Diagonalised Lattices and the Steinhaus Chessboard Theorem

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Abstract

A diagonalised lattice is a two dimensional grid, where we add exactly one arbitrary diagonal in each square, and color each vertex black or white. We show that for every diagonalised lattice there is a walk from the left to the right, using only black vertices, if and only if there is no walk from the top to the bottom, using only white vertices.

1 Introduction

The following chessboard problem, which is also called Steinhaus Chessboard Theorem, is due to Steinhaus [4, p. 36]. Let there be a chessboard with n rows and m columns. The squares are colored arbitrarily black and white. Assume that a king is not able to step from the left column to the right column, by using only black squares. Then a rook can go from the top row to the bottom row, by using only white squares. Surowka [5] gives a combinatorial proof of a theorem that implies the Steinhaus Chessboard Theorem. In Section 4, we will prove the Steinhaus Chessboard Theorem with a theorem for *diagonalised lattices*, which we define in Section 2.

Gale [1] showed that the game of Hex can not end in a draw, and that this fact is equivalent to the Brouwer Fixed-Point Theorem. The dual graph of the game of Hex is a diagonalised lattice, where all diagonals are directed in the same way. One direction of our Theorem 1 in Section 3 says that, if in a diagonalised lattice there is no horizontal black walk then there is a vertical white walk. This statement is also equivalent to the Brouwer Fixed-Point Theorem, as in Gales proof [1, Section 3] the direction of the diagonals does not play any role. Nevertheless, we will give a proof by induction for both directions of Theorem 1 in Section 3, and notice that the Brouwer Fixed-Point Theorem occurs in the proof of both directions.

The following idea is contained in the proof of a basic result of plane topology [2, Lemma 2.1] of Luo *et al.*. Let there be a (plane) brick wall such that some of the bricks are colored black and the others are colored white. If there is no way from the left to the right using only black bricks, then there is a way from the top to the bottom using only white bricks. We notice that the dual graph of such a brick wall is a diagonalised lattice.

2 Definition of Diagonalised Lattices

An *undirected graph* G is a pair (V, E), where $V = V(G)$ is a finite set, and $E = E(G)$ is a subset of $\{\{a, b\} \mid a, b \in V \text{ and } a \neq b\}$. We will write $a \sim b$ if $\{a, b\} \in E$. A *bimarked* graph $G \equiv (V, E, M)$ is an undirected graph $G \equiv (V, E)$ together with a function $M : V \to \{0, 1\}$. We say a vertex $u \in V$ is *black* if $M(u) = 0$ and it is *white* if $M(u) = 1$.

A *diagonalised lattice* $G \equiv (V, E, M)$ is a bimarked graph, where $x_a, x_b, y_a, y_b \in \mathbb{Z}$, $\mathbb{Z}$ is the set of integers, $x_a \leq x_b$, $y_a \leq y_b$, and $V = \{(x, y) \in \mathbb{Z}^2 | x_a \leq x \leq x_b \text{ and } y_a \leq y \leq y_b\}$. The edges of G are defined as follows. For $x_1, x_2, y_1, y_2 \in \mathbb{Z}$ we have on the one hand $(x_1, y_1) \sim (x_2, y_2)$ if $|x_2 - x_1| = 0$ and $|y_2 - y_1| = 1$ or $|x_2 - x_1| = 1$ and $|y_2 - y_1| = 0$. On the other hand, if $x_2 - x_1 = 1$ and $y_2 - y_1 = 1$, then we have exactly one of the edges $(x_1, y_1) \sim (x_2, y_2)$ and $(x_1, y_2) \sim (x_2, y_1)$, i.e., the *diagonals* in G. The *width* of G is $x_b - x_a$ and the *height* of G is $y_b - y_a$. We now define

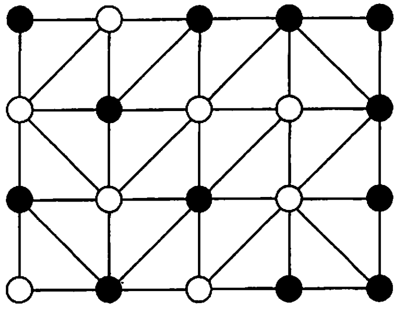

Figure 1: Example of a diagonalised lattice.

$L = \{(x, y) \in V | x = x_a\}$, $R = \{(x, y) \in V | x = x_b\}$, $T = \{(x, y) \in V | y = y_b\}$, and $B = \{(x, y) \in V | y = y_a\}$, i.e., the left, right, top, and bottom side of the diagonalised lattice.

Let $G \equiv (V, E)$ be an undirected graph. A *walk* of length $n \geq 1$ from a to b in G is a sequence $(a_0, a_1, \ldots, a_n)$ of vertices in V such that $a_0 = a$, $a_n = b$, and $a_{i-1} \sim a_i$ for $i = 1, \ldots, n$. If all vertices of a walk are distinct then it is a *path*. A walk is *black* if all its vertices are black, and it is *white* if all its vertices are white. We call a walk from a to b *horizontal* if $a \in L$ and $b \in R$ and we call it *vertical* if $a \in T$ and $b \in B$. By a *Jordan arc* J we denote a continuous map from $[0, 1]$ into $\mathbb{R}^2$ such that $J(x) \neq J(y)$ for $x \neq y$. A graph G is *planar* if there is a drawing of G in the plane such that the vertices are drawn as points at pairwise different positions, each edge is represented by a Jordan arc between the corresponding points, and two Jordan arcs may only intersect at points of common vertices.

