Unimodality of independence polynomials of very well-covered graphs

Shih-Yan Chen* and Hsin-Ju Wang†

Abstract. In this paper, we show that the independence polynomial $I(G^*;x)$ of G^* is unimodal for any graph G^* whose skeleton G has stability number $\alpha(G) \leq 8$. In addition, we show that the independence polynomial of $K_{2,n}^*$ is log-concave with unique mode.

1 Introduction

In this paper, all graphs are undirected and simple. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *order* of a graph G is the cardinality of $V(G)$. A *stable set* in a graph G is a set of pairwise non-adjacent vertices. The *stability number* $\alpha(G)$ of G is the cardinality of a maximum stable set in G. We use s_k for the number of stable sets in G of cardinality k ($s_0 = 1$). The sequence $\{s_0, \ldots, s_\alpha\}$ is called the *independence sequence* of G. The polynomial $I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is called the *independence polynomial* of G (Gutman and Harry, [3]). A number of general properties of independence polynomial of a graph are shown in [1] and [3]. A sequence $\{a_0, \ldots, a_n\}$ of real numbers is said to be:

- *unimodal* if there exists some $k \in \{1, 2, \ldots, n\}$, called the *mode* of the sequence, such that $0 \leq a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$.
- *log-concave* if $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ holds for $i \in \{1, 2, \ldots, n-1\}$.

An independence polynomial of G is called unimodal (log-concave) if the independence sequence formed by its coefficients is unimodal (log-concave). We use *mode*(G) for the mode of the independence sequence. A graph G is *well-covered* if all its maximal stable sets are of the same size. A well-covered graph is called *very well-covered* if it has no isolated vertices and its order equals $2\alpha(G)$. For instance, the graph G^*, which is obtained

*Department of Applied Mathematics, Chung Yuan Christian University, Taiwan.
†Department of Mathematics, National Chung Cheng University, Taiwan.

from G by appending a single pendant edge to each vertex of G, is a very well-covered graph.

Recently, there has been a lot of investigation on the unimodality of independence polynomials of graphs. J.I. Brown, K. Dilcher and R.J. Nowakowski [2] conjectured that $I(G;x)$ is unimodal for any well-covered graph G. However, T.S. Michael and W.N. Traves [9] provided examples of well-covered graphs with non-unimodal independence polynomials. Nevertheless, the conjecture of Brown *et al.* is still open for very well-covered graphs.

In [4], [5], [7] and [8], Levit and Mandrescu investigated some properties of very well-covered graphs. For example, they showed in [5] that the independence polynomial $I(G^*;x)$ of G^* is unimodal for any G with $\alpha(G) \leq 4$. They also showed in [7] and [8], that the independence polynomial of $K_{1,n}^*$ is log-concave with unique mode. The goal of this paper is to show that the independence polynomial $I(G^*;x)$ of G^* is unimodal for any G with $\alpha(G) \leq 8$ and the independence polynomial of $K_{2,n}^*$ is log-concave with unique mode.

2 Preliminaries

Throughout, let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The neighbor of a subset S of V is the set $N(S) = \{w \mid w \in V, wv \in E \textit{ for some } v \in S\}$, while $N[S] = N(S) \cup S$. By $G_1 \sqcup G_2$ we denote the disjoint union of the graphs G_1, G_2. That is, the graph with $V = V(G_1) \cup V(G_2)$ and $E = E(G_1) \cup E(G_1)$. In particular, $\sqcup nG$ means the disjoint union of n copies of the graph G. A graph G is a *complete bipartite graph* with vertex classes V_1 and V_2 if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and each edge joins a vertex of V_1 to a vertex of V_2. If $|V_1| = m$ and $|V_2| = n$, then we use the symbol $K_{m,n}$ for the complete bipartite graph. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph.* We use K_n for the complete graph with n vertices.

We state some useful results in this section. In the sequel, we use $\lceil n \rceil$ for the smallest integer that is greater than or equal to n and $\lfloor n \rfloor$ for the largest integer that is smaller than or equal to n.

Theorem 2.1 *[7] If G is a very well-covered graph of order n with $\alpha(G) = \alpha$, then $s_0 \leq s_1 \leq \cdots \leq s_{\lceil \alpha/2 \rceil}$ and $s_{\lceil (2\alpha-1)/3 \rceil} \geq \cdots \geq s_{\alpha-1} \geq s_\alpha$.*

Theorem 2.2 *[6] Let G be a graph of order n and $I(G;x) = \sum_{i=0}^{\alpha(G)} s_i x^i$.*

If $I(G^*;x) = \sum_{i=0}^{\alpha(G^*)} t_i x^i$, *then*

$$t_k = \sum_{j=0}^{k} s_j \cdot \binom{n-j}{k-j}$$

for $0 \le k \le n = \alpha(G^*)$.

Lemma 2.3 *If G is a graph of order n with stability number $\alpha(G) = \alpha$, then*

$$\binom{\alpha}{k} \le s_k \le \binom{\alpha}{k} \cdot 2^{n-\alpha}.$$

Proof. Since the stability number of G is α, there exists a stable set, say S_1, consisting of α vertices. It is clear that the number of stable sets in the subgraph induced by S_1 of cardinality k is $\binom{\alpha}{k}$, therefore, $s_k \ge \binom{\alpha}{k}$.
To obtain the upper bound, let $S_2 = V(G) - S_1$. Notice that to choose k vertices from $V(G)$ to form a stable set one can choose i vertices from S_2 and $k-i$ vertices from S_1. Notice also that if we choose i vertices, say $v_1, \ldots, v_i$, from S_2, then there are at most $\binom{\alpha-i}{k-i}$ ways to choose $k-i$ vertices from S_1 to form a stable set of cardinality k. For if we let $S_2' = \{v_1, \ldots, v_i\}$, then $|N(S_2') \cap S_1| \ge i$. Otherwise, there are vertices, say $u_1, u_2, \ldots, u_j$ ($j > \alpha - i$) of $S_1 - N(S_2')$ such that $\{u_1, u_2, \ldots, u_j, v_1, v_2, \ldots, v_i\}$ is a stable set of cardinality larger than α. This yields a contradiction. Thus $|N(S_2') \cap S_1| \ge i$. That is, if we choose i vertices from S_2 then there are at most $\binom{\alpha-i}{k-i}$ ways to choose $k-i$ vertices from S_1 to form a stable set of cardinality k. Since $|S_2| = n - \alpha$, there are at most $\binom{n-\alpha}{i}$ ways to choose i vertices from S_2, we see that

$$\begin{aligned} s_k &\le \textstyle\sum_{i=0}^{k} \binom{n-\alpha}{i}\binom{\alpha-i}{k-i} \\ &\le \textstyle\sum_{i=0}^{k} \binom{n-\alpha}{i}\binom{\alpha}{k} \\ &\le \textstyle\sum_{i=0}^{n-\alpha} \binom{n-\alpha}{i}\binom{\alpha}{k} \\ &= \textstyle\binom{\alpha}{k} \cdot 2^{n-\alpha}. \end{aligned}$$

□

3 The unimodality of G^*

Throughout, let G be a graph of order n with $\alpha(G) \ge 5$ and G^* be the very well-covered graph obtained from G by appending a single pendant edge to each vertex of G. Let $I(G;x) = \sum_{i=0}^{\alpha(G)} s_i x^i$ and $I(G^*;x) = \sum_{i=0}^{n} t_i x^i$.

Theorem 3.1 *Let G be a graph of order n with $\alpha(G) = 5$. Then $I(G^*; x)$ is unimodal with*

$$\left\lceil \frac{n+1}{2} \right\rceil \leq mode(G^*) \leq \left\lceil \frac{n+1}{2} \right\rceil + 2.$$

Moreover, if n is even, then

$$\left\lceil \frac{n+1}{2} \right\rceil \leq mode(G^*) \leq \left\lceil \frac{n+1}{2} \right\rceil + 1.$$

Proof. Case 1. n is odd, say $n = 2m + 1$. Since $\alpha(G) = 5$, we only consider the case that $m \geq 2$. We first show that $t_0 \leq t_1 \leq \cdots \leq t_{m+1}$ and $t_{m+3} \geq t_{m+4} \geq \cdots \geq t_{2m+1}$. By Theorem 2.1, we have that $t_0 \leq t_1 \leq \cdots \leq t_{m+1}$. On the other hand,

$$\begin{aligned} t_i - t_{i+1} &= \textstyle\sum_{j=0}^{5}[\binom{2m+1-j}{i-j} - \binom{2m+1-j}{i+1-j}]s_j \\ &\geq 0 \end{aligned}$$

holds for any $m + 3 \leq i \leq 2m$. Therefore $t_{m+3} \geq t_{m+4} \geq \cdots \geq t_{2m+1}$. Moreover, observe that if $m \geq 3$, then

