On infinite families of optimal double-loop networks with non-unit steps *

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Abstract

Double-loop networks have been widely studied as architecture for local area networks. A double-loop network $G(N; s_1, s_2)$ is a digraph with N vertices $0, 1, \ldots, N-1$ and $2N$ edges of two types:
s_1-edge: $i \rightarrow i + s_1(\mod N); i = 0, 1, \ldots, N-1$.
s_2-edge: $i \rightarrow i + s_2(\mod N); i = 0, 1, \ldots, N-1$.
for some fixed steps $1 \leq s_1 < s_2 < N$ with $\gcd(N, s_1, s_2) = 1$. Let $D(N; s_1, s_2)$ be the diameter of G and let us define $D(N) = \min\{D(N; s_1, s_2) | 1 \leq s_1 < s_2 < N \text{ and } \gcd(N, s_1, s_2) = 1\}$, and $D_1(N) = \min\{D(N; 1, s) | 1 < s < N\}$. If N is a positive integer and $D(N) < D_1(N)$, then N is called a non-unit step integer or a nus integer. Xu and Aguiló et al. gave some infinite families of 0-tight nus integers with $D_1(N) - D(N) \geq 1$.

In this work, we give a method for finding infinite families of nus integers. As application examples, we give one infinite family of 0-tight nus integers with $D_1(N) - D(N) \geq 5$, one infinite family of 2-tight nus integers with $D_1(N) - D(N) \geq 1$ and one infinite family of 3-tight nus integers with $D_1(N) - D(N) \geq 1$.

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1 Introduction

Double-loop digraphs $G = G(N; s_1, s_2)$, with $1 \le s_1 < s_2 < N$ and $\gcd(N, s_1, s_2) = 1$, have the vertex set $V = \{0, 1, \dots, N-1\}$ and the adjacencies are defined by $v \to v + s_i(\mod N)$ for $v \in V$ and $i = 1, 2$. The hops s_1 and s_2 between vertices are called steps. These kinds of digraphs have been widely studied as architecture for local area networks, known as double-loop networks (DLN). For surveys about these networks, see[3,7].

From the metric point of view, the minimization of the diameter of G corresponds to a faster transmission of messages in the network. The diameter of G is denoted by $D(N; s_1, s_2)$. As G is vertex symmetric, its diameter can be computed from the expression $max\{d(0; i) | i \in V\}$, where $d(u; v)$ is the distance from u to v in G. For a fixed integer $N > 0$, the optimal value of the diameter is denoted by

$$D(N) = \min\{D(N; s_1, s_2) | 1 \le s_1 < s_2 < N \text{ and } \gcd(N, s_1, s_2) = 1\}.$$

Several works studied the minimization of the diameter (for a fixed N) with $s_1 = 1$. Let us denote $D_1(N) = \min\{D(N; 1; s) | 1 < s < N\}$.

Since the work of Wong and Coppersmith [12], a sharp lower bound is known for $D_1(N)$:

$$D_1(N) \ge \lceil \sqrt{3N}\, \rceil - 2 = lb(N).$$

Fiol et al. in [8] showed that $lb(N)$ is also a sharp lower bound for $D(N)$. A given DLN $G(N; s_1, s_2)$ is called k-tight if $D(N; s_1, s_2) = lb(N) + k(k \ge 0)$. A k-tight DLN is called optimal if $D(N) = lb(N) + k(k \ge 0)$. The 0-tight DLN are known as tight ones and they are also optimal. A given DLN $G(N; 1, s_2)$ is called k-tight if $D(N; 1, s_2) = lb(N) + k(k \ge 0)$. A k-tight DLN $G(N; 1, s_2)$ is called optimal if $D_1(N) = lb(N) + k(k \ge 0)$.

The metrical properties of $G(N; s_1, s_2)$ are fully contained in its related L-shaped tile $L(N; l, h, x, y)$ where $N = lh - xy, l > y \text{ and } h \ge x$. In Figure 1, we illustrate generic dimensions of an L-shaped tile.

Let $D(L) = D(L(N; l, h, x, y)) = max\{l + h - x - 2, l + h - y - 2\}$. For obvious reasons, the value $D(L)$ is called the diameter of the tile L. It is known that an L-shaped tile $L(N; l, h, x, y)$ can be assigned to a $G(N; s_1, s_2)$ without any confusion. However, we can not find double-

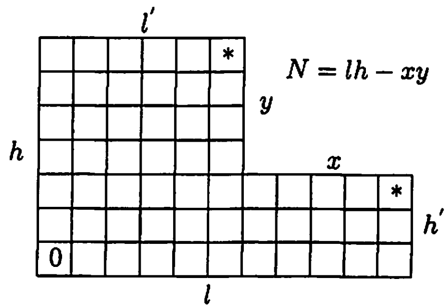

Figure 1: Generic dimensions of an L-shaped tile

loop network $G(N; s_1, s_2)$ from some L-shaped tiles. When an L-shaped tile $L(N; l, h, x, y)$ has diameter $lb(N) + k$, we say it is k-tight.

