PROOFS OF RAMANUJAN'S $_1\psi_1$-SUMMATION FORMULA

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ABSTRACT. Ramanujan's $_1\psi_1$-summation formula is one of the fundamental identities in basic hypergeometric series. We review proofs of this identity and clear its connections with other basic hypergeometric series transformations and formulae. In particular, we shall put our main emphasis on methods that can be used not only to provide deeper insight into Ramanujan's $_1\psi_1$-summation formula, but also to derive new transformations and identities for basic hypergeometric series.

1. INTRODUCTION AND NOTATION

For two indeterminate q and x, define the shifted-factorial by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad \text{for} \quad n \in \mathbb{N}.$$

When $|q| < 1$, the shifted factorial of infinite order is well-defined

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k) \quad \text{and} \quad (x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty} \quad \text{for} \quad n \in \mathbb{Z}.$$

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Its product and fraction forms are abbreviated compactly to

$$\begin{aligned}[a, b, \cdots, c; q]_n &= (a;q)_n(b;q)_n \cdots (c;q)_n, \\ \begin{bmatrix} a, b, \cdots, c \\ \alpha, \beta, \cdots, \gamma \end{bmatrix} q \Bigg]_n &= \frac{(a;q)_n(b;q)_n \cdots (c;q)_n}{(\alpha;q)_n(\beta;q)_n \cdots (\gamma;q)_n}.\end{aligned}$$

Following Bailey [9], Gasper-Rahman [21] and Slater [34], the unilateral and bilateral basic hypergeometric series are defined, respectively, by

$${}_{1+r}\phi_s \left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix} \middle| q; z\right] = \sum_{n=0}^{\infty} z^n \left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{matrix} \middle| q\right]_n,$$

$${}_r\psi_s \left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix} \middle| q; z\right] = \sum_{n=-\infty}^{\infty} z^n \left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix} \middle| q\right]_n.$$

Throughout the paper, the base q will be confined to $|q| < 1$ for non-terminating q-series. One of the fundamental basic hypergeometric series identities is Ramanujan's ${}_1\psi_1$-summation formula (cf. [21, II-5]):

$${}_1\psi_1 \left[\begin{matrix} a \\ c \end{matrix} \middle| q; z\right] = \left[\begin{matrix} q, & c/a, & az, & q/az \\ c, & q/a, & z, & c/az \end{matrix} \middle| q\right]_\infty \quad \text{where } |c/a| < |z| < 1. \qquad (1)$$

The objective of the present paper is to review the hypergeometric proofs of this identity and clear its connections with other basic hypergeometric series transformations and formulae. In particular, we shall put our main emphasis on methods that can be used not only to provide deeper insight into Ramanujan's ${}_1\psi_1$-summation formula, but also to derive new transformations and identities for basic hypergeometric series.

The identity (1) is not only the bilateral extension of the q-binomial theorem (see (6) in §2.2), but also has many applications in classical analysis and number theory (cf. [5, §10.6]). For example, the celebrated Jacobi triple product identity (cf. [21, II-28])

$$[q, z, q/z; q]_\infty = \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} z^k \quad \text{for} \quad |q| < 1 \qquad (2)$$

follows from (1) as a limit. In fact, specializing with $c \to 0$ and $z \to z/a$ and then letting $a \to \infty$, we derive it immediately from (1). For the historical account of the identity (1), see Berndt [10, Entry 17], Chan [11] and Johnson [27]. Further applications and different proofs can be found from the references collected at the end of this paper.

2. Elementary Method for Proving ${}_1\psi_1$-Summation formula

In this section, we will prove Ramanujan's ${}_1\psi_1$-summation formula by the q-Gauß summation theorem, the Cauchy method and partial fraction decomposition. For the proofs via functional equations and Abel's lemma on summation by parts, refer to Gasper-Rahman [21, §5.2] and the recent paper due to Chu [15], respectively.

§2.1. **q-Gauß summation theorem.** One of the very useful q-series identities reads as the q-Gauß summation theorem (cf. [21, II-8])

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, q; \frac{c}{ab}\right] = \left[\begin{matrix} c/a, & c/b \\ c, & c/ab \end{matrix}\,\middle|\, q\right]_\infty \quad \text{where} \quad |c/ab| < 1. \tag{3} $$

Letting $a := aq^{-m}$, $b := c/az$ and $c := cq^{-m}$, we may reformulate it as

$$ {}_2\phi_1\left[\begin{matrix} c/az, & aq^{-m} \\ & cq^{-m} \end{matrix}\,\middle|\, q; z\right] = \left[\begin{matrix} c/a, & azq^{-m} \\ z, & cq^{-m} \end{matrix}\,\middle|\, q\right]_\infty . $$

According to the definition, we may write explicitly the last ${}_2\phi_1$-series as follows:

$$ {}_2\phi_1\left[\begin{matrix} c/az, aq^{-m} \\ cq^{-m} \end{matrix}\,\middle|\, q; z\right] = \sum_{k=0}^{+\infty} \left[\begin{matrix} c/az, aq^{-m} \\ q, \; cq^{-m} \end{matrix}\,\middle|\, q\right]_k z^k $$

$$ = z^m \left[\begin{matrix} c/az, \; aq^{-m} \\ q, \; cq^{-m} \end{matrix}\,\middle|\, q\right]_m \sum_{k=-m}^{+\infty} \left[\begin{matrix} a, \; cq^m/az \\ q^{1+m}, \; c \end{matrix}\,\middle|\, q\right]_k z^k $$

where the last line has been justified by changing the summation index $n \to n+m$. Under some routine simplification, we find the following relation:

