λ-Designs with $g = 7$

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Abstract

A λ-design on v points is a set of v distinct subsets (blocks) of a v-set such that any two different blocks meet in exactly λ points and not all of the blocks have the same size. Ryser's and Woodall's λ-design conjecture states that every λ-design can be obtained from a symmetric design by a certain complementation procedure. A result of Ryser and Woodall establishes that there exist two integers, r and r^*, such that each point in a λ-design is in exactly r or r^* blocks. The main result of the present paper is that the λ-design conjecture is true for λ-designs with $\gcd(r-1, r^*-1) = 7$.

1 Introduction

Definition 1.1. Given integers λ and v satisfying $0 < \lambda < v$, a λ-*design* on v points is a pair $(X, \mathcal{B})$, where X is a set of cardinality v whose elements are called *points* and $\mathcal{B}$ is a set of v distinct subsets of X whose elements are called *blocks*, such that

(i) For all $A, B \in \mathcal{B}$, $A \neq B$, $|A \cap B| = \lambda$, and

(ii) There exist $A, B \in \mathcal{B}$ such that $|A| \neq |B|$.

Remark 1.2. If X is a v-set and $\mathcal{B}$ is a set of subsets of X such that any two distinct subsets intersect in exactly $\lambda \geq 1$ elements (λ fixed), then the non-uniform Fisher inequality [14] asserts that $|\mathcal{B}| \leq v$. Thus, λ-designs are *extremal* set systems in this sense. They are the non-uniform analogue of symmetric designs.

λ-designs were first defined by Ryser [15] and Woodall [22]. The only known examples of λ-designs are obtained from symmetric designs by the following complementation procedure: Let $(X, \mathcal{A})$ be a symmetric (v, k, μ)-design with $\mu \neq k/2$ and fix $A \in \mathcal{A}$. Put $\mathcal{B} = \{A\} \cup \{A \Delta B | B \in \mathcal{A}, B \neq A\}$, where Δ denotes the symmetric difference of sets. Then elementary counting shows that $(X, \mathcal{B})$ is a λ-design with $\lambda = k - \mu$. Any λ-design obtained in this manner is called a *type*-1 λ-design.

The *λ-design conjecture* of Ryser [15] and Woodall [22] states that all λ-designs are type-1. The conjecture was proven for $\lambda = 1$ by deBruijn and Erdős [4], for $\lambda = 2$ by Ryser [15], for $3 \leq \lambda \leq 9$ by Bridges and Kramer [1], [3], [12], for $\lambda = 10$ by Seress [17], and for all remaining $\lambda \leq 34$ by Weisz [20]. S. S. Shrikhande and Singhi [19] proved the conjecture for prime λ and Seress [18] proved it when λ is twice a prime. Recently, Fiala [7], [8] proved the conjecture for λ-designs with only two block sizes for all $\lambda < 150$.

Investigating the conjecture as a function of v rather than λ, Ionin and M. S. Shrikhande [10], [11] proved the conjecture for $v = p+1, 2p+1, 3p+1$, and $4p + 1$, where p is any prime. Also, Hein and Ionin [9] proved the conjecture for $v = 5p + 1$, $p \not\equiv 2$ or $8 \pmod{15}$ prime, and Fiala [5], [6] proved it for $v = 6p+1$, p prime, and $v = 8p+1$, $p \equiv 1$ or $7 \pmod 8$ prime. The conjecture has also been verified by computer for all $v \leq 85$ [23].

2 Preliminary results

Definition 2.1. Given a λ-design $(X, \mathcal{B})$ and $x \in X$, the *replication number* of x is $|\{A \in \mathcal{B} | x \in A\}|$.

Ryser [15] and Woodall [22] independently proved the following theorem concerning these replication numbers.

Theorem 2.2. *If $(X, \mathcal{B})$ is a λ-design on v points, then there exists integers $r > 1$ and $r^* > 1$, $r \neq r^*$, such that every $x \in X$ has replication number r or r^* and $r + r^* = v + 1$. In addition, r and r^* satisfy the equation*

$$\frac{1}{\lambda} + \sum_{A \in \mathcal{B}} \frac{1}{|A| - \lambda} = \frac{(v-1)^2}{(r-1)(r^*-1)}. \tag{1}$$

We have the following results regarding these two replication numbers.

Theorem 2.3. [5], [6], [9], [10], [11] *Let D be a λ-design with replication numbers r and r^* and put $g = \gcd(r-1, r^*-1)$. If $g = 1, 2, 3, 4, 5, 6$, or 8, then D is type-1.*

This led to the following results.

Theorem 2.4. [5], [6], [9], [10], [11] *All λ-designs on $v = p+1, 2p+1, 3p+1, 4p+1$, and $6p+1$ points, p any prime, are type-1. All λ-designs on $v = 5p+1$ points, $p \not\equiv 2$ or 8 $\pmod{15}$ prime, are type-1. All λ-designs on $v = 8p+1$ points, $p \equiv 1$ or 7 $\pmod 8$ prime, are type-1.*

In the present paper, we prove the following result.

Theorem 2.5. *Let D be a λ-design on v points with replication numbers r and r^*. If $\gcd(r-1, r^*-1) = 7$, then D is type-1.*

We prove Theorem 2.5 using the method of Ionin and Shrikhande from [10] and [11]. However, whereas in [10] and [11] they were able to reduce to the case of designs with at most two block sizes and in [5] and [6] they were able to reduce to the case of designs with at most three block sizes, we will have to consider designs with possibly four block sizes.

We will also need the following theorem concerning the replication numbers. It was first stated without proof in [23]. For a proof see [16].

Theorem 2.6. *A λ-design on v points with replication numbers r and r^* is type-1 if and only if $r(r-1)/(v-1)$ or $r^*(r^*-1)/(v-1)$ is an integer.*

Additionally, we will need the following results concerning the validity of the λ-design conjecture for certain values of λ.

Theorem 2.7. [1], [3], [4], [12], [15], [17], [18], [19], [20] *All λ-designs with $\lambda \leq 34$, λ prime, or λ twice prime are type-1.*

3 The Ionin-Shrikhande method

Let $D = (X, \mathcal{B})$ be λ-design on v points. Then Theorem 2.2 implies that every point of D has replication number r or r^* for some integers $r \neq r^*$. Therefore, the underlying set X of our λ-design is partitioned into two subsets, E and E^*, of points having replication numbers r and r^*, respectively. Let $|E| = e$ and $|E^*| = e^*$, so $e + e^* = v$. Also, for each $A \in \mathcal{B}$, put $\tau_A = |A \cap E|$ and $\tau_A^* = |A \cap E^*|$, so $\tau_A + \tau_A^* = |A|$. We will frequently use the trivial inequalities $0 \leq \tau_A \leq e$ for all A.

The following simple relation among these parameters is the starting point of the Ionin-Shrikhande method developed in [10].

Lemma 3.1. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r and r^*. Then the following equation holds for all $A \in \mathcal{B}$:*

$$(r-1)(|A| - 2\tau_A) = (v-1)(|A| - \lambda - \tau_A). \qquad (2)$$

Proof. Fixing $A \in \mathcal{B}$, we will count in two different ways all the pairs (x, B), where $x \in X$, $B \in \mathcal{B}$, $B \neq A$, and $x \in A \cap B$. This gives us the equation $\tau_A(r-1) + \tau_A^*(r^*-1) = \lambda(v-1)$, which is easily transformed into equation (2). $\square$

Now, let $g = \gcd(r-1, r^*-1)$. Then, since $(r-1)+(r^*-1) = v-1$ by Theorem 2.2, we also have $g = \gcd(r-1, v-1) = \gcd(r^*-1, v-1)$. We put

$$q = \frac{v-1}{g}. \tag{3}$$

Then, since $\gcd((r-1)/g, q) = 1$, equation (2) implies that q divides $|A| - 2\tau_A$. Therefore, for each $A \in \mathcal{B}$ we define an integer σ_A by

$$q\sigma_A = |A| - 2\tau_A. \tag{4}$$

Next, we define

$$s = \sum_{A \in \mathcal{B}} \sigma_A. \tag{5}$$

Also, equations (2) and (4) imply that

$$\tau_A = \lambda - \frac{r^*-1}{g}\sigma_A \tag{6}$$

and

$$\tau_A^* = \lambda + \frac{r-1}{g}\sigma_A \tag{7}$$

for all A. Adding equations (6) and (7) we obtain

$$|A| = 2\lambda + \frac{r-r^*}{g}\sigma_A \tag{8}$$

for all A.

