Even and Odd Eulerian Paths

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Abstract. Improving on Domokos's improvement of Swan's theorem, we show that under certain conditions on a finite digraph, whenever p, q are vertices, then the number of even Eulerian paths from p to q is the same as the number of odd ones from p to q.

All digraphs Γ considered are finite and are allowed to have multiple edges and loops. We let $V(\Gamma)$ be the set of vertices of Γ, and let $E(\Gamma)$ be the set of its edges. For each $e \in E(\Gamma)$ there are $x, y \in V(\Gamma)$ such that e *goes from* x *to* y. If $p, q \in V(\Gamma)$ (with $p = q$ permitted), then an *Eulerian path* from p to q is a sequence $e_1, e_2, e_3, \ldots, e_n$ of edges of Γ, each edge of Γ appearing exactly once, such that e_1 goes from p, e_n goes to q, and, for $1 \leq i < n$, e_i goes to the vertex that e_{i+1} goes from. By considering a fixed given order of $E(\Gamma)$, we can view each Eulerian path as a permutation of $E(\Gamma)$, and, thus, each Eulerian path has a parity, which is either *even* or *odd*.

THEOREM: *Suppose Γ is a digraph such that there are nonempty subsets $A, B \subseteq V(\Gamma)$ with at least $|A| + |B|$ edges going from A to B. Then, for any $p, q \in V(\Gamma)$, the number of even Eulerian paths from p to q is the same as the number of odd ones.*

When $A = B = V(\Gamma)$, this theorem reduces to a theorem of Swan [3]. Swan proved his theorem in order to give a simpler and almost entirely graph-theoretic proof of the theorem of Amitsur and Levitzki [1] on polynomial identities for matrix algebras. It has been noted that Swan's theorem also follows from the Amitsur-Levitzki theorem. Domokos [2] extended Swan's theorem, using an algebraic method, to two special cases of the above Theorem: (1) when $A = B$ and (2) when $|A| = |B|$ and $A \cap B = \emptyset$. The above Theorem will be proved by reducing it to case (2) of Domokos's theorem.

Proof. We first prove the Theorem with the added condition that A and B are disjoint. Choose such A, B and a set E of at least $|A| + |B|$ edges going from A to B so as to minimize $|A| + |B| + |E|$. Then $|E| = |A| + |B|$, as otherwise an edge in E could be removed. For each vertex in A there is at least one edge in E going from it, as otherwise the offending vertex could be removed. Similarly, for each vertex in B there is at least one edge in E going to it. Also, $|A| = |B|$, for if, say, $|A| > |B|$ then there would be $a \in A$ with exactly one edge $e \in E$ going from a, and then a and e could both be removed. Thus, condition (2) of Domokos's theorem holds.

Now suppose that we have arbitrary (that is, not necessarily disjoint) $A, B \subseteq V(\Gamma)$ and a set E of at least $|A|+|B|$ edges going from A to B. Let $p, q \in V(\Gamma)$, and consider Eulerian paths from p to q.

We can assume that $p = q$. For if $p \neq q$, then just add a new vertex r and two edges, one going from r to p and the other from q to r. Then we can consider Eulerian paths from r to r in this larger graph instead of Eulerian paths from p to q in Γ. Thus, we do assume that $p = q$, and we will refer to Eulerian paths from p to p as *Eulerian cycles.*

We also can assume that Γ has at least one Eulerian cycle. Thus, for each vertex $x \in V(\Gamma)$, its outdegree and indegree are the same; let us denote this common value by $\delta(x)$. We now form a new digraph Γ' from Γ as follows. Each $x \in V(\Gamma)$ will determine $2 + \delta(x)$ vertices of $V(\Gamma')$, namely $x', x'', x_1, x_2, \ldots, x_{\delta(x)}$, and also will determine $2\delta(x)$ edges of Γ', namely $e'_{x,1}, e'_{x,2}, \ldots, e'_{x,\delta(x)}, e''_{x,1}, e''_{x,2}, \ldots, e''_{x,\delta(x)}$. Each edge e of Γ will also be an edge of Γ'. There are no other vertices and edges in Γ'. If $x \in V(\Gamma)$ and $1 \leq i \leq \delta(x)$, then the following hold: $e'_{x,i}$ goes from x' to x_i in Γ'; $e''_{x,i}$ goes from x_i to x'' in Γ'; and if e goes from x to y in Γ, then in Γ', e goes from x'' to y'. We will be considering Eulerian paths from p' to p', which we will also refer to as Eulerian cycles.

In order to refer to the parity of these Eulerian cycles, let us take as the given fixed ordering of $E(\Gamma')$ one which extends the ordering of $E(\Gamma)$ and has all pairs $e'_{x,i}, e''_{x,i}$ of new edges being adjacent in that order.

There are two observations to make about Γ'.

(1) Let $A' = \{x'' : x \in A\}$ and $B' = \{y' : y \in B\}$. Clearly, $|A'| = |A|$, $|B'| = |B|$, and A' and B' are disjoint. Then each $e \in E(\Gamma)$ that in Γ goes from $x \in A$ to $y \in B$ is also an edge of Γ' which goes from $x'' \in A'$ to $y' \in B'$. Thus, there are $|A'| + |B'|$ edges going from A' to B'. Then, by the first paragraph of this proof, the number of even Eulerian cycles in Γ' is the same as the number of odd ones.

(2) In any Eulerian cycle γ in Γ', each occurrence of an edge $e''_{x,i}$ immediately follows the edge $e'_{x,i}$. Also, if all the edges of types $e'_{x,i}$ and $e''_{x,i}$ are removed from γ, then the remaining subsequence is an Eulerian cycle in Γ having the same parity as γ. Furthermore, there is some fixed number, specifically $m = \prod(\delta(x)!)$, where the product is taken over all vertices $x \in V(\Gamma)$, such that each Eulerian cycle in Γ is a subsequence of exactly m Eulerian cycles in Γ'.

It now easily follows from (1) and (2) that Γ has the same number of even Eulerian cycles as it does odd Eulerian cycles. □

References

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[2] M. Domokos, Criteria for vanishing of Eulerian polynomials on $n \times n$ matrices, Lin. Alg. Appl. **234** (1996), 181–195.

[3] R.G. Swan, An application of graph theory to algebra, Proc. Amer. Math. Soc. **14** (1963), 367–380. (*Correction*: R.G. Swan, Correction to "An application of graph theory to algebra", Proc. Amer. Math Soc. **21** (1969), 379–380.)