$$ \sum_{k=-m}^{+\infty} \left[\begin{matrix} a, \; cq^m/az \\ q^{1+m}, \; c \end{matrix}\,\middle|\, q\right]_k z^k = z^{-m} \left[\begin{matrix} q, \; cq^{-m} \\ c/az, \; aq^{-m} \end{matrix}\,\middle|\, q\right]_m {}_2\phi_1\left[\begin{matrix} aq^{-m}, \; c/az \\ cq^{-m} \end{matrix}\,\middle|\, q; z\right] $$

$$ = z^{-m} \left[\begin{matrix} q, & cq^{-m} \\ c/az, & aq^{-m} \end{matrix}\,\middle|\, q\right]_m \left[\begin{matrix} c/a, & azq^{-m} \\ z, & cq^{-m} \end{matrix}\,\middle|\, q\right]_\infty $$

$$ = z^{-m} \left[\begin{matrix} q, & azq^{-m} \\ c/az, & aq^{-m} \end{matrix}\,\middle|\, q\right]_m \left[\begin{matrix} c/a, & az \\ z, & c \end{matrix}\,\middle|\, q\right]_\infty $$

$$ = \left[\begin{matrix} q, & q/az \\ c/az, & q/a \end{matrix}\,\middle|\, q\right]_m \left[\begin{matrix} c/a, & az \\ z, & c \end{matrix}\,\middle|\, q\right]_\infty . $$

Letting $m \to \infty$ and then appealing to the Weierstrass M-test on uniformly convergent series (cf. Stromberg [35, P 141]), we derive Ramanujan's formula (1) for ${}_1\psi_1$-series. The proof presented here is essentially the same as those found in [12], [16, F2] and [29] recently.

§2.2. **The Cauchy method.** Define $F(z)$ by the factorial fraction and then express it in terms of Laurent series

$$F(z) = \left[\begin{matrix} az, & q/az \\ z, & c/az \end{matrix} \middle| q\right]_\infty = \sum_{n=-\infty}^{+\infty} \Omega_n z^n \tag{4}$$

where Ω_n is independent of variable z. It is not hard to check that

$$F(qz) = F(z)\frac{(1-z)(1-1/az)}{(1-az)(1-c/qaz)}$$

which is equivalent to the following functional equation:

$$q(1-z)F(z) = (c-qaz)F(qz). \tag{5}$$

Extracting coefficients of z^n for $n \in \mathbb{Z}$ on both sides of (5), we find that

$$q\Omega_n - q\Omega_{n-1} = cq^n\Omega_n - aq^n\Omega_{n-1} \quad \text{for} \quad n \in \mathbb{Z}$$

which leads us to the following recurrence relation:

$$\Omega_n = \frac{1-aq^{n-1}}{1-cq^{n-1}}\Omega_{n-1} \quad \text{for} \quad n \in \mathbb{Z}.$$

Iterating the above recurrence relation for n-times, we get

$$\Omega_n = \frac{(a;q)_n}{(c;q)_n}\Omega_0 \quad \text{for} \quad n \in \mathbb{Z}.$$

Substituting the last relation into (4), we find the following summation formula

$$F(z) = \Omega_0 \times {}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix} \middle| q; z\right].$$

In order to compute Ω_0, we recall the q-binomial formula (cf. [21, II-3])

$${}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix} \middle| q; z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad \text{for} \quad |z| < 1. \tag{6}$$

We can expand the factorial fraction in (4) as follows

$$\begin{aligned} F(z) = \left[\begin{matrix} az, q/az \\ z, c/az \end{matrix} \middle| q\right]_\infty &= {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix} \middle| q; z\right] {}_1\phi_0\left[\begin{matrix} q/c \\ - \end{matrix} \middle| q; \frac{c}{az}\right] \\ &= \sum_{i=0}^{\infty} \frac{(a;q)_i}{(q;q)_i} z^i \sum_{j=0}^{\infty} \frac{(q/c;q)_j}{(q;q)_j}\left(\frac{c}{az}\right)^j. \end{aligned}$$

By means of the q-Gauß summation formula (3), we find the constant term as follows:

$$\Omega_0 = \sum_{k=0}^{\infty} \frac{(a;q)_k(q/c;q)_k}{(q;q)_k(q;q)_k}\left(\frac{c}{a}\right)^k = {}_2\phi_1\left[\begin{matrix} a, q/c \\ q \end{matrix} \middle| q; \frac{c}{a}\right] = \left[\begin{matrix} q/a, c \\ q, c/a \end{matrix} \middle| q\right]_\infty.$$

Therefore we have established the following identity

$$ {}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix}\,\middle|\, q; z\right] = F(z)/\Omega_0 = \left[\begin{matrix} q, c/a \\ q/a, c \end{matrix}\,\middle|\, q\right]_\infty \left[\begin{matrix} az, q/az \\ z, c/az \end{matrix}\,\middle|\, q\right]_\infty $$

which is exactly Ramanujan's ${}_1\psi_1$-summation formula (1).

The first part of this proof is motivated by Askey [7], where the constant term is determined in a different manner (cf. Berndt [10, Entry 17]). More applications of the Cauchy method can be found in the recent paper due to Johnson [27].