Remark 3.2. Note that equation (8) implies that for any $A, B \in \mathcal{B}$, $|A| = |B|$ if and only if $\sigma_A = \sigma_B$.

The next two equations are easily verified:

$$\sum_{A \in \mathcal{B}} |A| = er + e^*r^* \tag{9}$$

and

$$\sum_{A \in \mathcal{B}} \tau_A = er. \tag{10}$$

Equations (4), (9), and (10) then imply that $sq = \sum_{A \in \mathcal{B}}(|A| - 2\tau_A) = e^*r^* - er = (v-e)(v-r+1) - er$, which can be transformed into

$$sq = gq(gq - e - r + 3) - (2e + r - 2). \tag{11}$$

Equation (11) then implies that q divides $2e + r - 2$. Therefore, we define integers m and m^* by

$$qm = 2e + r - 2 \tag{12}$$

and

$$qm^* = 2e^* + r^* - 2. \tag{13}$$

Adding equations (12) and (13), we obtain

$$m + m^* = 3g. \tag{14}$$

Then equations (11), (12), and (14) imply that

$$s = g^2 q - g(e + r) + 3g - m. \tag{15}$$

Remark 3.3. Upon further manipulation of the above equations, we eventually arrive at

$$(r - r^*)(m^* - m) = g[v - (4\lambda - 1)]. \tag{16}$$

Note that equation (16) and the fact that $r \neq r^*$ imply that $v = 4\lambda - 1$ if and only if $m = m^*$.

The next lemma establishes formulas for e and r in terms of the parameters λ, g, q, and m. They follow easily from equations (12) and (16).

Lemma 3.4. [10] *If $v \neq 4\lambda - 1$, then*

$$e = \frac{g\lambda - (g - m)^2 q + g - m}{3g - 2m}$$

and

$$r = \frac{(2g - m)(gq + 2) - 2g\lambda}{3g - 2m}.$$

Our last result gives a way of constructing new λ-designs from old by *complementing* with respect to a fixed block. For a proof see [10].

Remark 3.5. In what follows, if we complement with respect to a block A, the parameters of the new design will be denoted by $\lambda(A)$, $r(A)$, $m(A)$, etc.

Lemma 3.6. *Let $D = (X, \mathcal{B})$ be a λ-design on v points with replication numbers r and r^*. Let $A \in \mathcal{B}$. Put*

$$\mathcal{B}(A) = \{A\} \cup \{A \Delta B | B \in \mathcal{B}, B \neq A\}.$$

Denote by $D(A)$ the complemented set system $(X, \mathcal{B}(A))$. Then we have

(i) *If* $A = E$ *or* E^*, *then* $D(A)$ *is a symmetric* $(v, |A|, |A| - \lambda)$-*design,*

(ii) *If* $A \neq E$ *and* $A \neq E^*$, *then* $D(A)$ *is a* $\lambda(A)$-*design on* v *points with* $r(A) = r$, $r^*(A) = r^*$, *and* $m(A) = m + 2\sigma_A$, *where* $\lambda(A) = |A| - \lambda$,

(iii) *If* $A \neq E$, $A \neq E^*$, *and* D *is type-*1, *then* $D(A)$ *is also type-*1, *and*

(iv) $(D(A))(A) = D$.

4 λ-designs with $g = 7$

We are now in a position to prove our main result, Theorem 2.5. In what follows, the computer program Mathematica [21] was used extensively to carry out computations.

Theorem 4.1. *Let* $D = (X, \mathcal{B})$ *be a* λ-*design on* v *points with replication numbers* r *and* r^*. *If* $g = \gcd(r-1, r^*-1) = 7$, *then* D *is type-*1.

Proof. If $\lambda \leq 34$, then Theorem 2.7 implies that D is type-1. Therefore, we shall assume that $\lambda \geq 35$. By equation (3), we may write $v = 7q + 1$. For each $i \in \mathbb{Z}$, let $a_i = |\{A \in \mathcal{B} | \sigma_A = i\}|$. We clearly have

$$\sum_{i \in \mathbf{Z}} a_i = 7q + 1. \tag{17}$$

Now, since 7 is odd, equation (14) implies that $m = m^*$ is impossible. Therefore, by equation (16), $v = 4\lambda - 1$ is also impossible. So, equations (5), (15), and (16) and the formulas of Lemma 3.4 imply that

$$\sum_{i \in \mathbf{Z}} i a_i = \frac{(7q+2)(m^2 - 21m + 98) + 49\lambda}{21 - 2m}. \tag{18}$$

Next, equation (4) implies that $|A| = 2\tau_A + q\sigma_A$ for all A. Using this and the formulas of Lemma 3.4, equation (1) is transformed into

$$\sum_{i \in \mathbf{Z}} \frac{(2m-21)a_i}{\lambda(2m-21) + i(4\lambda - 7q - 2)} = \frac{(2m-21)^2 q^2}{[q(m-7) - 2\lambda + 1][q(m-14) + 2\lambda - 1]} - \frac{1}{\lambda} \tag{19}$$

Also, equation (14) implies that $m + m^* = 21$. Without loss of generality, we may assume that $m \leq m^*$. Therefore, $m \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Case 1: $m = 1$.
In this case, Lemma 3.4 implies that $e = (7\lambda - 36q + 6)/19$, $r = (91q - 14\lambda + 26)/19$, and $r^* = 2(21q + 7\lambda + 6)/19$. Also, equation (6) implies that $\tau_A = \lambda - [(6q + 2\lambda - 1)/19]\sigma_A$ for all A. Then the inequalities $0 \leq \tau_A \leq e$

imply that $6 \leq \sigma_A \leq 19\lambda/(6q+2\lambda-1)$ for all A. Now, 7 divides $r-1$ and $r > 1$, so $r \geq 8$. This gives us $q \geq (2\lambda+18)/13$. Combining the last two inequalities, we obtain that $\sigma_A = 6$ for all A. Therefore, by Remark 3.2, all blocks have the same cardinality, a contradiction.

Case 2: $m = 2$.

In this case, Lemma 3.4 implies that $e = (7\lambda - 25q + 5)/17$, $r = 2(42q - 7\lambda + 12)/17$, and $r^* = (35q + 14\lambda + 10)/17$. Also, equation (6) implies that $\tau_A = \lambda - [(5q + 2\lambda - 1)/17]\sigma_A$ for all A. Then the inequalities $0 \leq \tau_A \leq e$ imply that $5 \leq \sigma_A \leq 17\lambda/(5q + 2\lambda - 1)$ for all A. Next, $r \geq 8$ gives us $q \geq (\lambda + 8)/6$. Combining the last two inequalities, we obtain that $\sigma_A = 5$ for all A, a contradiction.

Case 3: $m = 3$.