Although the identity $D(N) = D_1(N)$ holds for infinite values of N, there are also another infinite set of integers with $D(N) < D_1(N)$. These other integral values of N are called non-unit step integers or nus integers (see [2]). Xu [10] gave three infinite families of 0-tight nus integers with $D_1(N) - D(N) \geq 1$. Aguiló et al. [2] gave a method for finding infinite families of nus integers and then gave some infinite families of 0-tight nus integers with $D_1(N) - D(N) \geq 1$.

In this paper, we propose a method for finding infinite families of nus integers. As application examples, we give one infinite family of 0-tight nus integers with $D_1(N) - D(N) \geq 5$, one infinite family of 2-tight nus integers with $D_1(N) - D(N) \geq 1$ and one infinite family of 3-tight nus integers with $D_1(N) - D(N) \geq 1$.

2 Preliminary

The following Lemma 1, 2, 3 and 4 can be found in [6 or 8 or 9 or 10].
Lemma 1[8, 10]. Let $L(N; l, h, x, y)$ be an L-shaped tile, $N = lh - xy$. Then

(a) There exists $G(N; 1, s)$ realizing the L-shaped tile iff $l > y$, $h \geq x$ and $\gcd(h, y) = 1$, where $s \equiv \alpha l - \beta(l - x) (\mod N)$ for some integral values α and β satisfying $\alpha y + \beta(h - y) = 1$.

(b) There exists $G(N; s_1, s_2)$ realizing the L-shaped tile iff $l > y$, $h \geq x$ and $\gcd(l, h, x, y) = 1$, where $s_1 \equiv \alpha h + \beta y (\mod N)$, $s_2 \equiv \alpha x + \beta l($

mod N) for some integral values α and β satisfying $\gcd(N, s_1, s_2) = 1$.

Lemma 2[6, 9]. Let us denote by t a non-negative integer. We define $I_1(t) = [3t^2+1, 3t^2+2t]$, $I_2(t) = [3t^2+2t+1, 3t^2+4t+1]$ and $I_3(t) = [3t^2+4t+2, 3(t+1)^2]$. Then we have $[4, 3T^2+6T+3] = \bigcup_{t=1}^{T}\bigcup_{i=1}^{3} I_i(t)$, where $T > 1$, and $lb(N) = 3t+i-2$ if $N \in I_i(t)$ for $i = 1, 2, 3$.

Lemma 3[9]. Let $L(N; l, h, x, y)$ be an L-shaped tile, $N = lh - xy$. Then

(a) If $L(N; l, h, x, y)$ is realizable, then $|y - x| < \sqrt{N}$;

(b) If $x > 0$ and $|y - x| < \sqrt{N}$, then

$$D(L(N; l, h, x, y)) \geq \sqrt{3N - \tfrac{3}{4}(y-x)^2} + \tfrac{1}{2}|y - x| - 2\ ;$$

(c) Let $f(z) = \sqrt{3N - \frac{3}{4}z^2} + \frac{1}{2}z$. Then $f(z)$ is strictly increasing when $0 \leq z \leq \sqrt{N}$.

Proof. (a) In the case of $y - x \geq 0$, by Lemma 1, we know that $y < l$, thus $y - x \leq l'$.

Since $yl' < N$, thus $|y - x| \leq \min\{y, l'\} < \sqrt{N}$.

In the case of $x - y > 0$, by Lemma 1, we know that $x \leq h$, thus $x - y \leq h'$.

Since $xh' < N$, thus $|y - x| \leq \min\{x, h'\} < \sqrt{N}$.

(b) If $a > 0$ and $ax^2 + bx + c \leq 0$, it is well known that $b^2 - 4ac \geq 0$.

In the case of $y - x \geq 0$. As $y - x \geq 0 \Leftrightarrow h + l - x - 2 \geq l + h - y - 2$, so $D(L) = h + l - x - 2$.

Let $z = y - x$ and $d = D(L)$. Then $N = lh - x(x+z) = (d + x + 2 - h)h - x(x+z) \Leftrightarrow x^2 + (z-h)x - h(d+2-h) + N = 0$, thus

$$(z-h)^2 + 4h(d+2-h) - 4N \geq 0 \Leftrightarrow 3h^2 + (2z - 4d - 8)h + 4N - z^2 \leq 0.$$

Again we have $(2z - 4d - 8)^2 \geq 12(4N - z^2)$.