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = & \textstyle\sum_{i=0}^{5} s_i \cdot [2\binom{2m+1-i}{m+2-i} - \binom{2m+1-i}{m+1-i} - \binom{2m+1-i}{m+3-i}] \\ = & \textstyle\sum_{i=0}^{5} s_i \cdot \frac{(2m+1-i)!}{m!(m+3-i)!} \cdot (2m - i^2 + 5i - 6) \\ \geq & \ 0 \end{aligned} \tag{1}$$

as the coefficients near s_i are non-negative, therefore $2t_{m+2} - t_{m+1} - t_{m+3} \geq 0$. Thus (1) shows that either $t_{m+2} \geq t_{m+1}$ or $t_{m+2} \geq t_{m+3}$ if $m \geq 3$. If $m = 2$, then $n = 5$. Since $\alpha(G) = 5$, $G = \sqcup 5K_1$ and $G^* = \sqcup 5K_2$. It is easy to see that

$$I(\sqcup 5K_2; x) = (1 + 2x)^5 = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$$

is unimodal with $mode(\sqcup 5K_2) = 3$.
From the above, we conclude that the sequence $\{t_0, t_1, \ldots, t_{2m+1}\}$ is unimodal and the possible positions for its mode are $m + 1$, $m + 2$ and $m + 3$.

Case 2. n is even, say $n = 2m$. Since $\alpha(G) = 5$, $m \geq 3$. By Theorem 2.1, we have that $t_0 \leq t_1 \leq \ldots \leq t_m$. Moreover,

$$\begin{aligned} t_i - t_{i+1} &= \textstyle\sum_{j=0}^{5}[\binom{2m-j}{i-j} - \binom{2m-j}{i+1-j}]s_j \\ &\geq 0 \end{aligned}$$

holds for any $m+2 \le i \le 2m-1$, therefore $t_{m+2} \ge t_{m+3} \ge \cdots \ge t_{2m}$. Now, by Lemma 2.3,

$$\begin{aligned} t_{m+1} - t_m &\ge [\binom{2m}{m+1} - \binom{2m}{m}]s_0 + [\binom{2m-2}{m-1} - \binom{2m-2}{m-2}]s_2 \\ &\ge [\binom{2m}{m+1} - \binom{2m}{m}] \cdot 1 + [\binom{2m-2}{m-1} - \binom{2m-2}{m-2}] \cdot \binom{5}{2} \\ &= \frac{-(2m)!}{m!(m+1)!} + \frac{10(2m-2)!}{m!(m-1)!} \\ &= \frac{(2m-2)!}{m!(m+1)!}(6m^2 + 12m). \end{aligned}$$

Thus $t_{m+1} > t_m$ for $m \ge 3$. Therefore, the sequence $\{t_0, t_1, \ldots, t_{2m}\}$ is unimodal and the possible positions for its mode are $m+1$ and $m+2$. □

Theorem 3.2 *Let G be a graph of order n with $\alpha(G) = 6$. Then $I(G^*; x)$ is unimodal with*

$$\left\lceil \frac{n+1}{2} \right\rceil \le mode(G^*) \le \left\lceil \frac{n+1}{2} \right\rceil + 2.$$

Proof. Case 1. n is odd, say $n = 2m+1$. Since $\alpha(G) = 6$, $m \ge 3$. By Theorem 2.1, we have that $t_0 \le t_1 \le \cdots \le t_{m+1}$. On the other hand,

$$\begin{aligned} t_i - t_{i+1} &= \sum_{j=0}^{6}[\binom{2m+1-j}{i-j} - \binom{2m+1-j}{i+1-j}]s_j \\ &\ge 0 \end{aligned}$$

holds for any $m+3 \le i \le 2m$. Therefore $t_{m+3} \ge t_{m+4} \ge \cdots \ge t_{2m+1}$. Moreover, observe that if $m \ge 6$, then

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = & \sum_{i=0}^{6} s_i \cdot [2\binom{2m+1-i}{m+2-i} - \binom{2m+1-i}{m+1-i} - \binom{2m+1-i}{m+3-i}] \\ = & \sum_{i=0}^{6} s_i \cdot \frac{(2m+1-i)!}{m!(m+3-i)!} \cdot (2m - i^2 + 5i - 6) \\ \ge & \ 0 \end{aligned} \quad (2)$$

as the coefficients near s_i are non-negative.
If $3 \le m \le 5$, then by Lemma 2.3,

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = & \sum_{i=0}^{6} s_i \cdot \frac{(2m+1-i)!}{m!(m+3-i)!} \cdot (2m - i^2 + 5i - 6) \\ \ge & \frac{1}{m!}[\frac{(2m-2)!}{m!}(2m)s_3 + \frac{(2m-5)!}{(m-3)!}(2m-12)s_6] \\ = & \frac{1}{m!}\frac{(2m-5)!}{(m-3)!}[(2m-12)s_6 + \frac{2m(2m-2)(2m-3)(2m-4)}{m(m-1)(m-2)}s_3]. \\ \ge & \frac{1}{m!}\frac{(2m-5)!}{(m-3)!}[(2m-12)2^{2m+1-6} + 160(2m-3)] \\ > & \ 0. \end{aligned} \quad (3)$$

Therefore (2) and (3) show that either $t_{m+2} \ge t_{m+1}$ or $t_{m+2} \ge t_{m+3}$. From the above, we conclude that the sequence $\{t_0, t_1, \ldots, t_{2m+1}\}$ is unimodal and the possible positions for its mode are $m+1$, $m+2$ and $m+3$.

Case 2. n is even, say $n = 2m$. Since $\alpha(G) = 6$, $m \geq 3$. By Theorem 2.1, we have that $t_0 \leq t_1 \leq \cdots \leq t_m$. Moreover,

$$\begin{aligned} t_i - t_{i+1} &= \sum_{j=0}^{6}[\binom{2m-j}{i-j} - \binom{2m-j}{i+1-j}]s_j \\ &\geq 0 \end{aligned}$$

holds for any $m + 3 \leq i \leq 2m - 1$, therefore $t_{m+3} \geq t_{m+4} \geq \cdots \geq t_{2m}$. Now, by Lemma 2.3,

$$\begin{aligned} t_{m+1} - t_m &\geq [\binom{2m}{m+1} - \binom{2m}{m}]s_0 + [\binom{2m-2}{m-1} - \binom{2m-2}{m-2}]s_2 \\ &\geq [\binom{2m}{m+1} - \binom{2m}{m}] \cdot 1 + [\binom{2m-2}{m-1} - \binom{2m-2}{m-2}] \cdot \binom{6}{2} \\ &= \frac{-(2m)!}{m!(m+1)!} + \frac{15(2m-2)!}{m!(m-1)!} \\ &= \frac{(2m-2)!}{m!(m+1)!}(11m^2 + 17m). \end{aligned}$$

Thus $t_{m+1} > t_m$ for $m \geq 3$. Moreover, observe that if $m \geq 7$, then

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = \; & \sum_{i=0}^{6} s_i \cdot [2\binom{2m-i}{m+2-i} - \binom{2m-i}{m+1-i} - \binom{2m-i}{m+3-i}] \\ = \; & \sum_{i=0}^{6} s_i \cdot [\frac{2(2m-i)!}{(m+2-i)!(m-2)!} - \frac{(2m-i)!}{(m+1-i)!(m-1)!} - \frac{(2m-i)!}{(m+3-i)!(m-3)!}] \\ = \; & \sum_{i=0}^{6} s_i \cdot \frac{(2m-i)!}{(m-1)!(m+3-i)!} \cdot (2m - i^2 + 7i - 14) \\ > \; & 0 \end{aligned} \tag{4}$$

as the coefficients near s_i are non-negative.
If $m = 3$, then $n = 6$. Since $\alpha(G) = 6$, $G = \sqcup 6K_1$ and $G^* = \sqcup 6K_2$. It is easy to see that

$$I(\sqcup 6K_2; x) = (1 + 2x)^6 = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$$

is unimodal with $mode(\sqcup 6K_2) = 4 = m + 1$.
If $4 \leq m \leq 6$, then by Lemma 2.3

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = \; & \sum_{i=0}^{6} s_i \cdot \frac{(2m-i)!}{(m-1)!(m+3-i)!} \cdot (2m - i^2 + 7i - 14) \\ \geq \; & \frac{1}{(m-1)!}[\frac{(2m)!}{(m+3)!}(2m - 14)s_0 + \frac{(2m-3)!}{m!}(2m - 2)s_3] \\ = \; & \frac{1}{(m-1)!}\frac{(2m-3)!}{m!}[\frac{2m(2m-1)(2m-2)}{(m+3)(m+2)(m+1)}(2m - 14)s_0 + (2m - 2)s_3] \\ \geq \; & \frac{1}{(m-1)!}\frac{(2m-3)!}{m!}[\frac{2m(2m-1)(2m-2)}{(m+3)(m+2)(m+1)}(2m - 14) + (2m - 2)\binom{6}{3}] \\ > \; & 0. \end{aligned} \tag{5}$$