In this case, Lemma 3.4 implies that $e = (7\lambda - 16q + 4)/15$, $r = (77q - 14\lambda + 22)/15$, and $r^* = 2(14q + 7\lambda + 4)/15$. Also, equation (6) implies that $\tau_A = \lambda - [(4q + 2\lambda - 1)/15]\sigma_A$ for all A. Then the inequalities $0 \leq \tau_A \leq e$ imply that $4 \leq \sigma_A \leq 15\lambda/(4q + 2\lambda - 1)$ for all A. Next, $r \geq 8$ gives us $q \geq (2\lambda + 14)/11$. Combining the last two inequalities, we obtain that $\sigma_A = 4$ or 5 for all A. Hence, $a_i = 0$ for all i except possibly 4 and 5. Then equations (17), (18), and (19) become

$$a_4 + a_5 = 7q + 1, \tag{20}$$

$$4a_4 + 5a_5 = \frac{(308q + 49\lambda + 88)}{15}, \tag{21}$$

and

$$\sum_{i=4}^{5} \frac{15a_i}{15\lambda - i(4\lambda - 7q - 2)} = \frac{225q^2}{(1 - 4q - 2\lambda)(2\lambda - 11q - 1)} - \frac{1}{\lambda}. \tag{22}$$

Solving equations (20) and (21) yields

$$a_4 = \frac{217q - 49\lambda - 13}{15}$$

and

$$a_5 = \frac{7(7\lambda - 16q + 4)}{15}.$$

Inserting the above expressions for a_4 and a_5 into equation (22) and manipulating the result, we arrive at

$$(4\lambda - 7q - 2)^2(7\lambda - 16q + 4)(31\lambda q - 55q - 7\lambda^2 + 6\lambda - 5) = 0.$$

Now, $e > 0$ so $7\lambda - 16q + 4 \neq 0$. Also, $4\lambda - 7q - 2 \neq 0$ since $v \neq 4\lambda - 1$. Therefore, we obtain that

$$31\lambda q - 55q - 7\lambda^2 + 6\lambda - 5 = 0$$

which can be transformed into

$$(31\lambda - 55)(31q - 7\lambda) = 199\lambda + 155.$$

Now, $199\lambda + 155 > 0$ and $31\lambda - 55 > 0$, so $31q - 7\lambda \geq 1$. If $31q - 7\lambda \leq 6$, then $\lambda < 0$, a contradiction. If $31q - 7\lambda = 7$, then $r = (5q + 12)/5$ is not an integer, a contradiction. Therefore, we must have $31q - 7\lambda \geq 8$. Consequently, $8(31\lambda - 55) \leq 199\lambda + 155$, which implies that $\lambda \leq 12$, a contradiction.

Case 4: $m = 4$.
In this case, Lemma 3.4 implies that $e = (7\lambda - 9q + 3)/13$, $r = 2(35q - 7\lambda + 10)/13$, and $r^* = (21q + 14\lambda + 6)/13$. Also, equation (6) implies that $\tau_A = \lambda - [(3q + 2\lambda - 1)/13]\sigma_A$ for all A. Then the inequalities $0 \leq \tau_A \leq e$ imply that $3 \leq \sigma_A \leq 13\lambda/(3q + 2\lambda - 1)$ for all A. Also, $r \geq 8$ gives us $q \geq (\lambda + 12)/5$. Combining the last two inequalities, we obtain that $\sigma_A = 3$ or 4 for all A. Hence, $a_i = 0$ for all i except possibly 3 and 4. Then equations (17), (18), and (19) become

$$a_3 + a_4 = 7q + 1, \tag{23}$$

$$3a_3 + 4a_4 = \frac{210q + 49\lambda + 60}{13}, \tag{24}$$

and

$$\sum_{i=3}^{4} \frac{13a_i}{13\lambda - i(4\lambda - 7q - 2)} = \frac{169q^2}{(1 - 3q - 2\lambda)(2\lambda - 10q - 1)} - \frac{1}{\lambda}. \tag{25}$$

Solving equations (23) and (24) yields

$$a_3 = \frac{154q - 49\lambda - 8}{13}$$

and

$$a_4 = \frac{7(7\lambda - 9q + 3)}{13}.$$

Inserting the above expressions for a_3 and a_4 into equation (25) and manipulating the result, we arrive at

$$(4\lambda - 7q - 2)^2(7\lambda - 9q + 3)(22\lambda q - 40q - 7\lambda^2 + 5\lambda - 4) = 0.$$

Now, $e > 0$ so $7\lambda - 9q + 3 \neq 0$. Also, $4\lambda - 7q - 2 \neq 0$ since $v \neq 4\lambda - 1$. Therefore, we obtain that

$$22\lambda q - 40q - 7\lambda^2 + 5\lambda - 4 = 0$$

which can be transformed into

$$(22\lambda - 40)(22q - 7\lambda) = 170\lambda + 88.$$

Now, $170\lambda + 88 > 0$ and $22\lambda - 40 > 0$, so $22q - 7\lambda \geq 1$. If $22q - 7\lambda \leq 7$, then $\lambda < 0$, a contradiction. If $22q - 7\lambda = 8$, then $r = (26q + 36)/13$ is not an integer, a contradiction. If $22q - 7\lambda = 9$, then $r = (26q + 38)/13$ is not an integer, a contradiction. Therefore, we must have $22q - 7\lambda \geq 10$. Consequently, $10(22\lambda - 40) \leq 170\lambda + 88$, which implies that $\lambda \leq 9$, a contradiction.

Case 5: $m = 5$.

In this case, Lemma 3.4 implies that $e = (7\lambda - 4q + 2)/11$, $r = (63q - 14\lambda + 18)/11$, and $r^* = 2(7q + 7\lambda + 2)/11$. Also, equation (6) implies that $\tau_A = \lambda - [(2q + 2\lambda - 1)/11]\sigma_A$ for all A. Then the inequalities $0 \leq \tau_A \leq e$ imply that $2 \leq \sigma_A \leq 11\lambda/(2q + 2\lambda - 1)$ for all A. Also, $r \geq 8$ gives us $q \geq (2\lambda + 10)/9$. Combining the last two inequalities, we obtain that $\sigma_A = 2, 3$, or 4 for all A. Hence, $a_i = 0$ for all i except possibly 2, 3, and 4. Then equations (17), (18), and (19) become

$$a_2 + a_3 + a_4 = 7q + 1, \tag{26}$$

$$2a_2 + 3a_4 + 4a_4 = \frac{126q + 49\lambda + 36}{11}, \tag{27}$$

and

$$\sum_{i=2}^{4} \frac{11a_i}{11\lambda - i(4\lambda - 7q - 2)} = \frac{121q^2}{(1 - 2q - 2\lambda)(2\lambda - 9q - 1)} - \frac{1}{\lambda}. \tag{28}$$

Solving equations (26), (27), and (28) yields

$$a_2 = \frac{a_{23}q^3 + a_{22}q^2 + a_{21}q + a_{20}}{22\lambda(2\lambda - 9q - 1)(2\lambda + 2q - 1)},$$

where $a_{23} = -5460\lambda + 3024$, $a_{22} = 2848\lambda^2 - 4342\lambda - 312$, $a_{21} = 175\lambda^3 + 889\lambda^2 - 1386\lambda - 504$, and $a_{20} = -147\lambda^4 - 94\lambda^3 + 103\lambda^2 - 80\lambda - 48$,

$$a_3 = \frac{a_{33}q^3 + a_{32}q^2 + a_{31}q + a_{30}}{11\lambda(2\lambda - 9q - 1)(2\lambda + 2q - 1)},$$

where $a_{33} = 2184\lambda - 3024$, $a_{32} = -4514\lambda^2 + 5472\lambda + 312$, $a_{31} = 1239\lambda^3 - 2072\lambda^2 + 1624\lambda + 504$, and $a_{30} = -49\lambda^4 + 322\lambda^3 - 184\lambda^2 + 88\lambda + 48$, and

$$a_4 = \frac{a_{43}q^3 + a_{42}q^2 + a_{41}q + a_{40}}{22\lambda(2\lambda - 9q - 1)(2\lambda + 2q - 1)},$$

where $a_{43} = -1680\lambda + 3024$, $a_{42} = 4024\lambda^2 - 5920\lambda - 312$, $a_{41} = -2037\lambda^3 + 2331\lambda^2 - 1554\lambda - 504$, and $a_{40} = 254\lambda^4 - 462\lambda^3 + 177\lambda^2 - 74\lambda - 48$.