Since $z^2 < N$ and $z = y - x < h + l - x = d + 2$, thus $2d + 4 - z \geq \sqrt{3(4N - 3z^2)}$, which is

$$D(L) \geq \sqrt{3N - \tfrac{3}{4}(y-x)^2} + \tfrac{1}{2}|y - x| - 2.$$

In the case of $x - y > 0$. Note that $D(L) = l + h - y - 2$ and $N =$

$(d+y+2-h)h-y(y+z)$, the argument is similar.

(c) For $z(0 \le z < \sqrt{N})$, $f'(z) = \frac{-\frac{3}{2}z}{2\sqrt{3N-\frac{3}{4}z^2}} + \frac{1}{2} = \frac{\sqrt{3N-\frac{3}{4}z^2}-\frac{3}{2}z}{2\sqrt{3N-\frac{3}{4}z^2}} > 0$, thus $f(z)$ is strictly increasing when $0 \le z \le \sqrt{N}$.

We have this lemma. □

Lemma 4[9]. Let $N(t) = 3t^2 + At + B \in I_i(t)$ and L be the L-shaped tile $L(N(t); l, h, x, y)$, where A and B are integral values; $l = 2t + a$, $h = 2t + b$, $z = |y - x|$, a, b, x, y are all integral polynomials of variable t. Then L is k-tight iff $j = i + k (k \ge 0)$, and the following identity holds

$$(a+b-j)(a+b-j+z) - ab + (A+z-2j)t + B = 0. \qquad (1)$$

Proof. We only prove the case of $x \le y$. The others are similar.

L is k-tight iff $D(L) = max\{l+h-x-2, l+h-y-2\} = lb(N) + k = 3t+i-2+k = 3t+j-2$ and $N(t) = lh - xy$. Note that $y = x + z (z \ge 0)$, thus $D(L) = l + h - x - 2 = 4t + a + b - x - 2 = 3t + j - 2$, we have $x = t + a + b - j$.

Consider $y = x + z = t + a + b - j + z$, the identity (1) is equivalent to $N(t) = lh - xy$.

We have this lemma. □

The following Lemma 5 is the generalization of Theorem 2 in [11].

Lemma 5. Let $H(z, j) = (2j - z)^2 - 3[j(j - z) + (A + z - 2j)t + B]$, and the identity (1) be an equation of a and b. A necessary condition for the equation (1) to have integral solution is that $4H(z, j) = s^2 + 3m^2$, where s and m are integers.

Proof. Suppose that the equation (1) of variable a and b has an integral solution and rewrite it as the following

$$a^2 + (b - 2j + z)a + b^2 - (2j - z)b + c = 0$$

where $c = j(j - z) + (A + z - 2j)t + B$. Thus, there exists an integer m such that

$$(b - 2j + z)^2 - 4[b^2 - (2j - z)b + c] = m^2$$

Rewrite it as an equation of variable b

$$3b^2 - 2(2j - z)b + 4c + m^2 - (2j - z)^2 = 0$$

Thus, there exists an integer n such that

$4(2j - z)^2 - 12[4c + m^2 - (2j - z)^2] = n^2$

This implies that n is even. Let $n = 2s$.

We have $4(2j - z)^2 - 12c = s^2 + 3m^2$, hence $4H(z, j) = s^2 + 3m^2$.

We have this lemma. □

It is easy to show that the following Lemma 6 is equivalent to Theorem 1 in [11] .

Lemma 6. Let n, s and m be integers, $n = s^2 + 3m^2$. If n has a prime factor p, here $p \equiv 2(\mod 3)$, then there exists an even integer q, such that n is divisible by p^q , but not divisible by p^{q+1}.

Proof. We prove it by mathematical induction on p.

1. We first prove the case of $p = 2$. Suppose n is divisible by 2, then s and m are both even or odd. Consider s and m are both odd, let $s = 2i + 1$ and $m = 2j + 1$, thus $n = 2^2[i(i+1) + 3j(j+1)]$, note that $i(i+1) + 3j(j+1)$ is odd, so the lemma is true. Consider s and m are both even, let $s = 2i$ and $m = 2j$, thus $n = 2^2(i^2 + 3j^2)$, if $i^2 + 3j^2$ is divisible by 2, with the same argument, we know that $i^2 + 3j^2$is divisible by 2^2. Finally there exists an even integer q, such that n is divisible by 2^q , but not divisible by 2^{q+1}.

2. Suppose that n is divisible by a prime number p, where $p > 2$ and $p \equiv 2(\mod 3)$. Note that p is odd, so there exist i and j, such that $0 < i < p/2$, $0 < j < p/2$, $s = xp + i$(or $s = xp - i$) and $m = yp + j$(or $m = yp - j$), therefore $i^2 + 3j^2$ is divisible by p.

