$.

As discussed in Lemma 6, we have $d(v_3) = d(v_4) = 2$ and $v_3, v_4 \notin S$, $v_5 \in S$. Let $T' = T - \{v_0, v_1, v_2, v_3, v_4\}$. Then $S - \{v_1, v_2\}$ is a total dominating set of T', so, $\gamma_t(T') \leq \gamma_t(T) - 2$. Since $\gamma(T) \leq \gamma(T') + 2 \leq \gamma_t(T') + 2 \leq \gamma_t(T) = \gamma(T)$, we have $\gamma(T') = \gamma_t(T')$. Then $T' \in \mathcal{J}_t$. By inductive hypothesis, $T' \in \mathcal{F}_t$. For every γ-set X' of T', we claim that there is no vertex w such that $PN(w, X') = \{v_5\}$. Otherwise, $X = (X' \setminus \{w\}) \cup \{v_1, v_4\}$ is a dominating set of T, then $\gamma(T) \leq \gamma(T') + 1$, contradicts to the fact that $\gamma(T) = \gamma_t(T) \geq \gamma_t(T') + 2 = \gamma(T') + 2$. Thus T is obtained from T' by a Type-3 operation. Then $T \in \mathcal{F}_t$.

The proof is completed. □

Theorem 2 follows as an immediate consequence of Lemma 8 and Lemma 9.

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