Replacing q by a real variable x in the above expressions for a_2, a_3, and a_4, we obtain three functions, $a_2(x)$, $a_3(x)$, and $a_4(x)$. Now, we already know $q \geq (2\lambda + 10)/9$ and the inequality $e \geq 1$ implies that $q \leq (7\lambda - 9)/4$. This implies $(2\lambda - 9q - 1)(2\lambda + 2q - 1) < 0$. Therefore, $a_2(x)$, $a_3(x)$, and $a_4(x)$ are continuous functions of x on the interval $[(2\lambda + 10)/9, (7\lambda - 9)/4]$.

Now, $a_3(x)$ has zeros only at $(\lambda - 6)/21$, $(7\lambda + 2)/4$, and

$$z_{31} = \frac{7\lambda^2 - 6\lambda + 4}{2(13\lambda - 18)}.$$

Clearly, $(\lambda - 6)/21, (7\lambda + 2)/4 \notin [(2\lambda + 10)/9, (7\lambda - 9)/4]$, so $a_3(x)$ has at most one zero on this interval. However,

$$a_3(\frac{2\lambda + 10}{9}) = \frac{-5\lambda^4 + 72\lambda^3 + 672\lambda^2 - 1720\lambda + 576}{27\lambda(2\lambda + 1)} < 0$$

and

$$a_3(\frac{7\lambda - 9}{4}) = \frac{7(13\lambda^2 - 41\lambda + 30)}{5\lambda - 7} > 0.$$

Therefore, $a_3(x)$ has exactly one zero on $[(2\lambda+10)/9, (7\lambda-9)/4]$ at z_{31}. The above inequalities then imply $a_3(x) < 0$ on the interval $[(2\lambda + 10)/9, z_{31})$. Hence, we must have $q \in [z_{31}, (7\lambda - 9)/4]$.

Next, $a_4(x)$ has zeros only at $(5\lambda - 8)/28$, $(7\lambda + 2)/4$, and

$$z_{41} = \frac{7\lambda^2 - 4\lambda + 3}{3(5\lambda - 9)}.$$

Clearly, $(5\lambda - 8)/28, (7\lambda + 2)/4 \notin [z_{31}, (7\lambda - 9)/4]$, so $a_4(x)$ has at most one zero on this interval. However,

$$a_4(z_{31}) = \frac{7(7\lambda^2 - 8\lambda - 4)}{2(13\lambda - 18)} > 0$$

and

$$a_4(\frac{7\lambda - 9}{4}) = \frac{-7(\lambda - 3)(4\lambda - 5)}{\lambda(5\lambda - 7)} < 0.$$

Therefore, $a_4(x)$ has exactly one zero on the interval $[z_{31}, (7\lambda - 9)/4]$ at z_{41}. The above inequalities then imply $a_4(x) < 0$ on the interval $(z_{41}, (7\lambda - 4)/9]$. Hence, we must have $q \in [z_{31}, z_{41}]$.

Now, $a_2(x)$ has zeros only at $-(3\lambda + 4)/14$ and

$$z_{21}, z_{22} = \frac{287\lambda^2 - 245\lambda - 84 \mp 11\sqrt{49\lambda^4 - 374\lambda^3 - 287\lambda^2 + 264\lambda + 144}}{12(65\lambda - 36)}.$$

Clearly, $-(3\lambda+4)/14 \notin [z_{31}, z_{41}]$, so $a_2(x)$ has at most two zeros on this interval. However,

$$a_2(z_{31}) = \frac{10(2\lambda+1)}{13\lambda-18} > 0,$$

$$a_2(\frac{2\lambda}{5}) = \frac{-3655\lambda^4 + 29948\lambda^3 + 62665\lambda^2 + 35200\lambda + 6000}{110\lambda(112\lambda^2 + 30\lambda - 25)} < 0,$$

and

$$a_2(z_{41}) = \frac{2(13\lambda+6)}{5\lambda-9} > 0,$$

where $z_{31} < 2\lambda/5 < z_{41}$. Therefore, $a_2(x)$ has exactly two zeros on the interval $[z_{31}, z_{41}]$. The above inequalities then imply $a_2(x) < 0$ on the interval (z_{21}, z_{22}). Hence, we must have $q \in [z_{31}, z_{21}] \cup [z_{22}, z_{41}]$.

Next, we easily obtain the inequalities $z_{31} > (7\lambda+3)/26$ and $z_{41} < (7\lambda+11)/15$. Also, $49\lambda^4 - 374\lambda^3 - 287\lambda^2 + 264\lambda + 144 > (7\lambda^2 - 27\lambda - 98)^2$, which implies $z_{21} < (210\lambda^2 + 52\lambda + 994)/[12(65\lambda - 36)]$ and $z_{22} > (364\lambda^2 - 542\lambda - 1162)/[12(65\lambda - 36)]$. This implies $z_{21} < (7\lambda+9)/26$ and $z_{22} > (7\lambda-9)/15$. Therefore, we have $q \in ((7\lambda+3)/26, (7\lambda+9)/26) \cup ((7\lambda-9)/15, (7\lambda+11)/15)$. Thus, since q is an integer, we must have $q \in \{(7\lambda+j)/26 | 4 \le j \le 8\} \cup \{(7\lambda+k)/15 | -8 \le k \le 10\}$. If $q \in \{(7\lambda+j)/26 | 4 \le j \le 8\}$, then $r = (11q + 2j + 18)/11$ for $j = 4, 5, 6, 7$, or 8. But, clearly r is not an integer for any of these values. Therefore, $q \in \{(7\lambda+k)/15 | -8 \le k \le 10\}$.

Then $r = (33q + 2k + 18)/11$ for some integer $-8 \le k \le 10$. But, clearly r is an integer only for $k = 2$. So, we must have $q = (7\lambda+2)/15$. Then

$$a_4(q) = \frac{-3375\lambda^4 + 102487\lambda^3 + 280841\lambda^2 + 220946\lambda + 42592}{13310\lambda(\lambda+1)(4\lambda-1)} < 0$$

for all $\lambda \ge 33$. Therefore, we must have $\lambda \le 32$, a contradiction.

Case 6: $m = 7$.

In this case, Lemma 3.4 implies that $e = \lambda$, $r = 7q - 2\lambda + 2$, and $r^* = 2\lambda$. Also, equation (6) implies that $\tau_A = \lambda - [(2\lambda-1)/7]\sigma_A$ for all A. Then the inequalities $0 \le \tau_A \le e$ imply that $0 \le \sigma_A \le 7\lambda/(2\lambda-1)$ for all A. This implies that $\sigma_A = 0, 1, 2$, or 3 for all A. Hence, $a_i = 0$ for all i except possibly 0, 1, 2, and 3. Then equations (17), (18), and (19) become

$$a_0 + a_1 + a_2 + a_3 = 7q + 1, \tag{29}$$

$$a_1 + 2a_2 + 3a_3 = 7\lambda, \tag{30}$$

and

$$\sum_{i=0}^{3} \frac{7a_i}{7\lambda - i(4\lambda - 7q - 2)} = \frac{49q^2}{(1-2\lambda)(2\lambda - 7q - 1)} - \frac{1}{\lambda}. \tag{31}$$

Subcase 6a: $a_0 = 0$.