Note that $i^2+3j^2 < (p/2)^2+3(p/2)^2 = p^2$, by the induction assumption, we know that there is no prime number p_0 and odd integer k, where $p_0 < p$ and $p_0 \equiv 2(\mod 3)$, such that $i^2 + 3j^2$ is divisible by $(p_0)^k$. Suppose that there exists integer q_0 (we can suppose $q_0 = (p_1)^q$, where $p_1 < p, p_1 \equiv 2(\mod 3)$, and q is an even integer), where $q_0 \equiv 1(\mod 3)$, such that $i^2 + 3j^2 = pq_0$. Note that $i^2 + 3j^2 \equiv 0$ or $1(\mod 3)$ and $pq_0 \equiv 2(\mod 3)$, this is a contradiction. It follows that the identity $i^2 + 3j^2 = p$ does not hold.

Therefore $i^2 + 3j^2$ is only divisible by 3 and p, thus $3(i/3)^2 + j^2$ is divisible by p. So we can conclude that $i^2 + 3j^2 = 0$, thus $s = xp$ and $m = yp$, we obtain that n is divisible by p^2 . Finally there exists an even integer q, such that n is divisible by p^q , but not divisible by p^{q+1}.

By mathematical induction, we have this lemma. □

3 A method to generate infinite families of nus integers

We first prove that

Theorem 1. Let $N = N(t) = 3t^2 + At + B \in I_i(t)$, $D(N) = lb(N) + k(k \geq 0)$ and L-shaped tile $L(N; l, h, x, y)$ be k-tight and realizable. Let $z = |y - x|$. Then the following hold

Case 1. If $A = 0$ or $A = 2$(if $i = 2$) or $A = 4$(if $i = 3$), and $3N - \frac{3}{4}(2k+3)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k + 2$.

Case 2. If $A = 1$ or $A = 3$ or $A = 5$, and $3N - \frac{3}{4}(2k+2)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k + 1$.

Case 3. If $A = 2$(if $i = 1$) or $A = 4$(if $i = 2$) or $A = 6$, and $3N - \frac{3}{4}(2k+1)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k$.

Proof. We only prove case 1. The others are similar.

Let $L(N; l, h, x, y)$ be k-tight, then $D(L) = 3t + i - 2 + k$.

Note that $i = A/2 + 1$, by lemma 3, if $z \geq 2k + 3$, we have

$$\begin{aligned} D(L(N; l, h, x, y)) &\geq \sqrt{3N(t) - \tfrac{3}{4}(2k+3)^2} + \tfrac{2k+3}{2} - 2 \\ &> (3t + \tfrac{A-1}{2}) + \tfrac{2k+3}{2} - 2 \\ &= 3t + i - 2 + k \\ &= lb(N) + k. \end{aligned}$$

Therefore, all the k-tight L-shaped tile $L(N; l, h, x, y)$ must satisfy $0 \leq z \leq 2k + 2, z = |y - x|$.

We have this theorem. □

We now describe our method to generate infinite families of nus integers.

Step 1. Find an integer N_0 , such that $G(N_0; s_1, s_2)$ is k-tight optimal $(k \geq 0)$, and $D(N_0) < D_1(N_0)$.

Step 2. Find a polynomial $N(t) = 3t^2 + At + B$, such that $N(t_0) = N_0$ and $N(t) \in I_i(t)$,$1 \leq i \leq 3$.

Step 3. For any $H(z, j)$, $i \leq j \leq k$, $0 \leq z \leq 2k + z0$(if $A = 0$ or $A = 2(i = 2)$ or $A = 4(i = 3)$, then $z0 = 2$; if $A = 1$ or $A = 3$ or $A = 5$,then $z0 = 1$; if $A = 2(i = 1)$ or $A = 4(i = 2)$ or $A = 6$, then $z0 = 0$).

Case 1. If $4H(z,j)$ has not the form of s^2+3m^2 , where s and m are integers. From Lemma 6, when $t=t_0$, $4H(z,j)$ has a prime factor $p \equiv 2($ mod 3), and there exists an even integer q, such that $4H(z,j)$ is divisible by p^{q-1} , but not divisible by p^q. Suppose we have got the following factors:

$p_1^{q_1}, p_2^{q_2}, \ldots, p_l^{q_l}$.

Let $g_0 = lcm(p_1^{q_1}, p_2^{q_2}, \ldots, p_l^{q_l})$.

Case 2. If $j=k$, $A+z-2j=0$, and $4H(z,j)$ has the form of s^2+3m^2 , where s and m are integers. For any integral solution (a,b) of the equation: $(a+b-j)(a+b-j+z)-ab+(A+z-2j)t+B=0$.