Therefore (4) and (5) show that either $t_{m+2} \geq t_{m+1}$ or $t_{m+2} \geq t_{m+3}$ if $m \geq 4$.
From the above, we conclude that the sequence $\{t_0, t_1, \ldots, t_{2m}\}$ is unimodal and the possible positions for its mode are $m + 1$, $m + 2$ and $m + 3$. □

Theorem 3.3 *Let G be a graph of order n with $\alpha(G) = 7$. Then $I(G^*; x)$ is unimodal with*

$$\left\lceil \frac{n+1}{2} \right\rceil \leq mode(G^*) \leq \left\lceil \frac{n+1}{2} \right\rceil + 3.$$

Moreover, if n is even, then

$$\left\lceil \frac{n+1}{2} \right\rceil \leq mode(G^*) \leq \left\lceil \frac{n+1}{2} \right\rceil + 2.$$

Proof. Case 1. n is odd, say $n = 2m + 1$. Since $\alpha(G) = 7$, $m \geq 3$. By Theorem 2.1, we have that $t_0 \leq t_1 \leq \cdots \leq t_{m+1}$. On the other hand,

$$\begin{aligned} t_i - t_{i+1} &= \textstyle\sum_{j=0}^{7}[\binom{2m+1-j}{i-j} - \binom{2m+1-j}{i+1-j}]s_j \\ &\geq 0 \end{aligned}$$

holds for any $m + 4 \leq i \leq 2m$. Therefore $t_{m+4} \geq t_{m+5} \geq \cdots \geq t_{2m+1}$. To finish the proof we need only to show that $2t_{m+2} - t_{m+1} - t_{m+3} \geq 0$ and $2t_{m+3} - t_{m+2} - t_{m+4} \geq 0$. For this, observe that

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = & \textstyle\sum_{i=0}^{7} s_i \cdot [2\binom{2m+1-i}{m+2-i} - \binom{2m+1-i}{m+1-i} - \binom{2m+1-i}{m+3-i}] \\ = & \textstyle\sum_{i=0}^{7} s_i \cdot \frac{(2m+1-i)!}{m!(m+3-i)!} \cdot (2m - i^2 + 5i - 6). \end{aligned}$$

If $m \geq 10$, then $2t_{m+2} - t_{m+1} - t_{m+3} \geq 0$ as the coefficients near s_i are non-negative. If $m = 3$, then $n = 7$. Since $\alpha(G) = 7$, $G = \sqcup 7K_1$ and $G^* = \sqcup 7K_2$. It is easy to see that

$$I(\sqcup 7K_2; x) = (1+2x)^7 = 1+14x+84x^2+280x^3+560x^4+672x^5+448x^6+128x^7 \tag{6}$$

and $2t_5 - t_4 - t_6 \geq 0$.
If $4 \leq m \leq 5$, then by Lemma 2.3

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = & \textstyle\sum_{i=0}^{7} s_i \cdot \frac{(2m+1-i)!}{m!(m+3-i)!} \cdot (2m - i^2 + 5i - 6). \\ \geq & \frac{1}{m!}[\frac{(2m-5)!}{(m-3)!}(2m-12)s_6 + \frac{(2m-6)!}{(m-4)!}(2m-20)s_7 + \frac{(2m-2)!}{(m)!}(2m)s_3] \\ \geq & \frac{1}{m!}[\frac{(2m-5)!}{(m-3)!}(2m-12)\binom{7}{6}2^{2m-6} + \frac{(2m-6)!}{(m-4)!}(2m-20)2^{2m-6} \\ & + \frac{(2m-2)!}{(m)!}(2m)\binom{7}{3}] \\ > & 0. \end{aligned}$$

If $6 \le m \le 9$, then by Lemma 2.3

$$\begin{array}{rl} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = & \sum_{i=0}^{7} s_i \cdot \frac{(2m+1-i)!}{m!(m+3-i)!} \cdot (2m - i^2 + 5i - 6). \\ \geq & \frac{1}{m!}[\frac{(2m-6)!}{(m-4)!}(2m-20)s_7 + \frac{(2m-2)!}{m!}(2m)s_3 + \frac{(2m-3)!}{(m-1)!}(2m-2)s_4] \\ = & \frac{1}{m!}\frac{(2m-6)!}{(m-4)!}[(2m-20)s_7 + \frac{(2m-2)(2m-3)(2m-4)(2m-5)}{m(m-1)(m-2)(m-3)}(2m)s_3 \\ & + \frac{(2m-3)(2m-4)(2m-5)}{(m-1)(m-2)(m-3)}(2m-2)s_4] \\ \geq & \frac{1}{m!}\frac{(2m-6)!}{(m-4)!}[(2m-20)\cdot 2^{2m+1-7} + \frac{(2m-2)(2m-3)(2m-4)(2m-5)}{m(m-1)(m-2)(m-3)}(2m)\binom{7}{3} \\ & + \frac{(2m-3)(2m-4)(2m-5)}{(m-1)(m-2)(m-3)}(2m-2)\binom{7}{4}] \\ > & 0. \end{array}$$

On the other hand, observe that

$$\begin{array}{rl} & 2t_{m+3} - t_{m+2} - t_{m+4} \\ = & \sum_{i=0}^{7} s_i \cdot [2\binom{2m+1-i}{m+3-i} - \binom{2m+1-i}{m+2-i} - \binom{2m+1-i}{m+4-i}] \\ = & \sum_{i=0}^{7} s_i \cdot \frac{(2m+1-i)!}{(m-1)!(m+4-i)!} \cdot (2m - i^2 + 9i - 22). \end{array}$$

If $m \geq 11$, then $2t_{m+3} - t_{m+2} - t_{m+4} \geq 0$ as the coefficients near s_i are non-negative. If $m = 3$, then by (6), $2t_6 - t_5 - t_7 \geq 0$.
If $4 \le m \le 6$, then by Lemma 2.3

$$\begin{array}{rl} & 2t_{m+3} - t_{m+2} - t_{m+4} \\ = & \sum_{i=0}^{7} s_i \cdot \frac{(2m+1-i)!}{(m-1)!(m+4-i)!} \cdot (2m - i^2 + 9i - 22). \\ \geq & \frac{1}{(m-1)!}[\frac{(2m+1)!}{(m+4)!}(2m-22)s_0 \\ & + \frac{(2m)!}{(m+3)!}(2m-14)s_1 + \frac{(2m-2)!}{(m+1)!}(2m-4)s_3] \\ \geq & \frac{1}{(m-1)!}\frac{(2m-2)!}{(m+1)!}[\frac{(2m+1)(2m)(2m-1)}{(m+4)(m+3)(m+2)}(2m-22) \\ & + \frac{(2m)(2m-1)}{(m+3)(m+2)}(2m-14)(2m+1) + (2m-4)\binom{7}{3}] \\ > & 0. \end{array}$$

If $7 \le m \le 10$, then by Lemma 2.3

$$\begin{array}{rl} & 2t_{m+3} - t_{m+2} - t_{m+4} \\ = & \sum_{i=0}^{7} s_i \cdot \frac{(2m+1-i)!}{(m-1)!(m+4-i)!} \cdot (2m - i^2 + 9i - 22). \\ \geq & \frac{1}{(m-1)!}[\frac{(2m+1)!}{(m+4)!}(2m-22)s_0 + \frac{(2m-1)!}{(m+2)!}(2m-8)s_2] \\ \geq & \frac{1}{(m-1)!}\frac{(2m-1)!}{(m+2)!}[\frac{(2m+1)(2m)}{(m+4)(m+3)}(2m-22)\cdot 1 + (2m-8)\binom{7}{2}] \\ > & 0. \end{array}$$

From the above, we conclude that $I(G^*; x)$ is unimodal and the possible positions for its mode are $m+1$, $m+2$, $m+3$ and $m+4$.