Solving equations (29), (30), and (31) yields

$$a_1 = \frac{a_{13}q^3 + a_{12}q^2 + a_{11}q + a_{10}}{14\lambda(2\lambda-1)(2\lambda-7q-1)},$$

where $a_{13} = -2058\lambda$, $a_{12} = 833\lambda^2 - 1715\lambda - 294$, $a_{11} = 392\lambda^3 + 329\lambda^2 - 525\lambda - 126$, and $a_{10} = -147\lambda^4 + 40\lambda^3 + 59\lambda^2 - 40\lambda - 12$,

$$a_2 = \frac{a_{23}q^3 + a_{22}q^2 + a_{21}q + a_{20}}{7\lambda(2\lambda-1)(2\lambda-7q-1)},$$

where $a_{23} = 2058\lambda$, $a_{22} = -2891\lambda^2 + 2744\lambda + 294$, $a_{21} = 882\lambda^3 - 1554\lambda^2 + 819\lambda + 126$, and $a_{20} = -49\lambda^4 + 240\lambda^3 - 192\lambda^2 + 61\lambda + 12$, and

$$a_3 = \frac{a_{33}q^3 + a_{32}q^2 + a_{31}q + a_{30}}{14\lambda(2\lambda-1)(2\lambda-7q-1)},$$

where $a_{33} = -2058\lambda$, $a_{32} = 3577\lambda^2 - 3087\lambda - 294$, $a_{31} = -1764\lambda^3 + 2191\lambda^2 - 917\lambda - 126$, and $a_{30} = 245\lambda^4 - 464\lambda^3 + 269\lambda^2 - 68\lambda - 12$.

Replacing q by a variable x in the above expressions for a_1, a_2, and a_3, we obtain three functions, $a_1(x)$, $a_2(x)$, and $a_3(x)$. Now, the inequality $r \geq 8$ implies $q \geq (2\lambda+6)/7$. This implies $(2\lambda-1)(2\lambda-7q-1) < 0$. Therefore, $a_1(x)$, $a_2(x)$, and $a_3(x)$ are continuous functions on the interval $[(2\lambda+6)/7, \infty)$.

Now, $a_2(x)$ has zeros only at $(\lambda-4)/14$ and

$$z_{21}, z_{22} = \frac{28\lambda^2 - 22\lambda - 3 \mp \sqrt{196\lambda^4 - 704\lambda^3 + 124\lambda^2 + 9\lambda + 9}}{42\lambda}.$$

Clearly, $(\lambda-4)/14 \notin [(2\lambda+6)/7, \infty)$, so $a_2(x)$ has at most two zeros on this interval. However,

$$a_2\left(\frac{2\lambda+6}{7}\right) = \frac{-15\lambda^4 + 208\lambda^3 + 750\lambda^2 - 4255\lambda - 336}{49\lambda(2\lambda-1)} < 0,$$

$$a_2\left(\frac{\lambda}{2}\right) = \frac{49\lambda^3 - 132\lambda^2 - 106\lambda - 12}{7\lambda(2\lambda-1)} > 0,$$

and

$$a_2(\lambda) = \frac{-(286\lambda^2 + 127\lambda + 12)}{7\lambda(2\lambda-1)} < 0,$$

where $(2\lambda+6)/7 < \lambda/2 < \lambda$. Therefore, $a_2(x)$ has exactly two zeros on the interval $[(2\lambda+6)/7, \infty)$ at z_{21} and z_{22}. The above inequalities then imply $a_2(x) < 0$ on $[(2\lambda+6)/7, z_{21}) \cup (z_{22}, \infty)$. Hence, we must have $q \in [z_{21}, z_{22}]$.

Next, $a_3(x)$ has zeros only at $(5\lambda-6)/21$ and

$$z_{31}, z_{32} = \frac{21\lambda^2 - 17\lambda - 2 \mp \sqrt{49\lambda^4 - 442\lambda^3 + 101\lambda^2 + 52\lambda + 4}}{28\lambda}.$$

Clearly, $(5\lambda-6)/21 \notin [z_{21}, z_{22}]$, so $a_3(x)$ has at most two zeros on this interval. However,

$$a_3(z_{21}) = \frac{(18\lambda^2 - 10\lambda - 3 - \sqrt{b_2})[14\lambda^3 + 38\lambda^2 - 13\lambda - 3 + (\lambda - 1)\sqrt{b_2}]}{12\lambda(2\lambda - 1)(16\lambda^2 - 16\lambda - 3 - \sqrt{b_2})} > 0,$$

$$a_3(\frac{3\lambda}{5}) = \frac{-(19\lambda + 15)(98\lambda^3 - 935\lambda^2 - 535\lambda - 50)}{175\lambda(2\lambda - 1)(11\lambda + 5)} < 0,$$

and

$$a_3(z_{22}) = \frac{(18\lambda^2 - 10\lambda - 3 + \sqrt{b_2})[14\lambda^3 + 38\lambda^2 - 13\lambda - 3 - (\lambda - 1)\sqrt{b_2}]}{12\lambda(2\lambda - 1)(16\lambda^2 - 16\lambda - 3 + \sqrt{b_2})} > 0,$$

where $z_{21} < 3\lambda/5 < z_{22}$ and $b_2 = 196\lambda^4 - 704\lambda^3 + 124\lambda^2 + 96\lambda + 9$. Therefore, $a_3(x)$ has exactly two zeros on the interval $[z_{21}, z_{22}]$ at z_{31} and z_{32}. The above inequalities then imply $a_3(x) < 0$ on the interval (z_{31}, z_{32}). Hence, we must have $q \in [z_{21}, z_{31}] \cup [z_{32}, z_{22}]$.

Suppose first that $q \in [z_{32}, z_{22}]$. Now, we easily obtain the inequalities $49\lambda^4 - 442\lambda^3 + 101\lambda^2 + 52\lambda + 4 > (7\lambda^2 - 32\lambda - 94)^2$ and $196\lambda^4 - 704\lambda^3 + 124\lambda^2 + 96\lambda + 9 < (14\lambda^2 - 25\lambda - 22)^2$. These imply that $z_{32} > (7\lambda - 14)/7$ and $z_{22} < (7\lambda - 7)/7$. Therefore, we have $q \in ((7\lambda - 14)/7, (7\lambda - 7)/7)$. However, there are clearly no integers in this interval, a contradiction. Therefore, we must have $q \in [z_{21}, z_{31}]$.

Now, $a_1(x)$ has zeros only at $-(3\lambda + 2)/7$ and

$$z_{11}, z_{12} = \frac{35\lambda^2 - 23\lambda - 6 \mp 5\sqrt{49\lambda^4 - 506\lambda^3 - 155\lambda^2 + 132\lambda + 36}}{84\lambda}.$$

Clearly, $-(3\lambda + 2)/7 \notin [z_{21}, z_{31}]$, so $a_1(x)$ has at most two zeros on this interval. However,

$$a_1(z_{21}) > 0, \quad a_1(\tfrac{2\lambda}{5}) < 0, \quad \text{and} \quad a_1(z_{31}) > 0,$$

where $z_{21} < 2\lambda/5 < z_{31}$. Therefore, $a_1(x)$ has exactly two zeros on the interval $[z_{21}, z_{31}]$. The above inequalities then imply $a_1(x) < 0$ on the interval (z_{11}, z_{12}). Hence, we must have $q \in [z_{21}, z_{11}] \cup [z_{12}, z_{31}]$.