Let $l(t) = 2t+a = 2(t-t_0)+l_0$, $h(t) = 2t+b = 2(t-t_0)+h_0$, $x(t) = t+a+b-j$(or $y(t)+z) = (t-t_0)+x_0$, $y(t) = x(t)+z$ (or $(t+a+b-j) = (t-t_0)+y_0$. From Lemma 1, as $L(N;l,h,x,y)$ can not be realized by $G(N;1,s)$, hence $\gcd(h_0,y_0)>1$, so there exists a prime factor p of $\gcd(h_0,y_0)$. Suppose we have got the following prime factors:

$p_1, p_2, \ldots, p_r$.

Let $g_1 = lcm(p_1, p_2, \ldots, p_r)$.

Step 4. Suppose $\{G(N(t);s_1(t),s_2(t)) : t = g_2e+t_0, e \geq 0\}$ is an infinite family of k-tight DLN(not necessarily optimal), then $\{N(t) : t = ge+t_0, e \geq 0\}$, where $g = lcm(g_0,g_1,g_2)$, is an infinite family of k-tight nus integers.

From Step 3, we know that our method can only deal with a part of nus integers.

4 Some application examples

We now apply our method to some initial nus integers.

Example 1. Take $N(t) = 3t^2+2t-2414$, and $N(14762) = 653777042$. For $D(N;19,44380) = lb(653777042) = 44285$, then $G(N;19,44380)$ is 0-tight optimal. For $D(N;1,27202990) = 44290 = lb(N)+5$, and it is checked by computer that $D_1(653777042) = 44290$, so 653777042 is a nus integer with $D_1(N)-D(N) = 5$.

In fact, the following proof will show that $D_1(653777042) \geq 44290$. With a similar argument as $G(10606260;1,414351)$ in section 5, it can be proved that $D(653777042;1,27202990) = 44290$.

For $A = 2$, $B = -2414$, $j = 1$, $z = 0$, then $A + z - 2j = 0$, so the equation (1) becomes

$$(a+b-1)(a+b-1) - ab - 2414 = 0,$$

which has integral solutions:

$$S = \{(-55, 19), (-55, 38), (19, -55), (19, 38), (38, -55), (38, 19)\}.$$

For $(-55, 19)$, $\gcd(h, y) = \gcd(2t+b, t+a+b-1) = \gcd(2t+19, t-37) = 31$ if $t \equiv 6(\mod 31)$.

For $(-55, 38)$, $\gcd(h, y) = \gcd(2t+38, t-18) = 2$ if $t \equiv 0(\mod 2)$.

For $(19, -55)$, $\gcd(h, y) = \gcd(2t-55, t-37) = 19$ if $t \equiv 18(\mod 19)$.

For $(19, 38)$, $\gcd(h, y) = \gcd(2t+38, t+56) = 2$ if $t \equiv 0(\mod 2)$.

For $(38, -55)$, $\gcd(h, y) = \gcd(2t-55, t-18) = 19$ if $t \equiv 18(\mod 19)$.

For $(38, 19)$, $\gcd(h, y) = \gcd(2t+19, t+56) = 31$ if $t \equiv 6(\mod 31)$.

By Lemma 1(a) and consider the symmetry of L-shaped tile, for $t = 2 \times 19 \times 31 \times e + 14762 (e \geq 0)$, there is no $G(N; 1, s)$ realizing the 0-tight L-shaped tile $L(N(t); l, h, x, y)$ where $|y - x| = 0$.

Some L-shaped tiles can not be realized by $G(N; 1, s)$, but can be realized by $G(N; s_1, s_2)$.

Take $j = 1$, $z = 0$, $(38, -55) \in S$, let $l = 2t+a$, $h = 2t+b$, $x = t+a+b-1$, $y = x$, $\alpha = -1$, $\beta = 2$, $s_1 \equiv \alpha h + \beta y(\mod N) = 19$, $s_2 \equiv \alpha x + \beta l(\mod N) = 3t + 94$, then for $t = 19e + 14762 (e \geq 0)$, $\gcd(l, h, x, y) = 1$ and $\gcd(N, s_1, s_2) = 1$. Hence, $\{G(N(t); s_1, s_2) : t = 19e + 14762, e \geq 0\}$ is an infinite family of 0-tight DLN.

For $2 \leq j \leq 5$, $0 \leq z \leq 2(j-1)$, $t = 14762$, $H(z, j)$ has the following factors:

$H(0, 2) = 95818 = 2 \times 47909$, where the power of 2 is 1.

$H(1, 2) = 51531 = 89 \times 579$, where the power of 89 is 1.

$H(2, 2) = 7246 = 2 \times 3623$, where the power of 2 is 1.

$H(0, 3) = 184395 = 5 \times 36879$, where the power of 5 is 1.

$H(1, 3) = 140107 = 11 \times 12737$, where the power of 11 is 1.

$H(2, 3) = 95821 = 11 \times 8711$, where the power of 11 is 1.

$H(3, 3) = 51537 = 41 \times 1257$, where the power of 41 is 1.