§2.3. **Partial fraction decomposition.** Consider rational function in partial fractions

$$ G(z) := \frac{[az, b/z; q]_m}{[cz, d/z; q]_{m+1}} = \sum_{i=0}^{m} \frac{A_i}{1 - q^i cz} + \sum_{j=0}^{m} \frac{B_j/z}{1 - q^j d/z}. $$

Computing the coefficients

$$ A_i = \lim_{z \to q^{-i}/c} (1 - q^i cz) G(z) = (a/c)^i \left[\begin{matrix} cd, & qc/a \\ q, & bc \end{matrix}\,\middle|\, q\right]_i \frac{(a/c; q)_{m-i} (bc; q)_{m+i}}{(q; q)_{m-i} (cd; q)_{m+i+1}}, $$

$$ B_j = \lim_{z \to q^j d} (z - q^j d) G(z) = d(qb/d)^j \left[\begin{matrix} cd, & qd/b \\ q, & ad \end{matrix}\,\middle|\, q\right]_j \frac{(b/d; q)_{m-j} (ad; q)_{m+j}}{(q; q)_{m-j} (cd; q)_{m+j+1}}; $$

we find the following identity

$$ \begin{aligned} \frac{[az, b/z; q]_m}{[cz, d/z; q]_{m+1}} &= \sum_{i=0}^{m} \frac{(a/c)^i}{1 - q^i cz} \left[\begin{matrix} cd, & qc/a \\ q, & bc \end{matrix}\,\middle|\, q\right]_i \frac{(a/c; q)_{m-i} (bc; q)_{m+i}}{(q; q)_{m-i} (cd; q)_{m+i+1}} \\ &+ \sum_{j=0}^{m} \frac{d/z (qb/d)^j}{1 - q^j d/z} \left[\begin{matrix} cd, & qd/b \\ q, & ad \end{matrix}\,\middle|\, q\right]_j \frac{(b/d; q)_{m-j} (ad; q)_{m+j}}{(q; q)_{m-j} (cd; q)_{m+j+1}}. \end{aligned} $$

Letting $m \to \infty$, we get the limiting case

$$ \left[\begin{matrix} az, & b/z \\ cz, & d/z \end{matrix}\,\middle|\, q\right]_\infty = \left[\begin{matrix} a/c, & bc \\ q, & cd \end{matrix}\,\middle|\, q\right]_\infty \sum_{i=0}^{+\infty} \frac{(a/c)^i}{1 - q^i cz} \left[\begin{matrix} cd, & qc/a \\ q, & bc \end{matrix}\,\middle|\, q\right]_i \quad (7a) $$

$$ + \left[\begin{matrix} ad, & b/d \\ q, & cd \end{matrix}\,\middle|\, q\right]_\infty \sum_{j=0}^{+\infty} \frac{d/z (qb/d)^j}{1 - q^j d/z} \left[\begin{matrix} cd, & qd/b \\ q, & ad \end{matrix}\,\middle|\, q\right]_j \quad (7b) $$

which is equivalent to Chan [11, Lemma 2.2].

When $|d| < |z| < 1/|c|$, we have two geometric series

$$\frac{1}{1-q^icz} = \sum_{k=0}^{+\infty} z^k(q^ic)^k \quad \text{and} \quad \frac{q^jd/z}{1-q^jd/z} = \sum_{k=-\infty}^{-1} z^k(q^jd)^{-k}.$$

Interchanging the summation order, we derive the following Laurent series expansion:

$$\frac{[az,b/z;q]_m}{[cz,d/z;q]_{m+1}} = \sum_{k=0}^{+\infty}(cz)^k\sum_{i=0}^{m}\frac{(a/c;q)_{m-i}(bc;q)_{m+i}}{(q;q)_{m-i}(cd;q)_{m+i+1}}\left[\begin{matrix}cd,\ qc/a\\ q,\ bc\end{matrix}\,\middle|\,q\right]_i(q^ka/c)^i$$
$$+\sum_{k=-\infty}^{-1}(z/d)^k\sum_{j=0}^{m}\frac{(b/d;q)_{m-j}(ad;q)_{m+j}}{(q;q)_{m-j}(cd;q)_{m+j+1}}\left[\begin{matrix}cd,\ qd/b\\ q,\ ad\end{matrix}\,\middle|\,q\right]_j(q^{-k}b/d)^j$$

which may further be reformulated in terms of balanced series:

$$\frac{[az,b/z;q]_m}{[cz,d/z;q]_{m+1}}$$
$$=\left[\begin{matrix}a/c,bc\\ q,qcd\end{matrix}\,\middle|\,q\right]_m\sum_{k=0}^{+\infty}\frac{(cz)^k}{1-cd}\,{}_4\phi_3\left[\begin{matrix}q^{-m},\ cd,\ qc/a,\ q^mbc\\ bc,\ q^{1-m}c/a,\ q^{1+m}cd\end{matrix}\,\middle|\,q;q^{1+k}\right]$$
$$+\left[\begin{matrix}ad,b/d\\ q,qcd\end{matrix}\,\middle|\,q\right]_m\sum_{k=-\infty}^{-1}\frac{(z/d)^k}{1-cd}\,{}_4\phi_3\left[\begin{matrix}q^{-m},\ cd,\ qd/b,\ q^mad\\ ad,q^{1-m}d/b,\ q^{1+m}cd\end{matrix}\,\middle|\,q;q^{1-k}\right].$$

When $m\to\infty$, the limit reads as

$$\left[\begin{matrix}az,b/z\\ cz,d/z\end{matrix}\,\middle|\,q\right]_\infty = \left[\begin{matrix}a/c,bc\\ q,cd\end{matrix}\,\middle|\,q\right]_\infty\sum_{k=0}^{+\infty}(cz)^k{}_2\phi_1\left[\begin{matrix}cd,\ qc/a\\ bc\end{matrix}\,\middle|\,q;q^ka/c\right]$$
$$+\left[\begin{matrix}ad,b/d\\ q,cd\end{matrix}\,\middle|\,q\right]_\infty\sum_{k=-\infty}^{-1}\left(\frac{z}{d}\right)^k{}_2\phi_1\left[\begin{matrix}cd,\ qd/b\\ ad\end{matrix}\,\middle|\,q;q^{-k}b/d\right].$$

For $ab=q$, both ${}_2\phi_1$-series reduce to ${}_1\phi_0$-series and therefore can be evaluated by q-binomial theorem (6). This leads to the following identity

$$\left[\begin{matrix}az, & b/z\\ cz, & d/z\end{matrix}\,\middle|\,q\right]_\infty = \left[\begin{matrix}ad, & bc\\ q, & cd\end{matrix}\,\middle|\,q\right]_\infty\sum_{k=-\infty}^{+\infty}\frac{(a/c;q)_k}{(ad;q)_k}(cz)^k$$

which is equivalent to Ramanujan's ${}_1\psi_1$-summation formula.