Next, we easily obtain the inequalities $z_{21} > \lambda/3$ and $z_{31} < (\lambda + 2)/2$. Also, $49\lambda^4 - 506\lambda^3 - 155\lambda^2 + 132\lambda + 36 > (7\lambda^2 - 37\lambda - 172)^2$, which implies $z_{11} < (\lambda + 1)/3$ and $z_{12} > (\lambda - 2)/2$. Therefore, we have $q \in (\lambda/3, (\lambda + 1)/3) \cup ((\lambda - 2)/2, (\lambda + 2)/2)$. But, there are no integers in the interval $(\lambda/3, (\lambda+1)/3)$. Thus, we must have $q = (\lambda-1)/2$, $q = \lambda/2$, or $q = (\lambda+1)/2$. If $q = (\lambda-1)/2$, then we have $r = 3(\lambda-1)/2$, so 2 divides $\lambda - 1$. Therefore, 14 divides $7\lambda - 7$. Also, 7 divides $r^* - 1 = 2\lambda - 1$, which implies that 14 divides $4\lambda - 2$. Thus, 14 divides $7\lambda - 7 - (4\lambda - 2) = 3\lambda - 5$. But, then $r(r-1)/(v-1) = 3(3\lambda-5)/14$ is an integer and D is type-1 by Theorem 2.6. If $q = \lambda/2$, then $r^*(r^*-1)/(v-1) = 4(2\lambda-1)/7$, which is an integer since 7 divides $2\lambda - 1$. Therefore, D is type-1 by Theorem 2.6. If $q = (\lambda+1)/2$, then we have

$$a_3(q) = \frac{11(\lambda^2 + 104\lambda + 9)}{42\lambda(2\lambda - 1)} < 1$$

for $\lambda \geq 17$. So, since a_3 is a non-negative integer and $\lambda \geq 35$, we must have $a_3 = 0$. This means that $(\lambda+1)/2$ is a zero of $a_3(x)$. Now, clearly $(\lambda+1)/2 > (5\lambda-6)/21$. Also, the inequality $49\lambda^4 - 442\lambda^3 + 101\lambda^2 + 52\lambda + 4 < (7\lambda^2 - 31\lambda - 92)^2$ implies that $z_{31} > (\lambda+1)/2$, a contradiction.

Subcase 6b: $a_0 = 1$.

Solving equations (29), (30), and (31) yields

$$a_1 = \frac{a_{13}q^3 + a_{12}q^2 + a_{11}q + a_{10}}{14\lambda(2\lambda-1)(2\lambda-7q-1)},$$

where $a_{13} = -2058\lambda$, $a_{12} = 833\lambda^2 - 1127\lambda - 588$, $a_{11} = 392\lambda^3 + 413\lambda^2 - 315\lambda - 252$, and $a_{10} = -147\lambda^4 - 32\lambda^3 + 83\lambda^2 - 10\lambda - 24$,

$$a_2 = \frac{a_{23}q^3 + a_{22}q^2 + a_{21}q + a_{20}}{7\lambda(2\lambda-1)(2\lambda-7q-1)},$$

where $a_{23} = 2058\lambda$, $a_{22} = -2891\lambda^2 + 2156\lambda + 588$, $a_{21} = 882\lambda^3 - 1344\lambda^2 + 462\lambda + 252$, and $a_{20} = -49\lambda^4 + 228\lambda^3 - 132\lambda^2 + 10\lambda + 24$, and

$$a_3 = \frac{a_{33}q^3 + a_{32}q^2 + a_{31}q + a_{30}}{14\lambda(2\lambda-1)(2\lambda-7q-1)},$$

where $a_{33} = -2058\lambda$, $a_{32} = 3577\lambda^2 - 2499\lambda - 588$, $a_{31} = -1764\lambda^3 + 1883\lambda^2 - 511\lambda - 252$, and $a_{30} = 245\lambda^4 - 424\lambda^3 + 181\lambda^2 - 10\lambda - 24$.

Replacing q by a variable x in the above expressions for a_1, a_2, and a_3, we obtain three continuous functions of x on the interval $[(2\lambda+6)/7, \infty)$.

Now, $a_2(x)$ has zeros only at $(\lambda-4)/14$ and

$$z_{21}, z_{22} = \frac{14\lambda^2 - 8\lambda - 3 \mp \sqrt{49\lambda^4 - 128\lambda^3 - 32\lambda^2 + 30\lambda + 9}}{21\lambda}.$$

Clearly, $(\lambda-4)/14 \notin [(2\lambda+6)/7, \infty)$, so $a_2(x)$ has at most two zeros on this interval. However,

$$a_2(\tfrac{2\lambda+6}{7}) < 0, \quad a_2(\tfrac{\lambda}{2}) > 0, \quad \text{and} \quad a_2(\lambda) < 0,$$

where $(2\lambda+6)/7 < \lambda/2 < \lambda$. Therefore, $a_2(x)$ has exactly two zeros on the interval $[(2\lambda+6)/7, \infty)$ at z_{21} and z_{22}. The above inequalities then imply $a_2(x) < 0$ on $[(2\lambda+6)/7, z_{21}) \cup (z_{22}, \infty)$. Hence, we must have $q \in [z_{21}, z_{22}]$.

Next, $a_3(x)$ has zeros only at $(5\lambda-6)/21$ and

$$z_{31}, z_{32} = \frac{21\lambda^2 - 13\lambda - 2 \mp \sqrt{49\lambda^4 - 338\lambda^3 - 39\lambda^2 + 72\lambda + 16}}{28\lambda}.$$

Clearly, $(5\lambda-6)/21 \notin [z_{21}, z_{22}]$, so $a_3(x)$ has at most two zeros on this interval. However,

$$a_3(z_{21}) > 0, \quad a_3(\tfrac{3\lambda}{5}) < 0, \quad \text{and} \quad a_3(z_{22}) > 0,$$

where $z_{21} < 3\lambda/5 < z_{22}$. Therefore, $a_3(x)$ has exactly two zeros on the interval $[z_{21}, z_{22}]$ at z_{31} and z_{32}. The above inequalities then imply $a_3(x) < 0$ on the interval (z_{31}, z_{32}). Hence, we must have $q \in [z_{21}, z_{31}] \cup [z_{32}, z_{22}]$.

Suppose first that $q \in [z_{32}, z_{22}]$. Now, we easily obtain the inequalities $49\lambda^4 - 338\lambda^3 - 39\lambda^2 + 72\lambda + 16 > (7\lambda^2 - 25\lambda - 41)^2$ and $49\lambda^4 - 128\lambda^3 - 32\lambda^2 + 30\lambda + 9 < (7\lambda^2 - 9\lambda - 11)^2$. These imply that $z_{32} > \lambda - 2$ and $z_{22} < \lambda$. Therefore, we have $q \in (\lambda - 2, \lambda)$. So we must have $q = \lambda - 1$. Then $r - 1 = 5\lambda - 6$, so 7 divides $5\lambda - 6$. But, then $r(r-1)/(v-1) = 5(5\lambda-6)/7$ is an integer and D is type-1 by Theorem 2.6. Therefore, we may assume $q \in [z_{21}, z_{31}]$.

Now, $a_1(x)$ has zeros only at $-(3\lambda+2)/7$ and

$$z_{11}, z_{12} = \frac{35\lambda^2 - 11\lambda - 12 \mp \sqrt{49\lambda^4 - 242\lambda^3 - 407\lambda^2 - 24\lambda + 144}}{84\lambda}.$$

Clearly, $-(3\lambda+2)/7 \notin [z_{21}, z_{31}]$, so $a_1(x)$ has at most two zeros on this interval. However,

$$a_1(z_{21}) > 0, \quad a_1(\tfrac{2\lambda}{5}) < 0, \quad \text{and} \quad a_1(z_{31}) > 0,$$

where $z_{21} < 2\lambda/5 < z_{31}$. Therefore, $a_1(x)$ has exactly two zeros on the interval $[z_{21}, z_{31}]$ at z_{11} and z_{12}. The above inequalities then imply $a_1(x) < 0$ on the interval (z_{11}, z_{12}). Hence, we must have $q \in [z_{21}, z_{11}] \cup [z_{12}, z_{31}]$.