$H(4,3) = 7255 = 5 \times 1451$, where the power of 5 is 1.

$H(0,4) = 272974 = 2 \times 136487$, where the power of 2 is 1.

$H(1,4) = 228685 = 5 \times 45737$, where the power of 5 is 1.

$H(2,4) = 184398 = 2 \times 92199$, where the power of 2 is 1.

$H(3,4) = 140113 = 167 \times 839$, where the power of 167 is 1.

$H(4,4) = 95830 = 2 \times 47915$, where the power of 2 is 1.

$H(5,4) = 51549 = 17183 \times 3$, where the power of 17183 is 1.

$H(6,4) = 7270 = 2 \times 3635$, where the power of 2 is 1.

$H(0,5) = 361555 = 5 \times 72311$, where the power of 5 is 1.

$H(1,5) = 317265 = 5 \times 63453$, where the power of 5 is 1.

$H(2,5) = 272977 = 29 \times 9413$, where the power of 29 is 1.

$H(3,5) = 228691 = 71 \times 3221$, where the power of 71 is 1.

$H(4,5) = 184407 = 61469 \times 3$, where the power of 61469 is 1.

$H(5,5) = 140125 = 5^3 \times 1121$, where the power of 5 is 3.

$H(6,5) = 95845 = 5 \times 19169$, where the power of 5 is 1.

$H(7,5) = 51567 = 17189 \times 3$, where the power of 17189 is 1.

$H(8,5) = 7291 = 23 \times 317$, where the power of 23 is 1.

Let $g_0 = 2^2 \times 5^4 \times 11^2 \times 29^2 \times 41^2 \times 71^2 \times 89^2 \times 167^2 \times 17183^2 \times 17189^2 \times 61469^2$, and $t = 19 \times 31 \times g_0 \times e + 14762 (e \geq 0)$.

For $t \geq 14762$, $A = 2, B = -2414, 0 \leq k \leq 4$, we know that $3N(t) - \frac{3}{4}(2k+1)^2 > (3t + \frac{A-1}{2})^2$ is equivalent to $3t + 3B > \frac{3}{4}(2k+3)^2 + \frac{1}{4}(A-1)^2$, which is true. From Theorem 1, if L-shaped tile $L(N; l, h, x, y)$ is k-tight, $z = |y - x|$, then $0 \leq z \leq 2k$. We know that there is no 0-tight L-shaped tile $L(N; l, h, x, y)$, $|y - x| = 0$, which can be realized by $G(N; 1, s)$, for $t = 19 \times 31 \times g_0 \times e + 14762 (e \geq 0)$.

For $1 \leq k \leq 4 (2 \leq j \leq 5), 0 \leq z = |y - x| \leq 2k$, by Lemma 6, $H(z, j)$ has no the form of $s^2 + 3m^2$. By lemma 5, the equation (1) has no integral solutions of a and b. By lemma 4, there is no k-tight L-shaped tile $L(N(t); l, h, x, y)$ for (z, j).

As a conclusion, the nodes $N(t) = 3t^2 + 2t - 2414, t = 19 \times 31 \times g_0 \times e + 14762 (e \geq 0)$, of an infinite family of 0-tight optimal DLN correspond to nus integers with $D_1(N) - D(N) \geq 5$.

Example 2. Take $N(t) = 3t^2 + 2t - 700$, and $N(1880) = 10606260$. For $D(N; 33, 5696) = lb(10606260) + 2 = 5641$, then $G(N; 33, 5696)$ is 2-tight optimal. For $D(N; 1, 414351) = 5642 = lb(N) + 3$, and it is checked by computer that $D_1(10606260) = 5642$, so 10606260 is a 2-tight nus integer with $D_1(N) - D(N) = 1$.

For $1 \leq j \leq 3, 0 \leq z \leq 2(j-1), t = 1880$, $H(z, j)$ has the following factors(except $H(4, 3)$):

$H(0,1) = 2101 = 11 \times 191$, where the power of 11 is 1.

$H(0,2) = 13384 = 2^3 \times 1673$, where the power of 2 is 3.

$H(1,2) = 7743 = 29 \times 267$, where the power of 29 is 1.

$H(2,2) = 2104 = 2^3 \times 263$, where the power of 2 is 3.

$H(0,3) = 24669 = 2741 \times 9$, where the power of 2741 is 1.

$H(1,3) = 19027 = 53 \times 359$, where the power of 53 is 1.

$H(2,3) = 13387 = 11 \times 1217$, where the power of 11 is 1.

$H(3,3) = 7749 = 41 \times 189$, where the power of 41 is 1.

$H(4,3) = 2113 = 41^2 + 3 \times 12^2$.

Let $g_0 = 2^4 \times 11^2 \times 29^2 \times 41^2 \times 53^2 \times 2741^2$.