This is essentially what has been done recently by Chan [11]. Further applications of the partial fraction method to basic hypergeometric series identities can be found in Chu [13, 14].

3. Proofs by Heine's Transformations

Heine's transformations (cf. [21, II-1-2-3]) on ${}_2\phi_1$-series read as

$$ {}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;z\right] = \left[\begin{matrix}b, & az\\ c, & z\end{matrix}\,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix}c/b, & z\\ & az\end{matrix}\,\middle|\, q;b\right] \tag{8a}$$

$$ = \left[\begin{matrix}c/b, bz\\ c, z\end{matrix}\,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix}abz/c, & b\\ & bz\end{matrix}\,\middle|\, q;c/b\right] \tag{8b}$$

$$ = \left[\begin{matrix}abz/c\\ z\end{matrix}\,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix}c/a, & c/b\\ & c\end{matrix}\,\middle|\, q;abz/c\right]. \tag{8c}$$

§3.1. With a and b being exchanged, the second transformation (8b) reads as

$$ {}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\,\middle|\, q;z\right] = \left[\begin{matrix}c/a, & az\\ c, & z\end{matrix}\,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix}abz/c, & a\\ & az\end{matrix}\,\middle|\, q;c/a\right]. \tag{9}$$

Performing the replacements

$$ a\to aq^{-n}, \quad b\to q \quad \text{and} \quad c\to cq^{-n} \tag{10}$$

we may reformulate the left hand side as follows:

$$ \text{LHS(9)} \Rightarrow {}_2\phi_1\left[\begin{matrix}q, & aq^{-n}\\ & cq^{-n}\end{matrix}\,\middle|\, q;z\right] = \sum_{k=0}^{+\infty}\frac{(aq^{-n};q)_k}{(cq^{-n};q)_k}z^k \tag{11a}$$

$$ = \left(\frac{az}{c}\right)^n\frac{(q/a;q)_n}{(q/c;q)_n}\sum_{k=-n}^{+\infty}\frac{(a;q)_k}{(c;q)_k}z^k \tag{11b}$$

where the summation index has been shifted by $k\to n+k$. The right hand side may be manipulated correspondingly as

$$ \text{RHS(9)} \Rightarrow {}_2\phi_1\left[\begin{matrix}qaz/c, aq^{-n}\\ azq^{-n}\end{matrix}\,\middle|\, q;\frac{c}{a}\right]\left[\begin{matrix}c/a, & azq^{-n}\\ z, & cq^{-n}\end{matrix}\,\middle|\, q\right]_\infty \tag{12a}$$

$$ = {}_2\phi_1\left[\begin{matrix}qaz/c, aq^{-n}\\ azq^{-n}\end{matrix}\,\middle|\, q;\frac{c}{a}\right]\left(\frac{az}{c}\right)^n\left[\begin{matrix}q/az\\ q/c\end{matrix}\,\middle|\, q\right]_n\left[\begin{matrix}c/a, az\\ c, z\end{matrix}\,\middle|\, q\right]_\infty. \tag{12b}$$

Comparing the identities (11b) and (12b), we get

$$ \sum_{k=-n}^{+\infty}\frac{(a;q)_k}{(c;q)_k}z^k = \left[\begin{matrix}q/az\\ q/a\end{matrix}\,\middle|\, q\right]_n\left[\begin{matrix}c/a, az\\ c, z\end{matrix}\,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix}qaz/c, aq^{-n}\\ azq^{-n}\end{matrix}\,\middle|\, q;\frac{c}{a}\right]. \tag{13}$$

Applying the q-binomial theorem (6), we have

$$ \lim_{n\to\infty} {}_2\phi_1\left[\begin{matrix}qaz/c, & aq^{-n}\\ & azq^{-n}\end{matrix}\,\middle|\, q;c/a\right] = {}_1\phi_0\left[\begin{matrix}qaz/c\\ -\end{matrix}\,\middle|\, q;\frac{c}{az}\right] = \frac{(q;q)_\infty}{(c/az;q)_\infty}.$$

Hence the limiting case $n \to \infty$ of (13) leads us to Ramanujan's ${}_1\psi_1$-summation formula (1) thanks again to the Weierstrass M-test on uniformly convergent series (cf. Stromberg [35, P 141]).

§3.2. Ramanujan's ${}_1\psi_1$-summation formula can also be obtained by Heine's third transformation (8c). Making the same replacements (10) and recalling (11b), we may restate the result as follows:

$$\left(\frac{az}{c}\right)^n \frac{(q/a;q)_n}{(q/c;q)_n} \sum_{k=-n}^{+\infty} \frac{(a;q)_k}{(c;q)_k} z^k = \frac{(qaz/c;q)_\infty}{(z;q)_\infty} \sum_{k=0}^{+\infty} \begin{bmatrix} c/a, cq^{-n-1} \\ q, cq^{-n} \end{bmatrix} q \Bigg]_k \left(\frac{qaz}{c}\right)^k. \quad (14)$$