Case 2. n is even, say $n = 2m$. Since $\alpha(G) = 7$, $m \geq 4$. By Theorem 2.1, we have that $t_0 \leq t_1 \leq \cdots \leq t_m$. Moreover,

$$\begin{aligned} t_i - t_{i+1} &= \textstyle\sum_{j=0}^{7}[\binom{2m-j}{i-j} - \binom{2m-j}{i+1-j}]s_j \\ &\geq 0 \end{aligned}$$

holds for any $m + 3 \leq i \leq 2m - 1$, therefore $t_{m+3} \geq t_{m+4} \geq \cdots \geq t_{2m}$. Now, by Lemma 2.3,

$$\begin{aligned} & t_{m+1} - t_m \\ \geq\ & [\binom{2m}{m+1} - \binom{2m}{m}]s_0 + [\binom{2m-2}{m-1} - \binom{2m-2}{m-2}]s_2 \\ \geq\ & [\binom{2m}{m+1} - \binom{2m}{m}] \cdot 1 + [\binom{2m-2}{m-1} - \binom{2m-2}{m-2}] \cdot \binom{7}{2} \\ =\ & \tfrac{-(2m)!}{(m+1)!m!} + \tfrac{21(2m-2)!}{m!(m-1)!} \\ =\ & \tfrac{(2m-2)!}{(m+1)!m!}(17m^2 + 23m) \\ >\ & 0. \end{aligned}$$

Moreover, observe that

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ =\ & \textstyle\sum_{i=0}^{7} s_i[2\binom{2m-i}{m+2-i} - \binom{2m-i}{m+1-i} - \binom{2m-i}{m+3-i}] \\ =\ & \textstyle\sum_{i=0}^{7} s_i[\frac{2(2m-i)!}{(m+2-i)!(m-2)!} - \frac{(2m-i)!}{(m+1-i)!(m-1)!} - \frac{(2m-i)!}{(m+3-i)!(m-3)!}] \\ =\ & \textstyle\sum_{i=0}^{7} s_i \cdot \frac{(2m-i)!}{(m-1)!(m+3-i)!} \cdot (2m - i^2 + 7i - 14). \end{aligned}$$

If $m \geq 7$, then $2t_{m+2} - t_{m+1} - t_{m+3} \geq 0$ as the coefficients near s_i are non-negative.
If $4 \leq m \leq 6$, then by Lemma 2.3

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ =\ & \textstyle\sum_{i=0}^{7} s_i \cdot \frac{(2m-i)!}{(m-1)!(m+3-i)!} \cdot (2m - i^2 + 7i - 14) \\ \geq\ & \tfrac{1}{(m-1)!}[\tfrac{(2m)!}{(m+3)!}(2m - 14)s_0 \\ & + \tfrac{(2m-7)!}{(m-4)!}(2m - 14)s_7 + \tfrac{(2m-3)!}{m!}(2m - 2)s_3] \\ \geq\ & \tfrac{1}{(m-1)!}[\tfrac{(2m)!}{(m+3)!}(2m - 14) \\ & + \tfrac{(2m-7)!}{(m-4)!}(2m - 14)2^{2m-7} + \tfrac{(2m-3)!}{m!}(2m - 2)\binom{7}{3}] \\ >\ & 0. \end{aligned}$$

From the above, we conclude that $I(G^*; x)$ is unimodal and the possible positions for its mode are $m + 1$, $m + 2$ and $m + 3$. □

Theorem 3.4 *Let G be a graph of order n with $\alpha(G) = 8$. Then $I(G^*; x)$ is unimodal with*

$$\left\lceil \frac{n+1}{2} \right\rceil \leq mode(G^*) \leq \left\lceil \frac{n+1}{2} \right\rceil + 3.$$

Proof. Case 1. n is odd, say $n = 2m+1$. Since $\alpha(G) = 8$, $m \geq 4$. By Theorem 2.1, we have that $t_0 \leq t_1 \leq \cdots \leq t_{m+1}$. On the other hand,

$$\begin{aligned} t_i - t_{i+1} &= \sum_{j=0}^{8}[\binom{2m+1-j}{i-j} - \binom{2m+1-j}{i+1-j}]s_j \\ &\geq 0 \end{aligned}$$

holds for any $m+4 \leq i \leq 2m$. Therefore $t_{m+4} \geq t_{m+5} \geq \cdots \geq t_{2m+1}$.
To prove the assertion, we first observe that $t_{m+2} > t_{m+1}$ for $4 \leq m \leq 14$. This is because of the following:

$$\begin{aligned} & t_{m+2} - t_{m+1} \\ \geq\ & [\binom{2m+1}{m+2} - \binom{2m+1}{m+1}]s_0 + [\binom{2m}{m+1} - \binom{2m}{m}]s_1 \\ & +[\binom{2m-2}{m-1} - \binom{2m-2}{m-2}]s_3 + [\binom{2m-3}{m-2} - \binom{2m-3}{m-3}]s_4 \\ \geq\ & \frac{-2(2m+1)!}{m!(m+2)!} \cdot 1 + \frac{-(2m)!}{m!(m+1)!} \cdot (2m+1) + \frac{(2m-2)!}{m!(m-1)!} \cdot \binom{8}{3} \\ & + \frac{2(2m-3)!}{m!(m-2)!} \cdot \binom{8}{4} \\ =\ & \frac{(2m-3)!}{(m+2)!m!}[-4m(2m+1)(2m-1)(2m-2) \\ & -2m(2m+1)(2m-1)(2m-2)(m+2) \\ & +56m(2m-2)(m+1)(m+2) + 140m(m-1)(m+1)(m+2)] \\ =\ & \frac{(2m-3)!}{(m+2)!(m-2)!}[-8(2m+1)(2m-1) - 4(2m+1)(2m-1)(m+2) \\ & +252(m+1)(m+2)] \\ >\ & 0. \end{aligned}$$

If $4 \leq m \leq 5$, then $t_{m+2} \geq t_{m+3} \geq \cdots$ by Theorem 2.1, so that $I(G^*;x)$ is unimodal and the mode of it is $m+2$.
If $6 \leq m \leq 8$, then $t_{m+3} \geq t_{m+4} \geq \cdots$ by Theorem 2.1, so that $I(G^*;x)$ is unimodal and the possible positions for its mode are $m+2$ and $m+3$.
If $9 \leq m \leq 14$, then $t_{m+4} \geq t_{m+5} \geq \cdots$ from the above and

$$\begin{aligned} & 2t_{m+3} - t_{m+2} - t_{m+4} \\ =\ & \sum_{i=0}^{8} s_i \cdot \frac{(2m+1-i)!}{(m-1)!(m+4-i)!} \cdot (2m - i^2 + 9i - 22) \\ \geq\ & \frac{(2m+1)!}{(m-1)!(m+4)!} \cdot (2m-22) \cdot s_0 + \frac{(2m)!}{(m-1)!(m+3)!} \cdot (2m-14) \cdot s_1 \\ =\ & \frac{(2m+1)!}{(m-1)!(m+4)!} \cdot (2m-22) + \frac{(2m)!}{(m-1)!(m+3)!} \cdot (2m-14)(2m+1) \\ =\ & \frac{(2m+1)!}{(m-1)!(m+4)!}[(2m-22) + (2m-14)(m+4)] \\ >\ & 0, \end{aligned}$$

so that $I(G^*;x)$ is unimodal and the possible positions for its mode are $m+2$, $m+3$ and $m+4$.
Finally, if $m \geq 15$, then

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ =\ & \sum_{i=0}^{8} s_i \cdot [2\binom{2m+1-i}{m+2-i} - \binom{2m+1-i}{m+1-i} - \binom{2m+1-i}{m+3-i}] \\ =\ & \sum_{i=0}^{8} s_i \cdot \frac{(2m+1-i)!}{m!(m+3-i)!} \cdot (2m - i^2 + 5i - 6) \\ \geq\ & 0 \end{aligned}$$

and

$$\begin{aligned} & 2t_{m+3} - t_{m+2} - t_{m+4} \\ = \ & \sum_{i=0}^{8} s_i \cdot [2\binom{2m+1-i}{m+3-i} - \binom{2m+1-i}{m+2-i} - \binom{2m+1-i}{m+4-i}] \\ = \ & \sum_{i=0}^{8} s_i \cdot \frac{(2m+1-i)!}{(m-1)!(m+4-i)!} \cdot (2m - i^2 + 9i - 22) \\ \geq \ & 0, \end{aligned}$$

so that $I(G^*; x)$ is unimodal and the possible positions for its mode are $m+1$, $m+2$, $m+3$ and $m+4$.