For $A = 2, B = -700, j = 3, z = 4$, then $A + z - 2j = 0$, so the equation (1) becomes

$$(a + b - 3)(a + b + 1) - ab - 700 = 0,$$

which has integral solutions:

$$S = \{(-25, -1), (-25, 28), (-1, -25), (-1, 28), (28, -25), (28, -1)\}.$$

For $(-25, -1)$, $\gcd(h, y) = \gcd(2t + b, t + a + b - 3 + 4) = \gcd(2t - 1, t - 25) = 7$ if $t \equiv 4(\mod 7)$.

For $(-25, 28)$, $\gcd(h, y) = \gcd(2t + 28, t + 4) = 2$ if $t \equiv 0(\mod 2)$.

For $(-1, -25)$, $\gcd(h, y) = \gcd(2t - 25, t - 25) = 5$ if $t \equiv 0(\mod 5)$.

For $(-1, 28)$, $\gcd(h, y) = \gcd(2t + 28, t + 28) = 2$ if $t \equiv 0(\mod 2)$.

For $(28, -25)$, $\gcd(h, y) = \gcd(2t - 25, t + 4) = 3$ if $t \equiv 2(\mod 3)$.

For $(28, -1)$, $\gcd(h, y) = \gcd(2t - 1, t + 28) = 3$ if $t \equiv 2(\mod 3)$.

By Lemma 1(a) and consider the symmetry of L-shaped tile, for $t = 2 \times 3 \times 5 \times 7 \times e + 1880 (e \geq 0)$, there is no $G(N; 1, s)$ realizing the 2-tight L-shaped tile $L(N(t); l, h, x, y)$ where $|y - x| = 4$.

With a similar argument as Example 1, let $t = 3 \times 5 \times 7 \times g_0 \times e + 1880 (e \geq 0)$, there is no k-tight$(0 \leq k \leq 1)$ L-shaped tile $L(N(t); l, h, x, y)$, and there is no k-tight$(0 \leq k \leq 2)$ realizable L-shaped tile $L(N(t); l, h, x, y)$ by $G(N(t); 1, s)$.

Take $j = 3, z = 4, (28, -25) \in S$, let $l = 2t + a, h = 2t + b, x = t + a + b - 3, y = x + 4, \alpha = -1, \beta = 2, s_1 \equiv \alpha h + \beta y (\mod N) = 33, s_2 \equiv \alpha x + \beta l (\mod N) = 3t + 56$, then for $t = g_0 \times e + 1880 (e \geq 0), \gcd(l, h, x, y) = 1$ and $\gcd(N, s_1, s_2) = 1$. Hence, $\{G(N(t); s_1, s_2) : t = g_0 \times e + 1880, e \geq 0\}$ is an infinite family of 2-tight DLN.

As a conclusion, the nodes $N(t) = 3t^2 + 2t - 700, t = 3 \times 5 \times 7 \times g_0 \times e + 1880 (e \geq 0)$, of an infinite family of 2-tight optimal DLN correspond to nus integers with $D_1(N) - D(N) \geq 1$.

Example 3. Take $N(t) = 3t^2 + 2t - 927$, and $N(9714) = 283103889$. For $D(N; 6, 29093) = lb(283103889) + 3 = 29144$, then $G(N; 6, 29093)$ is 3-tight optimal. For $D(N; 1, 593449) = 29145 = lb(N) + 4$, and it is checked by computer that $D_1(283103889) = 29145$, so 283103889 is a 3-tight nus integer with $D_1(N) - D(N) = 1$.

For $1 \leq j \leq 4, 0 \leq z \leq 2(j - 1), t = 9714$, $H(z, j)$ has the following factors(unless $H(6, 4)$):

$H(0, 1) = 2782 = 2 \times 1391$, where the power of 2 is 1.

$H(0, 2) = 61069 = 173 \times 353$, where the power of 173 is 1.

$H(1, 2) = 31926 = 2 \times 15963$, where the power of 2 is 1.

$H(2, 2) = 2785 = 5 \times 557$, where the power of 5 is 1.

$H(0, 3) = 119358 = 2 \times 59679$, where the power of 2 is 1.

$H(1, 3) = 90214 = 2 \times 45107$, where the power of 2 is 1.

$H(2, 3) = 61072 = 11 \times 5552$, where the power of 11 is 1.

$H(3, 3) = 31932 = 887 \times 36$, where the power of 887 is 1.

$H(4, 3) = 2794 = 2 \times 1397$, where the power of 2 is 1.

$H(0, 4) = 177649 = 59 \times 3011$, where the power of 59 is 1.

$H(1, 4) = 148504 = 8 \times 18563$, where the power of 2 is 3.

$H(2,4) = 119361 = 11 \times 10851$, where the power of 11 is 1.

$H(3,4) = 90220 = 5 \times 18044$, where the power of 5 is 1.