3 The Main Result

Theorem 1 *In every diagonalised lattice there is a horizontal black walk if and only if there is no vertical white walk.*

Proof. Let $G \equiv (V, E, M)$ be a diagonalised lattice. To prove the first direction, we indirectly assume that there is a horizontal black walk and a vertical white walk in G. Let G be drawn planar in the plane such that a vertex $(x, y) \in V$ is represented by the point (x, y) in the plane and the edges are represented by straight lines. There is a shortest horizontal black walk and a shortest vertical white walk in G, which means that both walks are paths. The representations of those paths are continuous paths in the rectangular set $\{(x, y) \in \mathbb{R}^2 | x_a \leq x \leq x_b \text{ and } y_a \leq y \leq y_b\}$. By Lemma 2 of Maehara[3] the continuous paths must have a common point. The only allowed common points of Jordan arcs in a planar drawing are points of common vertices. Thus, the two paths have a common vertex, which is a contradiction. We notice that, for the proof of Lemma 2, Maehara uses the Brouwer Fixed-Point Theorem, which he then applies to prove the Jordan Curve Theorem.

To prove the second direction, we have to show that there is a horizontal black walk in G or there is a vertical white walk in G. If the height of G is 0, we have a vertical white walk in G if any of the vertices is white and otherwise we have a horizontal black walk in G. The same argument works if the width of G is 0.

If the height of G is 1, we use induction on n, the width of G. The case $n = 0$ was discussed before. Let $n \geq 1$ and G' be G without the right two vertices, indicated by the gray color in Figure 2 and Figure 3, where the diagonals are not shown. We indirectly assume that there is no horizontal black walk and no vertical white walk in G. Thus, there cannot be a vertical white walk in G', which implies the existence of a horizontal black walk in G', by the induction hypothesis. Up to isomorphism, there are only two possibilities, of which the first is shown in Figure 2. Here, the upper gray vertex has to be white and this

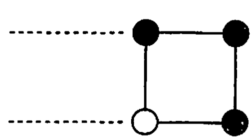

Figure 2:

implies that the lower gray vertex is black, and therefore, one diagonal yields a vertical white walk in G and the other produces a horizontal black walk in

G, contradicting our assumption. Figure 3 shows the second possibility, where both gray vertices have to be white, which also is a contradiction. If the width

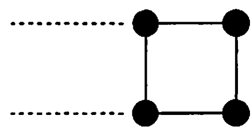

Figure 3:

of G is 1, the proof works in the same way.

Now, we indirectly assume that G is a diagonalised lattice with minimum number of vertices, such that there is no horizontal black walk and no vertical white walk in G. We denote G without L, R, T, and B by M, as in Figure 4. M

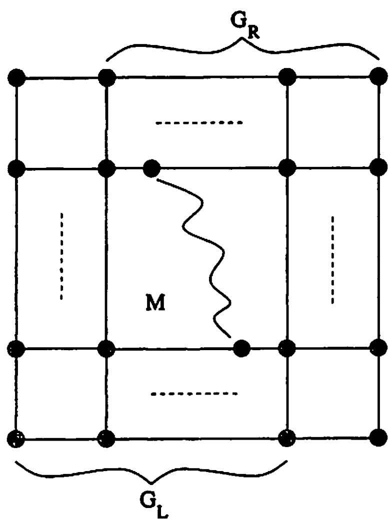

Figure 4:

is not empty, since the width and the height of G must be greater than 1, which we proved before. In M there has to be a horizontal white walk or a vertical black walk. Since otherwise we could rotate M by 90 degrees and obtain a diagonalised lattice, with no horizontal black walk and no vertical white walk in it, which is a contradiction to the minimality of G. We assume without loss of generality that there is a vertical black walk in M, as shown in Figure 4, since the proof works analogously if we assume a horizontal white walk in M. We denote the graph without R by G_L and the graph without L by G_R. In G_L and G_R we have a horizontal black walk, since by our assumption there is no vertical white walk. The two horizontal black walks of G_L and G_R combined with the vertical black walk of M produce a horizontal black walk of G, which is a contradiction. Actually, at this point, we again use Lemma 2 of Maehara[3], in a similar way as above, and therefore the Brouwer Fixed-Point Theorem also occurs implicitely in the proof of the second direction of our theorem. □

4 Steinhaus Chessboard Theorem

To prove the Steinhaus Chessboard Theorem with Theorem 1, we assume, that the rook has no possibility to go from the top row to the bottom row, by using only white squares on an arbitrarily black and white marked chessboard. We will show that the king has a walk from the left to the right.

The dual graph of the chessboard is a lattice. We mark the vertices of this lattice with the color of the corresponding squares of the chessboard. Now, we will add diagonals to the squares of the lattice to obtain a diagonalised lattice. If all vertices of a square S are white, we may add any diagonal to S, without producing a vertical white walk, as this vertical white walk would have been in the lattice before, also. If at least one vertex of S is black, we add a diagonal to S which leads into at least one black vertex. Again no vertical white walk can arise. After we have added a diagonal to each square of the lattice, we apply Theorem 1 and obtain that there is a horizontal black walk. This horizontal black walk is a walk for the king on black squares of the chessboard from left to right. □

5 Acknowledgement

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