Case 2. n is even, say $n = 2m$. Since $\alpha(G) = 8$, $m \geq 4$. By Theorem 2.1, we have that $t_0 \leq t_1 \leq \cdots \leq t_m$. Moreover,

$$\begin{aligned} t_i - t_{i+1} &= \sum_{j=0}^{8} [\binom{2m-j}{i-j} - \binom{2m-j}{i+1-j}] s_j \\ &\geq 0 \end{aligned}$$

holds for any $m+4 \leq i \leq 2m-1$, therefore $t_{m+4} \geq t_{m+5} \geq \cdots \geq t_{2m}$. Now, by Lemma 2.3,

$$\begin{aligned} & t_{m+1} - t_m \\ \geq \ & [\binom{2m}{m+1} - \binom{2m}{m}] s_0 + [\binom{2m-2}{m-1} - \binom{2m-2}{m-2}] s_2 \\ \geq \ & [\binom{2m}{m+1} - \binom{2m}{m}] \cdot 1 + [\binom{2m-2}{m-1} - \binom{2m-2}{m-2}] \cdot \binom{8}{2} \\ = \ & \frac{-(2m)!}{(m+1)!m!} + \frac{28(2m-2)!}{m!(m-1)!} \\ = \ & \frac{(2m-2)!}{(m+1)!m!} (24m^2 + 30m) \\ > \ & 0. \end{aligned}$$

If $4 \leq m \leq 7$, then $t_{m+2} \geq t_{m+3} \geq \cdots$ by Theorem 2.1, so that $I(G^*; x)$ is unimodal and the possible positions for its mode are $m+1$ and $m+2$. If $8 \leq m \leq 10$, then $t_{m+3} \geq t_{m+4} \geq \cdots$ by Theorem 2.1 and

$$\begin{aligned} & 2t_{m+2} - t_{m+1} - t_{m+3} \\ = \ & \sum_{i=0}^{8} s_i \cdot \frac{(2m-i)!}{(m-1)!(m+3-i)!} \cdot (2m - i^2 + 7i - 14) \\ \geq \ & \frac{(2m-1)!}{(m-1)!(m+2)!} \cdot (2m-8) \cdot s_1 + \frac{(2m-8)!}{(m-1)!(m-5)!} \cdot (2m-22) \cdot s_8 \\ \geq \ & \frac{(2m-1)!}{(m-1)!(m+2)!} \cdot (2m-8) \cdot (2m) + \frac{(2m-8)!}{(m-1)!(m-5)!} \cdot (2m-22) \cdot 2^{2m-8} \\ = \ & \frac{(2m-8)!}{(m-1)!(m-5)!} \cdot [\frac{(2m-1)(2m-2)\cdots(2m-7)}{(m+2)(m+1)\cdots(m-4)} 2m(2m-8) \\ & +(2m-22) \cdot 2^{2m-8}] \\ > \ & 0 \end{aligned}$$

by Lemma 2.3, so that $I(G^*; x)$ is unimodal and the possible positions for its mode are $m+1$, $m+2$ and $m+3$.

If $m \geq 11$, then

$$
\begin{aligned}
& 2t_{m+3} - t_{m+2} - t_{m+4} \\
= \; & \textstyle\sum_{i=0}^{8} s_i \cdot [2\binom{2m-i}{m+3-i} - \binom{2m-i}{m+2-i} - \binom{2m-i}{m+4-i}] \\
= \; & \textstyle\sum_{i=0}^{8} s_i \cdot \frac{(2m-i)!}{(m-2)!(m+4-i)!} \cdot (2m - i^2 + 11i - 34) \\
\geq \; & \tfrac{(2m)!}{(m-2)!(m+4)!} \cdot (2m-34) \cdot s_0 + \tfrac{(2m-1)!}{(m-2)!(m+3)!} \cdot (2m-24) \cdot s_1 \\
& + \tfrac{(2m-2)!}{(m-2)!(m+2)!} \cdot (2m-16) \cdot s_2 \\
\geq \; & \tfrac{(2m)!}{(m-2)!(m+4)!} \cdot (2m-34) \cdot 1 + \tfrac{(2m-1)!}{(m-2)!(m+3)!} \cdot (2m-24)(2m) \\
& + \tfrac{(2m-2)!}{(m-2)!(m+2)!} \cdot (2m-16) \cdot \binom{8}{2} \\
= \; & \tfrac{(2m-2)!}{(m-2)!(m+2)!} [\tfrac{2m(2m-1)}{(m+4)(m+3)}(2m-34) \\
& + \tfrac{2m-1}{m+3}(2m-24)(2m) + 28(2m-16)] \\
> \; & 0
\end{aligned}
$$

and

$$
\begin{aligned}
& 2t_{m+2} - t_{m+1} - t_{m+3} \\
= \; & \textstyle\sum_{i=0}^{8} s_i \cdot \frac{(2m-i)!}{(m-1)!(m+3-i)!} \cdot (2m - i^2 + 7i - 14) \\
> \; & 0
\end{aligned}
$$

by Lemma 2.3, so that $I(G^*; x)$ is unimodal and the possible positions for its mode are $m+1$, $m+2$, $m+3$ and $m+4$. □

4 Unimodality of $K_{2,n}^*$

The well-covered spider S_n, $n \geq 2$, has n vertices of degree 2, one vertex of degree $n+1$, and $n+1$ vertices of degree 1 (see Figure 1). It is well-known that $S_n = K_{1,n}^*$.

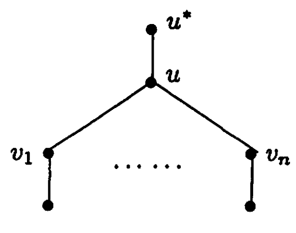

Figure 1 : The graph $S_n = K_{1,n}^*$.

In [7, Theorem 3.1], Levit and Mandrescu proved the following:

Theorem 4.1 *The independence polynomial of $K_{1,n}^*$, $n \geq 2$, is unimodal, moreover, $I(K_{1,n}^*; x) = (1+x) \cdot \{\sum_{k=0}^{n} [\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1}] \cdot x^k\}$, and its mode is unique and equals $\lceil \frac{2n+1}{3} \rceil$.*

In this section, we prove a similar result for $I(K_{2,n}^*; x)$. For this, we need some lemmas.

Lemma 4.2 *[3] If G is a graph and $u \in V(G)$, then $I(G; x) = I(G - u; x) + x \cdot I(G - N[u]; x)$. If G_1 and G_2 are graphs, then $I(G_1 \sqcup G_2; x) = I(G_1; x) \cdot I(G_2; x)$.*

One can obtain the following result by applying Lemma 4.2.

Theorem 4.3 *The independence polynomial of $K_{2,n}^*$, $n \geq 2$, is*

$$I(K_{2,n}^*; x) = (1+x)^2 \cdot \left\{ \sum_{k=0}^{n} \left[\binom{n}{k} \cdot 2^k + \sum_{i=1}^{2} \binom{2}{i} \binom{n-i}{k-i} \right] \cdot x^k \right\}.$$

Proof.

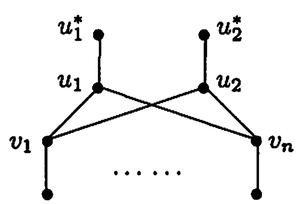

Figure 2 : The graph $K_{2,n}^*$.

Let $K_{2,n}$ be the complete bipartite graph with vertex classes $U = \{u_1, u_2\}$ and $V = \{v_1, \ldots, v_n\}$ (see Figure 2). Then by Lemma 4.2, the independent polynomial of $K_{2,n}^*$, $n \geq 2$, is

$$\begin{aligned}
& I(K_{2,n}^*; x) \\
= \; & I(K_{2,n}^* - \{u_1\}; x) + x \cdot I(K_{2,n}^* - N[u_1]; x) \\
= \; & I(S_n \sqcup \{u_1^*\}; x) + x \cdot I(K_2 \sqcup nK_1; x) \\
= \; & (1+x) \cdot \{\textstyle\sum_{k=0}^{n} [\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1}] \cdot x^k\} \cdot (1+x) + x(1+2x)(1+x)^n \\
= \; & (1+x)^2 \cdot \{\textstyle\sum_{k=0}^{n} [\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1}] \cdot x^k\} \\
& +(1+x)^2 [x(1+2x)(1+x)^{n-2}] \\
= \; & (1+x)^2 \cdot \{\textstyle\sum_{k=0}^{n} [\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} + \binom{n-2}{k-1} + 2\binom{n-2}{k-2}] \cdot x^k\} \\
= \; & (1+x)^2 \cdot \{\textstyle\sum_{k=0}^{n} [\binom{n}{k} \cdot 2^k + \sum_{i=1}^{2} \binom{2}{i}\binom{n-i}{k-i}] \cdot x^k\}
\end{aligned}$$

as $\binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$. This shows the assertion. □

Proposition 4.4 *Let*

$$c_k = \binom{n}{k} 2^k + \sum_{i=1}^{2} \binom{2}{i} \binom{n-i}{k-i}$$

and $h = \lfloor \frac{2n+1}{3} \rfloor$, *where* $0 \le k \le n$ *and* $n \ge 2$. *Then* $c_0 \le \cdots \le c_{h-1} \le c_h \ge c_{h+1} \ge \cdots \ge c_n$.