$H(4,4) = 61081 = 17 \times 3593$, where the power of 17 is 1.

$H(5,4) = 31944 = 8 \times 3993$, where the power of 2 is 3.

$H(6,4) = 2809 = 53^2$.

Let $g_0 = 2^4 \times 5^2 \times 11^2 \times 17^2 \times 59^2 \times 173^2 \times 887^2$.

For $A = 2, B = -927, j = 4, z = 6$, then $A + z - 2j = 0$, so the equation (1) becomes

$(a+b-4)(a+b+2) - ab - 927 = 0,$

which has integral solutions:

$S = \{(-17,-17),(-17,36),(36,-17)\}.$

For $(-17,-17)$, $\gcd(h,y) = \gcd(2t+b, t+a+b-4+6) = \gcd(2t-17, t-32) = 47$ if $t \equiv 32 (\mod 47)$.

For $(-17,36), \gcd(h,y) = \gcd(2t+36, t+21) = 3$ if $t \equiv 0 (\mod 3)$.

For $(36,-17), \gcd(h,y) = \gcd(2t-17, t+21) = 59$ if $t \equiv 38 (\mod 59)$.

By Lemma 1(a) and consider the symmetry of L-shaped tile, for $t = 3 \times 47 \times 59 \times e + 9714 (e \geq 0)$, there is no $G(N;1,s)$ realizing the 3-tight L-shaped tile $L(N(t);l,h,x,y)$ where $|y-x| = 6$.

With a similar argument as Example 1, let $t = 3 \times 47 \times g_0 \times e + 9714 (e \geq 0)$, there is no k-tight$(0 \leq k \leq 2)$ L-shaped tile $L(N(t);l,h,x,y)$, and there is no k-tight$(0 \leq k \leq 3)$ realizable L-shaped tile $L(N(t);l,h,x,y)$ by $G(N(t);1,s)$.

Take $j = 4, z = 6, (-17,36) \in S$, let $l = 2t+a, h = 2t+b, x = t+a+b-4, y = x+6, \alpha = -1, \beta = 2, s_1 \equiv \alpha h + \beta y (\mod N) = 6, s_2 \equiv \alpha x + \beta l (\mod N) = 3t - 49$, then for $t = g_0 \times e + 9714 (e \geq 0), \gcd(l,h,x,y) = 1$ and $\gcd(N, s_1, s_2) = 1$. Hence, $\{G(N(t); s_1, s_2) : t = g_0 \times e + 9714, e \geq 0\}$ is an infinite family of 3-tight DLN.

As a conclusion, the nodes $N(t) = 3t^2 + 2t - 927, t = 3 \times 47 \times g_0 \times e + 9714 (e \geq 0)$, of an infinite family of 3-tight optimal DLN correspond to nus integers with $D_1(N) - D(N) \geq 1$.

5 Remarks

We continue to discuss Example 2. For $t = 1880, A = 2, B = -700, j = 4, z = 3$, the equation (1) becomes

$$(a+b-4)(a+b-1) - ab - 3t - 700 = 0,$$

which has a solution $(41, 54)$.

Let $a = 41, b = 3f + 54$. From the above equation, we get $t = 3f^2 + 144f + 1880$.

Let

$$l(f) = 2t + a = 6f^2 + 288f + 3801,$$

$$h(f) = 2t + b = 6f^2 + 291f + 3814,$$

$$x(f) = y(f) + z = 3f^2 + 147f + 1974,$$

$$y(f) = t + a + b - j = 3f^2 + 147f + 1971,$$

$$l'(f) = l(f) - x(f) = 3f^2 + 141f + 1827,$$

$$h'(f) = h(f) - y(f) = 3f^2 + 144f + 1843.$$

When $f = 3481g, \alpha(g) = 3fg + 17421g + 72, \beta(g) = -3fg - 17424g - 77$, it is easy to show that $\alpha(g)y(f) + \beta(g)h'(f) = 1$.

From Lemma 1, we know that $G(N(t); 1, s(g))$ is an infinite family of 3-tight optimal DLN, where $N(t) = 3t^2 + 2t - 700, t = 3f^2 + 144f + 1880, f = 3481g, g = 3 \times 5 \times 7 \times g_0 \times e (e \geq 0), s(g) = \alpha(g)l(f) - \beta(g)l'(f)$. Therefore, $N(t(g))$ is an infinite family of nus integers with $D_1(N) - D(N) = 1$.

When $e = 0$, then $g = 0, f = 0, s(g) = \alpha(g)l(f) - \beta(g)l'(f) = 414351$. We know that $G(10606260; 1, 414351)$ is 3-tight optimal.

In a similar way to the above discussion, we can get infinite families of nus integers with a constant $D_1(N) - D(N)$ from an initial nus integer.

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