Next, we easily obtain the inequalities $z_{21} > \lambda/3$ and $z_{31} < (\lambda+1)/2$. Also, $49\lambda^4 - 242\lambda^3 - 407\lambda^2 - 24\lambda + 144 > (7\lambda^2 - 18\lambda - 58)^2$, which implies $z_{11} < (\lambda+1)/3$ and $z_{12} > (\lambda-1)/2$. Therefore, we have $q \in (\lambda/3, (\lambda+1)/3) \cup ((\lambda-1)/2, (\lambda+1)/2)$. But, there are no integers in the interval $(\lambda/3, (\lambda+1)/3)$. Thus, we must have $q = \lambda/2$. But, then $r^*(r^*-1)/(v-1) = 4(2\lambda-1)/7$ is an integer since 7 divides $2\lambda - 1$. Therefore, D is type-1 by Theorem 2.6.

Subcase 6c: $a_3 = 0$.

Solving equations (29), (30), and (31) yields

$$a_0 = \frac{a_{02}q^2 + a_{01}q + a_{00}}{2(2\lambda - 1)(2\lambda - 7q - 1)},$$

where $a_{02} = -98\lambda$, $a_{01} = 147\lambda^2 - 119\lambda - 14$, and $a_{00} = -49\lambda^3 + 34\lambda^2 - 13\lambda - 2$,

$$a_1 = \frac{a_{12}q^2 + a_{11}q + a_{10}}{(2\lambda - 1)(2\lambda - 7q - 1)},$$

where $a_{12} = -98\lambda + 98$, $a_{11} = 7\lambda^2 - 14\lambda + 42$, $a_{10} = 21\lambda^3 - 2\lambda + 4$, and

$$a_2 = \frac{a_{22}q^2 + a_{21}q + a_{20}}{2(2\lambda - 1)(2\lambda - 7q - 1)},$$

where $a_{22} = 98\lambda - 98$, $a_{21} = -105\lambda^2 + 63\lambda - 42$, and $a_{20} = 7\lambda^3 - 30\lambda^2 + 9\lambda - 4$.

Replacing q by a variable x in the above expressions for a_0, a_1, and a_2, we obtain three continuous functions on the interval $[(2\lambda + 6)/7, \infty)$.

Now, $a_1(x)$ has zeros only at $-(3\lambda + 2)/7$ and

$$z_{11} = \frac{7\lambda^2 - 4\lambda + 2}{14(\lambda - 1)}.$$

Clearly, $-(3\lambda + 2)/7 \notin [(2\lambda + 6)/7, \infty)$, so $a_1(x)$ has at most one zero on this interval. However,

$$a_1(\tfrac{2\lambda+6}{7}) < 0 \quad \text{and} \quad a_1(\lambda) > 0,$$

where $(2\lambda+6)/7 < \lambda$. Therefore, $a_1(x)$ has exactly one zero on the interval $[(2\lambda+6)/7, \infty)$ at z_{11}. The above inequalities then imply $a_1(x) < 0$ on the interval $[(2\lambda + 6)/7, z_{11})$. Hence, we must have $q \in [z_{11}, \infty)$.

Next, $a_2(x)$ has zeros only at $(\lambda - 4)/14$ and

$$z_{21} = \frac{7\lambda^2 - 2\lambda + 1}{7(\lambda - 1)}.$$

Clearly, $(\lambda - 4)/14 \notin [z_{11}, \infty)$, so $a_2(x)$ has at most one zero on this interval. However,

$$a_2(z_{11}) > 0 \quad \text{and} \quad a_2(\lambda + 1) < 0,$$

where $z_{11} < \lambda + 1$. Therefore, $a_2(x)$ has exactly one zero on the interval $[z_{11}, \infty)$ at z_{21}. The above inequalities then imply $a_2(x) < 0$ on the interval (z_{21}, ∞). Hence, we must have $q \in [z_{11}, z_{21}]$.

Now, $a_0(x)$ has zeros only at

$$z_{01}, z_{02} = \frac{21\lambda^2 - 17\lambda - 2 \mp \sqrt{49\lambda^4 - 442\lambda^3 + 101\lambda^2 + 52\lambda + 4}}{28\lambda}.$$

Also, we have

$$a_0(z_{11}) > 0, \quad a_0(\tfrac{3\lambda}{5}) < 0, \quad \text{and} \quad a_0(z_{21}) > 0,$$

where $z_{11} < 3\lambda/5 < z_{21}$. Therefore, $a_1(x)$ has exactly two zeros on the interval $[z_{11}, z_{21}]$ at z_{01} and z_{02}. The above inequalities then imply $a_1(x) < 0$ on the interval (z_{01}, z_{02}). Hence, we must have $q \in [z_{11}, z_{01}] \cup [z_{02}, z_{21}]$.

Next, we easily obtain the inequalities $z_{11} > \lambda/2$ and $z_{21} < (7\lambda + 6)/7$. Also, $4\lambda^4 - 442\lambda^3 + 101\lambda^2 + 52\lambda + 4 > (7\lambda^2 - 32\lambda - 76)^2$, which implies $z_{11} < (14\lambda^2 + 15\lambda + 74)/(28\lambda)$ and $z_{12} > (28\lambda^2 - 49\lambda - 78)/(28\lambda)$. This implies $z_{11} < (\lambda + 2)/2$ and $z_{12} > (7\lambda - 14)/7$. Therefore, we have $q \in (\lambda/2, (\lambda + 2)/2) \cup ((7\lambda - 14)/7, (7\lambda + 6)/7)$. Hence, we must have $q = (\lambda + 1)/2$, $\lambda - 1$, or λ. If $q = (\lambda + 1)/2$, then we have

$$a_0(q) = \frac{\lambda^2 + 104\lambda + 9}{3(2\lambda^2 + 5\lambda - 3)} < 1.$$

So, since a_0 is a non-negative integer, we must have $a_0 = 0$. This means that $(\lambda + 1)/2$ is a zero of $a_0(x)$. But, $49\lambda^4 - 442\lambda^3 + 101\lambda^2 + 52\lambda + 4 < (7\lambda^2 - 31\lambda - 92)^2$ implies that $z_{01} > (7\lambda^2 + 7\lambda + 45)/(14\lambda)$. This implies $z_{01} > (\lambda + 1)/2$, a contradiction. If $q = \lambda - 1$, then $r - 1 = 5\lambda - 6$, so 7 divides $5\lambda - 6$. But, then $r(r-1)/(v-1) = 5(5\lambda - 6)/7$ is an integer and D is type-1 by Theorem 2.6. If $q = \lambda$, then $r^*(r^* - 1)/(v-1) = 2(2\lambda - 1)/7$ is an integer since 7 divides $2\lambda - 1$. Therefore, by Theorem 2.6, D is type-1.

Subcase 6d: $a_3 = 1$.