Proof. Case 1. $n = 3m$. For $0 \le i \le m-1$,

$$\begin{aligned} & c_{2m+i} - c_{2m+i+1} \\ = \ & [\binom{3m}{2m+i} \cdot 2^{2m+i} + 2\binom{3m-1}{2m+i-1} + \binom{3m-2}{2m+i-2}] \\ & -[\binom{3m}{2m+i+1} \cdot 2^{2m+i+1} + 2\binom{3m-1}{2m+i} + \binom{3m-2}{2m+i-1}]. \end{aligned}$$

It is easy to see that for any $m \in \mathbb{N}$ and $0 \le i \le m-1$, $\binom{3m-1}{2m+i-1} \ge \binom{3m-1}{2m+i}$ and $\binom{3m-2}{2m+i-2} \ge \binom{3m-2}{2m+i-1}$. Moreover,

$$\begin{aligned} \binom{3m}{2m+i} \cdot 2^{2m+i} - \binom{3m}{2m+i+1} \cdot 2^{2m+i+1} &= \frac{(3m)! \cdot 2^{2m+i} \cdot (3i+1)}{(2m+i+1)!(m-i)!} \\ &> 0. \end{aligned}$$

Therefore, $c_{2m} \ge c_{2m+1} \ge \cdots \ge c_{3m}$. On the other hand, for any $m \in \mathbb{N}$ and $0 \le j \le 2m-1$, we have that

$$\begin{aligned} & c_{2m-j} - c_{2m-j-1} \\ = \ & [\binom{3m}{2m-j} \cdot 2^{2m-j} + 2\binom{3m-1}{2m-j-1} + \binom{3m-2}{2m-j-2}] \\ & -[\binom{3m}{2m-j-1} \cdot 2^{2m-j-1} + 2\binom{3m-1}{2m-j-2} + \binom{3m-2}{2m-j-3}] \\ = \ & \frac{(3m)! \cdot 2^{2m-j-1} \cdot (3j+2)}{(2m-j)!(m+j+1)!} - \frac{2(3m-1)! \cdot (m-2j-2)}{(2m-j-1)!(m+j+1)!} - \frac{(3m-2)! \cdot (m-2j-3)}{(2m-j-2)!(m+j+1)!} \\ = \ & \frac{(3m-2)!}{(2m-j)!(m+j+1)!} \cdot [3m(3m-1)(3j+2) \cdot 2^{2m-j-1} \\ & -2(3m-1)(m-2j-2)(2m-j) \\ & -(m-2j-3)(2m-j)(2m-j-1)]. \end{aligned}$$

Notice that

$$\begin{aligned} & 2(3m-1)(m-2j-2)(2m-j) + (m-2j-3)(2m-j)(2m-j-1) \\ \le \ & (m-2j-2)(2m-j)[2m + 2(3m-1)] \\ \le \ & (2m-j)m(8m-2) \\ < \ & 2^{2m-j-1} 3m(3m-1). \end{aligned}$$

Thus we conclude that $c_0 \le \cdots \le c_{2m-1} \le c_{2m} \ge c_{2m+1} \ge \cdots \ge c_n$.

Case 2. $n = 3m+1$. For $0 \le i \le m-1$,

$$\begin{aligned} & c_{2m+i+1} - c_{2m+i+2} \\ = \ & [\binom{3m+1}{2m+i+1} \cdot 2^{2m+i+1} + 2\binom{3m}{2m+i} + \binom{3m-1}{2m+i-1}] \\ & -[\binom{3m+1}{2m+i+2} \cdot 2^{2m+i+2} + 2\binom{3m}{2m+i+1} + \binom{3m-1}{2m+i}]. \end{aligned}$$

It is easy to see that for any $m \in \mathbb{N}$ and $0 \le i \le m-1$, $\binom{3m}{2m+i} \ge \binom{3m}{2m+i+1}$ and $\binom{3m-1}{2m+i-1} \ge \binom{3m-1}{2m+i}$. Moreover,

$$\begin{aligned} \binom{3m+1}{2m+i+1} \cdot 2^{2m+i+1} - \binom{3m+1}{2m+i+2} \cdot 2^{2m+i+2} &= \frac{(3m+1)! \cdot 2^{2m+i+1} \cdot (3i+2)}{(2m+i+2)!(m-i)!} \\ &> 0. \end{aligned}$$

Therefore, $c_{2m+1} \geq c_{2m+2} \geq \cdots \geq c_{3m+1}$. On the other hand, we have for any $m \in \mathbb{N}$ and $0 \leq j \leq 2m$

$$\begin{aligned} & c_{2m-j+1} - c_{2m-j} \\ = & [\binom{3m+1}{2m-j+1} \cdot 2^{2m-j+1} + 2\binom{3m}{2m-j} + \binom{3m-1}{2m-j-1}] \\ & -[\binom{3m+1}{2m-j} \cdot 2^{2m-j} + 2\binom{3m}{2m-j-1} + \binom{3m-1}{2m-j-2}] \\ = & \tfrac{(3m+1)!\cdot 2^{2m-j}\cdot(3j+1)}{(2m-j+1)!(m+j+1)!} - \tfrac{2(3m)!\cdot(m-2j-1)}{(2m-j)!(m+j+1)!} - \tfrac{(3m-1)!\cdot(m-2j-2)}{(2m-j-1)!(m+j+1)!} \\ = & \tfrac{(3m-1)!}{(2m-j+1)!(m+j+1)!} \cdot [3m(3m+1)(3j+1) \cdot 2^{2m-j} \\ & -2(3m)(m-2j-1)(2m-j+1) \\ & -(m-2j-2)(2m-j+1)(2m-j)]. \end{aligned}$$

Notice that

$$\begin{aligned} & 2(3m)(m-2j-1)(2m-j+1) + (m-2j-2)(2m-j+1)(2m-j) \\ \leq & (m-2j-1)(2m-j+1)[2m+2(3m)] \\ \leq & (2m-j+1)8m^2 \\ < & 2^{2m-j}3m(3m+1). \end{aligned}$$

Thus we conclude that $c_0 \leq \cdots \leq c_{2m} \leq c_{2m+1} \geq c_{2m+2} \geq \cdots \geq c_n$.

Case 3. $n = 3m+2$. For $0 \leq i \leq m$,

$$\begin{aligned} & c_{2m+i+1} - c_{2m+i+2} \\ = & [\binom{3m+2}{2m+i+1} \cdot 2^{2m+i+1} + 2\binom{3m+1}{2m+i} + \binom{3m}{2m+i-1}] \\ & -[\binom{3m+2}{2m+i+2} \cdot 2^{2m+i+2} + 2\binom{3m+1}{2m+i+1} + \binom{3m}{2m+i}]. \end{aligned}$$

It is easy to see that for any $m \in \mathbb{N}$ and $0 \leq i \leq m$, $\binom{3m+1}{2m+i} \geq \binom{3m+1}{2m+i+1}$ and $\binom{3m}{2m+i-1} \geq \binom{3m}{2m+i}$. Moreover,

$$\begin{aligned} \binom{3m+2}{2m+i+1} \cdot 2^{2m+i+1} - \binom{3m+2}{2m+i+2} \cdot 2^{2m+i+2} & = \tfrac{(3m+2)!\cdot 2^{2m+i+1}\cdot(3i)}{(2m+i+2)!(m-i+1)!} \\ & \geq 0. \end{aligned}$$

Therefore, $c_{2m+1} \geq c_{2m+2} \geq \cdots \geq c_{3m+2}$. On the other hand, we have for any $m \in \mathbb{N}$ and $0 \leq j \leq 2m$ that

$$\begin{aligned} & c_{2m-j+1} - c_{2m-j} \\ = & [\binom{3m+2}{2m-j+1} \cdot 2^{2m-j+1} + 2\binom{3m+1}{2m-j} + \binom{3m}{2m-j-1}] \\ & -[\binom{3m+2}{2m-j} \cdot 2^{2m-j} + 2\binom{3m+1}{2m-j-1} + \binom{3m}{2m-j-2}] \\ = & \tfrac{(3m+2)!\cdot 2^{2m-j}\cdot(3j+3)}{(2m-j+1)!(m+j+2)!} - \tfrac{2(3m+1)!\cdot(m-2j-2)}{(2m-j)!(m+j+2)!} - \tfrac{(3m)!\cdot(m-2j-3)}{(2m-j-1)!(m+j+2)!} \\ = & \tfrac{(3m)!}{(2m-j+1)!(m+j+2)!}[(3m+2)(3m+1)(3j+3) \cdot 2^{2m-j} \\ & -2(3m+1)(m-2j-2)(2m-j+1) \\ & -(m-2j-3)(2m-j+1)(2m-j)]. \end{aligned}$$

Notice that

$$\begin{array}{rl} & 2(3m+1)(m-2j-2)(2m-j+1) \\ & +(m-2j-3)(2m-j+1)(2m-j) \\ \leq & (m-2j-2)(2m-j+1)[2m+2(3m+1)] \\ \leq & (2m-j+1)m(8m+2) \\ < & 2^{2m-j}(3m+1)(3m+2). \end{array}$$

Thus we conclude that $c_0 \leq \cdots \leq c_{2m} \leq c_{2m+1} \geq c_{2m+2} \geq \cdots \geq c_n$. □

Lemma 4.5 *Let*

$$c_k = \binom{n}{k} 2^k + \sum_{i=1}^{2} \binom{2}{i} \binom{n-i}{k-i}$$

and $h = \lfloor \frac{2n+1}{3} \rfloor$, *where* $0 \leq k \leq n$ *and* $n \geq 2$. *Then* $c_{h-1} > c_{h+2}$ *and* $c_{h+1} > c_{h-2}$.