Replacing k by $n+k$ on the right side of the last equation and then simplifying the result, we obtain:

$$\sum_{k=-n}^{+\infty} \frac{(a;q)_k}{(c;q)_k} z^k = \frac{(qaz/c;q)_\infty}{(z;q)_\infty} \left[\begin{matrix} c/a, q^2/c \\ q, q/a \end{matrix} \Bigg| q\right]_n \sum_{k=-n}^{+\infty} \left[\begin{matrix} c/q, cq^n/a \\ c, q^{1+n} \end{matrix} \Bigg| q\right]_k \left(\frac{qaz}{c}\right)^k.$$

The limiting case $n \to \infty$ of the above identity results in the following relation:

$${}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix} \Bigg| q; z\right] = \left[\begin{matrix} q^2/c, & c/a, & qaz/c \\ q, & q/a, & z \end{matrix} \Bigg| q\right]_\infty {}_1\psi_1\left[\begin{matrix} c/q \\ c \end{matrix} \Bigg| q; \frac{qaz}{c}\right]. \quad (15)$$

§3.3. In order to derive Ramanujan's ${}_1\psi_1$-summation formula from the last transformation, we have to verify the following bilateral series identity

$${}_1\psi_1\left[\begin{matrix} c/q \\ c \end{matrix} \Bigg| q; \frac{qaz}{c}\right] = \left[\begin{matrix} q, & q, & az, & q/az \\ c, & q^2/c, & c/az, & qaz/c \end{matrix} \Bigg| q\right]_\infty \quad (16)$$

which is equivalent to the following formula:

$${}_1\psi_1\left[\begin{matrix} x \\ qx \end{matrix} \Bigg| q; y\right] = \left[\begin{matrix} q, & q, & xy, & q/xy \\ qx, & q/x, & y, & q/y \end{matrix} \Bigg| q\right]_\infty \quad \text{where} \quad |q| < |y| < 1. \quad (17)$$

We remark that for $|q| < |x|$ and $|y| < 1$, there is a curious symmetric relation

$${}_1\psi_1\left[\begin{matrix} x \\ qx \end{matrix} \Bigg| q; y\right](1-y) = {}_1\psi_1\left[\begin{matrix} y \\ qy \end{matrix} \Bigg| q; x\right](1-x) \quad (18)$$

which follows immediately from (17). However, the authors have not found a direct proof, even though it would be more desirable.

Interestingly, the identity (17) can be established by means of partial fraction method. Consider rational function in variable x and decompose it in partial fractions:

$$H(x)=\frac{1}{1-x}\left[\begin{matrix}xy, & q/xy\\ qx, & q/x\end{matrix}\,\middle|\, q\right]_m=\sum_{i=0}^{m}\frac{C_i}{1-xq^i}+\sum_{j=1}^{m}\frac{D_j}{1-xq^{-j}}$$

where the coefficients are determined as follows:

$$\begin{aligned}C_i &= \lim_{x\to q^{-i}}(1-xq^i)H(x)=\frac{(y;q)_{m-i}(q/y;q)_{m+i}}{(q;q)_{m-i}(q;q)_{m+i}}y^i,\\ D_j &= \lim_{x\to q^{j}}(1-xq^{-j})H(x)=\frac{(y;q)_{m+j}(q/y;q)_{m-j}}{(q;q)_{m-j}(q;q)_{m+j}}y^{-j}.\end{aligned}$$

We find the following finite summation formula

$$H(x)=\frac{1}{1-x}\left[\begin{matrix}xy, q/xy\\ qx, q/x\end{matrix}\,\middle|\, q\right]_m=\sum_{i=0}^{m}\frac{(y;q)_{m-i}(q/y;q)_{m+i}}{(q;q)_{m-i}(q;q)_{m+i}}\frac{y^i}{1-xq^i}$$

$$+\sum_{j=1}^{m}\frac{(y;q)_{m+j}(q/y;q)_{m-j}}{(q;q)_{m-j}(q;q)_{m+j}}\frac{y^{-j}}{1-xq^{-j}}.$$

Letting $m\to\infty$ and simplifying the result, we derive the identity:

$$\left[\begin{matrix}q, q, xy, q/xy\\ y, q/y, qx, q/x\end{matrix}\,\middle|\, q\right]_\infty=\sum_{i=0}^{+\infty}\frac{1-x}{1-xq^i}y^i+\sum_{j=1}^{+\infty}\frac{1-x}{1-xq^{-j}}y^{-j}.$$

It is trivial to see that the right side of the last identity is the restatement of the ${}_1\psi_1$-series in (17). This completes the proof of the bilateral series identity of (17) and also Ramanujan's ${}_1\psi_1$-summation formula.

4. Proof by Bailey's ${}_6\psi_6$-Series Identity

Among the classical hierarchy of basic hypergeometric identities, the most important one perhaps is Bailey's very well-poised bilateral ${}_6\psi_6$-series identity (Bailey [8], see also [21, II-33]):

$${}_6\psi_6\left[\begin{matrix}q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e\\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e\end{matrix}\,\middle|\, q; \frac{qa^2}{bcde}\right] \tag{19a}$$

$$=\left[\begin{matrix}q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de\\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde\end{matrix}\,\middle|\, q\right]_\infty \tag{19b}$$

provided that $|qa^2/bcde|<1$ for convergence.