Proof. Observe that

$$\begin{array}{rl} & c_{h-1} - c_{h+2} \\ = & [\binom{n}{h-1} \cdot 2^{h-1} - \binom{n}{h+2} \cdot 2^{h+2}] + 2[\binom{n-1}{h-2} - \binom{n-1}{h+1}] + [\binom{n-2}{h-3} - [\binom{n-2}{h}] \\ = & \frac{n! \cdot 2^{h-1}}{(h+2)!(n-h+1)!} \cdot Q_1(n,h) + \frac{2(n-1)!}{(h+1)!(n-h+1)!} \cdot Q_2(n,h) \\ & + \frac{(n-2)!}{h!(n-h+1)!} \cdot Q_3(n,h), \end{array}$$

where

$$\begin{array}{rl} Q_1(n,h) = & h(h+1)(h+2) - 8(n-h+1)(n-h)(n-h-1) \\ Q_2(n,h) = & h(h+1)(h-1) - (n-h+1)(n-h)(n-h-1) \\ Q_3(n,h) = & h(h-1)(h-2) - (n-h+1)(n-h)(n-h-1). \end{array}$$

If $n = 3m$, then $h = 2m$ and the following hold:

$$Q_1(3m, 2m) = 12m(m+1) > 0$$

and

$$Q_2(3m, 2m) \geq Q_3(3m, 2m) = m(m-1)(7m-5) \geq 0.$$

If $n = 3m+1$, then $h = 2m+1$ and the following hold:

$$Q_1(3m+1, 2m+1) = 2(m+1)(12m+3) > 0$$

and

$$Q_2(3m+1, 2m+1) \geq Q_3(3m+1, 2m+1) = m(7m^2-1) > 0.$$

If $n = 3m+2$, then $h = 2m+1$ and the following hold:

$$Q_1(3m+2, 2m+1) = 6(m+1) > 0$$

and

$$Q_2(3m+2, 2m+1) \geq Q_3(3m+2, 2m+1) = m(m-1)(7m+4) \geq 0.$$

From the above, we conclude that $c_{h-1} > c_{h+2}$. On the other hand, observe that

$$\begin{array}{ll} & c_{h+1} - c_{h-2} \\ \geq & \binom{n}{h+1} \cdot 2^{h+1} - [\binom{n}{h-2} \cdot 2^{h-2} + 2\binom{n-1}{h-3} + \binom{n-2}{h-4}] \\ = & [\frac{n!}{(h+1)!(n-h-1)!} \cdot 2^{h+1} - \frac{n!}{(h-2)!(n-h+2)!} \cdot 2^{h-2}] \\ & -[\frac{2(n-1)!}{(h-3)!(n-h+2)!} + \frac{(n-2)!}{(h-4)!(n-h+2)!}] \\ = & \frac{(n-2)!}{(h+1)!(n-h+2)!} \cdot R(n,h), \end{array}$$

where

$$\begin{array}{rl} R(n,h) = & n(n-1)[(n-h+2)(n-h+1)(n-h) \cdot 2^{h+1} \\ & -h(h+1)(h-1) \cdot 2^{h-2}] \\ & -h(h+1)(h-1)(h-2)[2(n-1)+(h-3)]. \end{array}$$

If $n = 3m$, then $h = 2m$ and

$$\begin{array}{ll} & R(3m, 2m) \\ = & (2m)(3m)(3m-1)(12m+9) \cdot 2^{2m-2} \\ & -(2m)(2m+1)(2m-1)(8m-5)(2m-2) \\ > & 0. \end{array}$$

If $n = 3m+1$, then $h = 2m+1$ and

$$\begin{array}{ll} & R(3m+1, 2m+1) \\ = & (2m)(2m+2)(3m+1)(9m) \cdot 2^{2m-1} \\ & -(2m)(2m+1)(2m+2)(8m-2)(2m-1) \\ > & 0. \end{array}$$

If $n = 3m+2$, then $h = 2m+1$ and

$$\begin{array}{ll} & R(3m+2, 2m+1) \\ = & (2m+2)(3m+1)(3m+2)(18m+24) \cdot 2^{2m-1} \\ & -(2m)(2m+1)(2m+2)(8m)(2m-1) \\ > & 0. \end{array}$$

Therefore, we conclude that $c_{h+1} > c_{h-2}$. □

Theorem 4.6 *The independence polynomial of $K^*_{2,n}$ is unimodal with mode equal to $\lfloor \frac{2n+4}{3} \rfloor$.*

Proof. If $n = 1$, then $I(K^*_{2,1}) = 1 + 6x + 10x^2 + 5x^3$, so that the mode is $2 = \lfloor \frac{6}{3} \rfloor$. Therefore, we may assume that $n \geq 2$. By Theorem 4.3, the independence polynomial of $K^*_{2,n}$ is

$$I(K^*_{2,n}; x) = (1+x)^2 \cdot \left\{ \sum_{k=0}^{n} \left[\binom{n}{k} 2^k + \sum_{i=1}^{2} \binom{2}{i} \binom{n-i}{k-i} \right] \cdot x^k \right\}.$$

Let

$$c_k = \binom{n}{k} 2^k + \sum_{i=1}^{2} \binom{2}{i} \binom{n-i}{k-i};$$

then $I(K^*_{2,n}; x) = (1+x)^2 \cdot \sum_{i=0}^{n} c_i x^i$. By Proposition 4.4, $c_0 \leq c_1 \leq \cdots \leq c_{h-1} \leq c_h \geq c_{h+1} \geq \cdots \geq c_n$ where $h = \lfloor \frac{2n+1}{3} \rfloor$. Write $I(K^*_{2,n}; x) = \sum_{i=0}^{n+2} a_i x^i$; then

$$\begin{aligned} & (1 + 2x + x^2)(c_0 + c_1 x + \cdots + c_n x^n) \\ = \; & a_0 + a_1 x + \cdots + a_{n+2} x^{n+2}, \end{aligned}$$

and therefore $a_0 = c_0$, $a_1 = c_1 + 2c_o$, $a_{n+1} = 2c_n + c_{n-1}$, $a_{n+2} = c_n$, and $a_k = c_k + 2c_{k-1} + c_{k-2}$ for $2 \leq k \leq n$. It is easy to see that $a_0 \leq a_1 \leq a_2$ and $a_n \geq a_{n+1} \geq a_{n+2}$. Notice that if $2 \leq k \leq h-1$, then $c_k + 2c_{k-1} + c_{k-2} \leq c_{k+1} + 2c_k + c_{k-1}$, so that $a_2 \leq a_3 \leq \cdots \leq a_h$. Similarly, we have that $a_{h+2} \geq a_{h+3} \geq \cdots \geq a_n$. Hence the possible modes of $I(K^*_{2,n}; x)$ are h, $h+1$ and $h+2$. Since $c_h \geq c_{h-1}$ and $c_h \geq c_{h+1}$, by Lemma 4.5,

$$\begin{aligned} & a_{h+1} - a_h \\ = \; & (c_{h+1} + 2c_h + c_{h-1}) - (c_h + 2c_{h-1} + c_{h-2}) \\ = \; & (c_{h+1} - c_{h-2}) + (c_h - c_{h-1}) \\ > \; & 0 \end{aligned}$$

and

$$\begin{aligned} & a_{h+1} - a_{h+2} \\ = \; & (c_{h+1} + 2c_h + c_{h-1}) - (c_{h+2} + 2c_{h+1} + c_h) \\ = \; & (c_h - c_{h+1}) + (c_{h-1} - c_{h+2}) \\ > \; & 0. \end{aligned}$$

Therefore, the mode of $K^*_{2,n}$ is $h + 1 = \lfloor \frac{2n+1}{3} \rfloor + 1 = \lfloor \frac{2n+4}{3} \rfloor$. □
In view of Theorem 4.6, we provide the following question:
Conjecture: $I(K^*_{t,n}; x)$ is unimodal for every t.

In [8, Theorem 5], Levit and Mandrescu proved that $I(K^*_{1,n}; x)$ is log-concave, we prove a similar result for $I(K^*_{2,n}; x)$ as follows.

Theorem 4.7 *The independence polynomial of $K^*_{2,n}$ is log-concave.*

Proof. If $n = 1$, then $I(K^*_{2,1}) = 1 + 6x + 10x^2 + 5x^3$. It is easy to see that the independence polynomial of $K^*_{2,1}$ is log-concave, so that the assertion holds for $n = 1$. Therefore, we may assume that $n \geq 2$. By Theorem 4.3,

$$I(K^*_{2,n}; x) = (1+x)^2 \cdot \left\{ \sum_{k=0}^{n} \left[\binom{n}{k} \cdot 2^k + \sum_{i=1}^{2} \binom{2}{i}\binom{n-i}{k-i} \right] \cdot x^k \right\}.$$

Let

$$c_k = \binom{n}{k} \cdot 2^k + \sum_{i=1}^{2} \binom{2}{i}\binom{n-i}{k-i};$$

then $I(K^*_{2,n}; x) = (1+x)^2 \cdot \sum_{i=0}^{n} c_i x^i$. Since the product of two log-concave polynomials is also log-concave and $(1+x)^2$ is log-concave, it is sufficient to show that $c_k^2 \geq c_{k-1}c_{k+1}$ for $1 \leq k \leq n-1$. For this, notice that $c_1^2 - c_0c_2 = 2n^2 + 8n + 5 > 0$ and $c_2^2 - c_1c_3 = \frac{1}{3}(4n^4 + 10n^3 + 2n^2 - 4n + 3) > 0$. For $3 \leq k \leq n-1$, let $A_k = \binom{n}{k} \cdot 2^k$ and $B_k = 2\binom{n-1}{k-1} + \binom{n-2}{k-2}$; then $c_k = A_k + B_k$. Moreover,

$$\begin{aligned} & c_k^2 - c_{k-1} \cdot c_{k+1} \\ = \; & A_k^2 + 2A_kB_k - A_{k-1}A_{k+1} - A_{k-1}B_{k+1} - A_{k+1}B_{k-1} \\ & + B_k^2 - B_{k-1}B_{k+1}. \end{aligned}$$

Observe that $\binom{n-i}{k-i}^2 \geq \binom{n-i}{k-i-1}\binom{n-i}{k-i+1}$ for i=1,2, therefore

$$\begin{aligned} & B_k^2 - B_{k-1}B_{k+1} \\ = \; & 4\binom{n-1}{k-1}^2 + 4\binom{n-1}{k-1}\binom{n-2}{k-2} + \binom{n-2}{k-2}^2 - 4\binom{n-1}{k-2}\binom{n-1}{k} \\ & -2\binom{n-1}{k-2}\binom{n-2}{k-1} - 2\binom{n-2}{k-3}\binom{n-1}{k} \\ & -\binom{n-2}{k-3}\binom{n-2}{k-1} \\ \geq \; & 4\binom{n-1}{k-1}\binom{n-2}{k-2} - 2\binom{n-1}{k-2}\binom{n-2}{k-1} - 2\binom{n-2}{k-3}\binom{n-1}{k} \\ = \; & 4\binom{n-1}{k-1}\binom{n-2}{k-2} - 2 \cdot \frac{k-1}{n-k+1} \cdot \binom{n-1}{k-1} \cdot \frac{n-k}{k-1} \cdot \binom{n-2}{k-2} \\ & -2 \cdot \frac{k-2}{n-k+1} \cdot \binom{n-2}{k-2} \cdot \frac{n-k}{k} \cdot \binom{n-1}{k-1} \\ = \; & 2\binom{n-1}{k-1}\binom{n-2}{k-2} \cdot [(1 - \frac{n-k}{n-k+1}) + (1 - \frac{n-k}{n-k+1} \cdot \frac{k-2}{k})] \\ > \; & 0 \end{aligned}$$

for $\frac{n-k}{n-k+1}, \frac{k-2}{k} \in (0,1)$. To finish the proof, it is enough to show that

$$A_k^2 + 2A_kB_k - A_{k-1}A_{k+1} - A_{k-1}B_{k+1} - A_{k+1}B_{k-1} \geq 0.$$

For this, notice that $A_{k+1} = \frac{n-k}{k+1}\binom{n}{k}2^{k+1}$, $A_{k-1} = \frac{k}{n-k+1}\binom{n}{k}2^{k-1}$, $B_k = [\frac{2k}{n} + \frac{k(k-1)}{n(n-1)}]\binom{n}{k}$, $B_{k+1} = [\frac{2(n-k)}{n} + \frac{(n-k)k}{n(n-1)}]\binom{n}{k}$ and $B_{k-1} = [\frac{2k(k-1)}{n(n-k+1)} + \frac{k(k-1)(k-2)}{n(n-1)(n-k+1)}]\binom{n}{k}$. It follows that

$$\begin{aligned} & A_k^2 + 2A_kB_k - A_{k-1}A_{k+1} - A_{k-1}B_{k+1} - A_{k+1}B_{k-1} \\ = \; & \frac{\binom{n}{k}^2 \cdot 2^{k-1}}{n(n-1)(k+1)(n-k+1)} D, \end{aligned}$$

where

$$
\begin{aligned}
D = \;& n(n-1)(k+1)(n-k+1)\cdot 2^{k+1} \\
& +8k(n-1)(k+1)(n-k+1) \\
& +4k(k-1)(k+1)(n-k+1) - n(n-1)k(n-k)\cdot 2^{k+1} \\
& -2k(n-k)(n-1)(k+1) - k^2(n-k)(k+1) \\
& -8k(n-k)(k-1)(n-1) - 4k(k-1)(k-2)(n-k) \\
= \;& [n(n-1)(n+1)\cdot 2^{k+1} - k^2(n-k)(k+1) \\
& -2k(k+1)(n-1)(n-k)] \\
& +[8k(n-1)(k+1)(n-k+1) - 8k(n-k)(k-1)(n-1)] \\
& +[4k(k-1)(k+1)(n-k+1) - 4k(k-1)(k-2)(n-k)].
\end{aligned}
$$

However,

$$8k(n-1)(k+1)(n-k+1) - 8k(n-k)(k-1)(n-1) \geq 0,$$

$$4k(k-1)(k+1)(n-k+1) - 4k(k-1)(k-2)(n-k) \geq 0$$

and

$$n(n-1)(n+1)\cdot 2^{k+1} - k^2(n-k)(k+1) - 2k(k+1)(n-1)(n-k) \geq 0.$$

This shows that $c_k^2 - c_{k-1}\cdot c_{k+1} \geq 0$ for $1 \leq k \leq n-1$. □
In view of Theorem 4.7, we provide the following question:
Conjecture: $I(K_{t,n}^*; x)$ is log-concave for every t.

References

[1] J.L. Arocha, *Propriedades del polinomio independiente de un grafo*, Revista Ciencias Mathematicas, vol V (1984) 103-110.

[2] J.I. Brown, K. Dilcher, R.J. Nowakowski, *Roots of independence polynomials of well-covered graphs*, Journal of Algebraic Combinatroics **11** (2000), 197-210.

[3] I. Gutman and F. Harary, *Generalizations of matching polynomial*, Utilitas Mathematica **24** (1983), 97-106.

[4] V.E. Levit and E. Mandrescu, *Trees with unimodal independence polynomials*, Congressus Numerantium **159** (2002), 193-202.

[5] V.E. Levit and E. Mandrescu, *A family of well-covered graphs with unimodal independence polynomials*, Congressus Numerantium **165** (2003), 195-207.

[6] V.E. Levit and E. Mandrescu, *On the roots of independence polynomials of almost all very well-covered graphs*, Los Alamos Archive, prE-print arXiv:math.CO/0305227, (2003), 17pp.

[7] V.E. Levit and E. Mandrescu, *On unimodity of independence polynomials of some well-covered trees*, Lecture Notes in Computer Science, LNCS **2731** Springer (2003), 237–256.

[8] V.E. Levit and E. Mandrescu, *Very well-covered graphs with log-concave independence polynomials*, Carpathian J. Math. **20**, no. 1, (2004), 73–80.

[9] T.S. Michael and W. N. Traves, *Independence sequences of well-covered graph: non-unimodality and the Roller-Coaster conjecture*, Graphs and Combinatorics **19** (2002) no.3., 403-411.

[10] M.D. Plummer, *Some covering concepts in graphs*, Journal of Combinatorial Theory **8** (1970) 91-98.