Performing the replacements $a=d=aq^{-2-2n}$ and $e=q^{-1-2n}$ in Bailey's identity (19a-19b) and then simplifying the result, we get the finite series formula:

$$ {}_6\phi_5\left[\begin{matrix} q^{-2-2n}a,\ q^{-n}\sqrt{a},\ -q^{-n}\sqrt{a}, & b,\ c,\ q^{-1-2n} \\ q^{-1-n}\sqrt{a},\ -q^{-1-n}\sqrt{a}, & q^{-1-2n}a/b,\ q^{-1-2n}a/c,\ a \end{matrix}\ \Big|\ q;\frac{a}{bc}\right] \tag{20a} $$

$$ = \left[\begin{matrix} q^{-1-2n}a, & q^{-1-2n}a/bc \\ q^{-1-2n}a/b, & q^{-1-2n}a/c \end{matrix}\ \Big|\ q\right]_{2n+1} = \left[\begin{matrix} q/a, & qbc/a \\ qb/a, & qc/a \end{matrix}\ \Big|\ q\right]_{2n+1}. \tag{20b} $$

The ${}_6\phi_5$-series displayed in (20a) consists of $2n+2$ terms. Splitting equally the sum into two parts with each of them having $n+1$ terms and then replacing the summation index for the second part by $k \to 1+2n-k$, we can reformulate (20a-20b) as follows:

$$ \left[\begin{matrix} q/a, & qbc/a \\ qb/a, & qc/a \end{matrix}\Big|\, q\right]_{1+2n} = \sum_{k=0}^{n}\left[\begin{matrix} q^{-2-2n}a,\, q^{-n}\sqrt{a},\, -q^{-n}\sqrt{a},\, b,\, c,\, q^{-1-2n} \\ q, q^{-1-n}\sqrt{a}, -q^{-1-n}\sqrt{a}, q^{-1-2n}a/b, q^{-1-2n}a/c, a \end{matrix}\Big|\, q\right]_k \left(\frac{a}{bc}\right)^k $$

$$ +\left[\begin{matrix} q/a,\ b,\ c \\ a/q, qb/a, qc/a \end{matrix}\Big|\, q\right]_{1+2n} \times \sum_{k=0}^{n}\left[\begin{matrix} q^{-2n}/a, q^{1-n}/\sqrt{a}, -q^{1-n}/\sqrt{a}, qb/a, qc/a, q^{-1-2n} \\ q,\, q^{-n}/\sqrt{a},\, -q^{-n}/\sqrt{a},\, q^{-2n}/b,\, q^{-2n}/c,\, q^2/a \end{matrix}\Big|\, q\right]_k \left(\frac{a}{bc}\right)^k. $$

In view of the Weierstrass M-test on uniformly convergent series, the limiting case $n \to \infty$ of this identity results in the following simpler relation:

$$ \left[\begin{matrix} q/a, qbc/a \\ qb/a, qc/a \end{matrix}\Big|\, q\right]_\infty = {}_2\phi_1\left[\begin{matrix} b,\ c \\ a \end{matrix}\Big|\, q; q\right] + \left[\begin{matrix} q/a,\ b,\ c \\ a/q, qb/a, qc/a \end{matrix}\Big|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} qb/a, qc/a \\ q^2/a \end{matrix}\Big|\, q; q\right]. \tag{21} $$

Applying Heine's transformation (8a) to both ${}_2\phi_1$-series:

$$ {}_2\phi_1\left[\begin{matrix} b,\ c \\ a \end{matrix}\Big|\, q; q\right] = \left[\begin{matrix} qb, c \\ q, a \end{matrix}\Big|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} q, a/c \\ qb \end{matrix}\Big|\, q; c\right], $$

$$ {}_2\phi_1\left[\begin{matrix} qb/a,\ qc/a \\ q^2/a \end{matrix}\Big|\, q; q\right] = \left[\begin{matrix} qb/a, q^2c/a \\ q, q^2/a \end{matrix}\Big|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} q, q/b \\ q^2c/a \end{matrix}\Big|\, q; \frac{qb}{a}\right]; $$

we can further reduce the right hand side of (21) as follows:

$$ \left[\begin{matrix} qb, c \\ q, a \end{matrix}\Big|\, q\right]_\infty \left\{ {}_2\phi_1\left[\begin{matrix} q, a/c \\ qb \end{matrix}\Big|\, q; c\right] - \frac{q(1-b)}{a(1-qc/a)} {}_2\phi_1\left[\begin{matrix} q, q/b \\ q^2c/a \end{matrix}\Big|\, q; \frac{qb}{a}\right] \right\} $$

$$ = \left[\begin{matrix} qb, c \\ q, a \end{matrix}\Big|\, q\right]_\infty \left\{ \sum_{k=0}^{\infty} \frac{(a/c;q)_k}{(qb;q)_k} c^k - \frac{q(1-b)}{a(1-qc/a)} \sum_{k=0}^{\infty} \frac{(q/b;q)_k}{(q^2c/a;q)_k} \left(\frac{bq}{a}\right)^k \right\} $$

$$ = \left[\begin{matrix} qb, c \\ q, a \end{matrix}\Big|\, q\right]_\infty \left\{ \sum_{k=0}^{\infty} \frac{(a/c;q)_k}{(qb;q)_k} c^k + \sum_{k=-1}^{-\infty} \frac{(a/c;q)_k}{(qb;q)_k} c^k \right\} $$

where the replacement $k \to -k-1$ has been made for the second sum. Equating the last result with the left hand side of (21), we have

$$ \left[\begin{matrix} q/a, qbc/a \\ qb/a, qc/a \end{matrix}\Big|\, q\right]_\infty = \left[\begin{matrix} qb, c \\ q, a \end{matrix}\Big|\, q\right]_\infty {}_1\psi_1\left[\begin{matrix} a/c \\ qb \end{matrix}\Big|\, q; c\right] $$

which is equivalent to the following identity

$$ {}_1\psi_1\left[\begin{matrix}a/c\\ qb\end{matrix}\,\middle|\, q;c\right] = \left[\begin{matrix}q/a, & qbc/a, & q, & a\\ qb/a, & qc/a, & qb, & c\end{matrix}\,\middle|\, q\right]_\infty . $$

Under the parameter replacements $a \to az$, $b \to c/q$ and $c \to z$, this identity becomes Ramnujan's ${}_1\psi_1$-summation formula (1).

In addition, we remark that the famous quintuple product identity (cf. [21, Ex 5.6] and Gordon [22] for example):

$$ [q,z,q/z;q]_\infty \; [qz^2,q/z^2;q^2]_\infty = \sum_{n=-\infty}^{+\infty} \left\{1 - z^{1+6n}\right\} q^{3\binom{n}{2}} \left(q^2/z^3\right)^n \tag{22} $$

follows from the case $a \to z^2$, $b \to -z$ and $c, d, e \to \infty$ of Bailey's ${}_6\psi_6$-series identity.

5. Proof by Jackson's q-Dougall Theorem

Jackson's terminating very well-poised ${}_8\phi_7$-series identity (cf. [21, II-22]) reads as

$$ {}_8\phi_7\left[\begin{matrix}a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n}\\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix}\,\middle|\, q;q\right] \tag{23a} $$

$$ = \left[\begin{matrix}aq, & aq/bc, & aq/bd, & aq/cd\\ aq/b, & aq/c, & aq/d, & aq/bcd\end{matrix}\,\middle|\, q\right]_n \quad \text{for} \quad a^2q^{n+1} = bcde. \tag{23b} $$

§5.1. Performing the replacements $a \to eq^{-2-2n}$, $d \to eq^{-1-2n}/d$, $e \to a$ and $n \to 1+2n$ in Jackson's sum (23a-23b), we obtain

$$ {}_8\phi_7\left[\begin{matrix}eq^{-2-2n}, q^{-n}\sqrt{e}, -q^{-n}\sqrt{e}, a, b, c, eq^{-1-2n}/d, q^{-1-2n}\\ q^{-1-n}\sqrt{e}, -q^{-1-n}\sqrt{e}, eq^{-1-2n}/a, eq^{-1-2n}/b, eq^{-1-2n}/c, d, e\end{matrix}\,\middle|\, q;q\right] $$

$$ = \left[\begin{matrix}d/a, & d/b, & d/c, & d/abc\\ d, & d/ab, & d/ac, & d/bc\end{matrix}\,\middle|\, q\right]_{1+2n} \quad \text{where} \quad qabc = de. $$

Following the same procedure as that for Bailey's identity (20a-20b) for ${}_6\psi_6$-series, we can split equally the last sum into two parts and reformulate the result as

$$\begin{aligned}
&\left[\begin{matrix} d/a, & d/b, & d/c, & d/abc \\ d, & d/ab, & d/ac, & d/bc \end{matrix}\middle| q\right]_{2n+1} \\
=& \sum_{k=0}^{n} \frac{1-q^{2k-2-2n}e}{1-q^{-2-2n}e} \left[\begin{matrix} a, b, c, eq^{-2-2n}, eq^{-1-2n}/d, q^{-1-2n} \\ q, d, e, eq^{-1-2n}/a, eq^{-1-2n}/b, eq^{-1-2n}/c \end{matrix}\middle| q\right]_k q^k \\
+& \left[\begin{matrix} a, & b, & c, & q/e, qd/e \\ qa/e, & qb/e, & qc/e, & d, \quad e/q \end{matrix}\middle| q\right]_{2n+1} \\
\times& \sum_{k=0}^{n} \frac{1-q^{2k-2n}/e}{1-q^{-2n}/e} \left[\begin{matrix} qa/e, qb/e, qc/e, q^{-2n}/e, q^{-2n}/d, q^{-1-2n} \\ q, qd/e, q^2/e, q^{-2n}/a, q^{-2n}/b, q^{-2n}/c \end{matrix}\middle| q\right]_k q^k.
\end{aligned}$$

The limiting case $n \to \infty$ of this identity results in the following non-terminating extension of the q-Saalschützian formula (Sears [32, Eq 5.2]), see also [21, II-24]:

$$\left[\begin{matrix} d/a, & d/b, & d/c, & d/abc \\ d, & d/ab, & d/ac, & d/bc \end{matrix}\middle| q\right]_\infty = {}_3\phi_2\left[\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}\middle| q; q\right] \tag{24a}$$

$$+\left[\begin{matrix} a, b, c, q/e, qd/e \\ qa/e, qb/e, qc/e, d, e/q \end{matrix}\middle| q\right]_\infty {}_3\phi_2\left[\begin{matrix} qa/e, & qb/e, & qc/e \\ & qd/e, & q^2/e \end{matrix}\middle| q; q\right]. \tag{24b}$$

Let the Ξ-function be defined by

$$\Xi[a, b, c, d] = \left[\begin{matrix} d, & abc/d \\ a, & b, \ c \end{matrix}\middle| q\right]_\infty {}_3\phi_2\left[\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}\middle| q; q\right] \quad \text{where} \quad de = qabc.$$

Then the Sears formula may be expressed in the following symmetric form:

$$\Xi[a, b, c, d] + \Xi[qa/e, qb/e, qc/e, qd/e] \tag{25a}$$

$$= \left[\begin{matrix} d/a, d/b, d/c, d/abc, abc/d \\ a, \ b, \ c, \ d/ab, \ d/ac, \ d/bc \end{matrix}\middle| q\right]_\infty. \tag{25b}$$

§5.2. Keeping in mind of $de = qabc$ and then specializing (25a-25b) by $c \to 0$ and $d \to 0$, we get the following relation:

$$\begin{aligned}
\left[\begin{matrix} e/q, \ q/e, \ qab/e \\ a, \ b, \ qa/e, \ qb/e \end{matrix}\middle| q\right]_\infty =& \left[\begin{matrix} e/q \\ a, \ b \end{matrix}\middle| q\right]_\infty {}_2\phi_1\left[\begin{matrix} a, & b \\ & e \end{matrix}\middle| q; q\right] \\
+& \left[\begin{matrix} q/e \\ qa/e, \ qb/e \end{matrix}\middle| q\right]_\infty {}_2\phi_1\left[\begin{matrix} qa/e, & qb/e \\ & q^2/e \end{matrix}\middle| q; q\right].
\end{aligned}$$

In view of Heine's transformation (8a), the last two ${}_2\phi_1$-series may be transformed into the following:

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a,\ b\\ e\end{matrix}\,\middle|\,q;q\right] &= \left[\begin{matrix}qa,b\\ q,e\end{matrix}\,\middle|\,q\right]_\infty {}_2\phi_1\left[\begin{matrix}q,e/b\\ qa\end{matrix}\,\middle|\,q;b\right]\\
{}_2\phi_1\left[\begin{matrix}qa/e,\ qb/e\\ q^2/e\end{matrix}\,\middle|\,q;q\right] &= \left[\begin{matrix}qa/e,q^2b/e\\ q,q^2/e\end{matrix}\,\middle|\,q\right]_\infty {}_2\phi_1\left[\begin{matrix}q,q/a\\ q^2b/e\end{matrix}\,\middle|\,q;\frac{qa}{e}\right]
\end{aligned}
$$

which lead us to the following reduction:

$$
\begin{aligned}
\left[\begin{matrix}q,e/q,q/e,qab/e\\ a,b,qa/e,qb/e\end{matrix}\,\middle|\,q\right]_\infty &= \frac{1-e/q}{1-a}{}_2\phi_1\left[\begin{matrix}q,e/b\\ qa\end{matrix}\,\middle|\,q;b\right]\\
&+ \frac{1-q/e}{1-qb/e}{}_2\phi_1\left[\begin{matrix}q,q/a\\ q^2b/e\end{matrix}\,\middle|\,q;\frac{qa}{e}\right].
\end{aligned}
$$

According to the definition, writing explicitly the two ${}_2\phi_1$-series and then inverting the summation order for the later, we have

$$
\begin{aligned}
\left[\begin{matrix}q,e/q,q/e,qab/e\\ a,b,qa/e,qb/e\end{matrix}\,\middle|\,q\right]_\infty &= \frac{1-e/q}{1-a}\sum_{k=0}^{\infty}\frac{(e/b;q)_k}{(qa;q)_k}b^k + \frac{1-q/e}{1-qb/e}\sum_{k=0}^{\infty}\frac{(q/a;q)_k}{(q^2b/e;q)_k}\left(\frac{qa}{e}\right)^k\\
&= \frac{1-e/q}{1-a}\sum_{k=0}^{\infty}\frac{(e/b;q)_k}{(qa;q)_k}b^k + \frac{1-e/q}{1-a}\sum_{k=-\infty}^{-1}\frac{(e/b;q)_k}{(qa;q)_k}b^k\\
&= \frac{1-e/q}{1-a}\sum_{k=-\infty}^{\infty}\frac{(e/b;q)_k}{(qa;q)_k}b^k = \frac{1-e/q}{1-a}{}_1\psi_1\left[\begin{matrix}e/b\\ qa\end{matrix}\,\middle|\,q;b\right].
\end{aligned}
$$

Therefore we have established

$$
{}_1\psi_1\left[\begin{matrix}e/b\\ qa\end{matrix}\,\middle|\,q;b\right] = \left[\begin{matrix}q,e,q/e,qab/e\\ qa,b,qa/e,qb/e\end{matrix}\,\middle|\,q\right]_\infty
$$

where $0<|b|<1$ and $0<|qa/e|<1$. Under the parameter replacements $a\to c/q$, $b\to z$ and $e\to az$, this becomes Ramanujan's ${}_1\psi_1$-summation formula.

Remark As we declared that our intention is to give a general coverage of the proofs for Ramanujan's ${}_1\psi_1$-series identity through basic hypergeometric series, but not to write a comprehensive review article. Among many other proofs, we invite the reader, in particular, to refer the following approaches:

- Analytical continuation by Ismail [25].
- The probabilistic proof by Kadell [28].
- The proof based on loop integral by Mimachi [30].
- Combinatorial proofs by Corteel-Lovejoy [18] and Yee [36].

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On infinite families of optimal double-loop networks with non-unit steps *

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Abstract

Double-loop networks have been widely studied as architecture for local area networks. A double-loop network $G(N; s_1, s_2)$ is a digraph with N vertices $0, 1, \ldots, N-1$ and $2N$ edges of two types:

s_1-edge: $i \rightarrow i + s_1(\mod N); i = 0, 1, \ldots, N-1$.

s_2-edge: $i \rightarrow i + s_2(\mod N); i = 0, 1, \ldots, N-1$.

for some fixed steps $1 \leq s_1 < s_2 < N$ with $\gcd(N, s_1, s_2) = 1$. Let $D(N; s_1, s_2)$ be the diameter of G and let us define $D(N) = \min\{D(N; s_1, s_2) | 1 \leq s_1 < s_2 < N \text{ and } \gcd(N, s_1, s_2) = 1\}$, and $D_1(N) = \min\{D(N; 1, s) | 1 < s < N\}$. If N is a positive integer and $D(N) < D_1(N)$, then N is called a non-unit step integer or a nus integer. Xu and Aguiló et al. gave some infinite families of 0-tight nus integers with $D_1(N) - D(N) \geq 1$.

In this work, we give a method for finding infinite families of nus integers. As application examples, we give one infinite family of 0-tight nus integers with $D_1(N) - D(N) \geq 5$, one infinite family of 2-tight nus integers with $D_1(N) - D(N) \geq 1$ and one infinite family of 3-tight nus integers with $D_1(N) - D(N) \geq 1$.

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