Solving equations (29), (30), and (31) yields

$$a_0 = \frac{a_{03}q^3 + a_{02}q^2 + a_{01}q + a_{00}}{2(2\lambda - 1)(2\lambda - 7q - 1)(5\lambda - 21q - 6)},$$

where $a_{03} = 2058\lambda$, $a_{02} = -3577\lambda^2 + 3087\lambda + 294$, $a_{01} = 1764\lambda^3 - 2387\lambda^2 + 1015\lambda + 126$, and $a_{00} = -245\lambda^4 + 520\lambda^3 - 325\lambda^2 + 82\lambda + 12$,

$$a_1 = \frac{a_{13}q^3 + a_{12}q^2 + a_{11}q + a_{10}}{2(2\lambda - 1)(2\lambda - 7q - 1)(5\lambda - 21q - 6)},$$

where $a_{13} = 2058\lambda - 2058$, $a_{12} = -637\lambda^2 + 1666\lambda - 1617$, $a_{11} = -406\lambda^3 - 112\lambda^2 + 441\lambda - 399$, and $a_{10} = 105\lambda^4 - 152\lambda^3 - 10\lambda^2 + 47\lambda - 30$, and

$$a_2 = \frac{a_{23}q^3 + a_{22}q^2 + a_{21}q + a_{20}}{2(2\lambda - 1)(2\lambda - 7q - 1)(5\lambda - 21q - 6)},$$

where $a_{23} = -2058\lambda + 2058$, $a_{22} = 2695\lambda^2 - 3577\lambda + 2058$, $a_{21} = -672\lambda^3 + 1995\lambda^2 - 1491\lambda + 588$, and $a_{20} = 35\lambda^4 - 216\lambda^3 + 345\lambda^2 - 176\lambda + 48$.

Replacing q by a variable x in the above expressions for a_0, a_1, and a_2, we obtain three continuous functions on the interval $[(2\lambda + 6)/7, \infty)$ since $5\lambda - 21q - 6 < 0$.

Now, $a_1(x)$ has zeros only at $-(3\lambda+2)/7$ and

$$z_{11}, z_{12} = \frac{31\lambda^2 - 40\lambda + 21 \mp \sqrt{121\lambda^4 + 136\lambda^3 + 22\lambda^2 - 216\lambda + 81}}{84(\lambda - 1)}.$$

Clearly, $-(3\lambda+2)/7 \notin [(2\lambda+6)/7, \infty)$. Also, $121\lambda^4+136\lambda^3+22\lambda^2-216\lambda+81 > (11\lambda^2+6\lambda+2)^2$ implies that $z_{11} < (20\lambda^2-46\lambda+19)/[84(\lambda-1)]$. This implies $z_{11} < (10\lambda - 13)/42$, which is strictly less than $(2\lambda+6)/7$. Thus, $a_1(x)$ has at most one zero on the interval $[(2\lambda+6)/7, \infty)$. However,

$$a_1(\tfrac{2\lambda+6}{7}) < 0 \quad \text{and} \quad a_1(\lambda) > 0,$$

where $(2\lambda+6)/7 < \lambda$. Therefore, $a_1(x)$ has exactly one zero on the interval $[(2\lambda+6)/7, \infty)$ at z_{12}. The above inequalities then imply $a_1(x) < 0$ on the interval $[(2\lambda+6)/7, z_{12})$. Hence, we must have $q \in [z_{12}, \infty)$.

Next, $a_2(x)$ has zeros only at $(\lambda-4)/14$ and

$$z_{21}, z_{22} = \frac{26\lambda^2 - 29\lambda + 15 \mp \sqrt{256\lambda^4 - 176\lambda^3 + 217\lambda^2 - 234\lambda + 81}}{42(\lambda - 1)}.$$

Clearly, $(\lambda-4)/14 \notin [z_{12}, \infty)$. Also, $256\lambda^4 - 176\lambda^3 + 217\lambda^2 - 234\lambda + 81 > (16\lambda^2 - 6\lambda + 15)^2$ implies that $z_{21} < \lambda(10\lambda - 23)/[42(\lambda-1)]$. This implies that $z_{21} < (10\lambda-13)/42$, which is strictly less than $(2\lambda+6)/7$. Thus, $a_2(x)$ has at most one zero on the interval $[z_{12}, \infty)$. However,

$$a_2(z_{12}) > 0 \quad \text{and} \quad a_3(\lambda+1) < 0,$$

where $z_{12} < \lambda + 1$. Therefore, $a_2(x)$ has exactly one zero on the interval $[z_{12}, \infty)$ at z_{22}. The above inequalities then imply $a_2(x) < 0$ on the interval (z_{22}, ∞). Hence, we must have $q \in [z_{12}, z_{22}]$.

Now, $a_0(x)$ has at most three zeros on the real line. Also, we have

$$a_0(\tfrac{5\lambda-7}{21}) < 0 \quad \text{and} \quad a_0(\tfrac{5\lambda-6.1}{21}) > 0.$$

Since $a_0(x)$ is continuous on the interval $[(5\lambda-7)/21, (5\lambda-6.1)/21]$, $a_0(x)$ must have a zero on this interval. But, clearly $(5\lambda-6.1)/21 < z_{12}$, so $a_0(x)$ has at most two zeros on the interval $[z_{12}, z_{22}]$. However,

$$a_0(z_{12}) > 0, \quad a_0(\tfrac{3\lambda}{5}) < 0, \quad \text{and} \quad a_0(z_{22}) > 0,$$

where $z_{12} < 3\lambda/5 < z_{22}$. Therefore, $a_0(x)$ has exactly two zeros on the interval $[z_{12}, z_{22}]$. Then the above inequalities imply $a_0(x) < 0$ between these two zeros.

Next, we easily obtain the inequality $z_{12} > (42\lambda^2-34\lambda+23)/[84(\lambda-1)]$, which implies that $z_{12} > \lambda/2$. Also, $256\lambda^4 - 176\lambda^3 + 217\lambda^2 - 234\lambda + 81 < (16\lambda^2 - 5\lambda - 4)^2$ implies $z_{22} < (42\lambda^2 - 34\lambda + 11)/[42(\lambda-1)]$. This implies that $z_{22} < \lambda + 1$. Also, we have

$$a_0(\tfrac{\lambda}{2}) > 0, \quad a_0(\tfrac{\lambda+1}{2}) < 0, \quad a_0(\lambda-2) < 0, \quad \text{and} \quad a_0(\lambda+1) > 0.$$

Therefore, we must have $q \in (\lambda/2, (\lambda+1)/2) \cup (\lambda-2, \lambda+1)$. Since q is an integer, we must have $q = \lambda - 1$ or λ. If $q = \lambda - 1$, then $r - 1 = 5\lambda - 6$, so 7 divides $5\lambda - 6$. But, then $r(r-1)/(v-1) = 5(5\lambda-6)/7$ is an integer and D is type-1 by Theorem 2.6. If $q = \lambda$, then $r^*(r^*-1)/(v-1) = 2(2\lambda-1)/7$ is an integer since 7 divides $2\lambda - 1$. Therefore, D is type-1 by Theorem 2.6.

Subcase 6e: $a_0, a_3 \geq 2$.

By hypothesis, there exist $A, B, C, D \in \mathcal{B}$ such that $\sigma_A = \sigma_B = 0$ and $\sigma_C = \sigma_D = 3$. Then we have $\tau_A = \tau_B = \lambda = e$ and $|A| = |B| = 2\lambda$. Put $A^* = A \cap E^*$ and $B^* = B \cap E^*$. Then we must have $|A^*| = |B^*| = \lambda$ and $|A^* \cap B^*| = 0$. Also, we have $\tau_C = \tau_D = \lambda - (6\lambda-3)/7 = (\lambda+3)/7$. Therefore, $|A^* \cap C| = |A^* \cap D| = \lambda - (\lambda+3)/7 = 3(2\lambda-1)/7$. Thus, we have $|A^* \cap C \cap D| \geq 2(6\lambda-3)/7 - \lambda = (5\lambda-6)/7$ and similarly $|B^* \cap C \cap D| \geq (5\lambda-6)/7$. However, then $|C \cap D| \geq 2(5\lambda-6)/7 = (10\lambda-12)/7 > \lambda$, a contradiction.

Case 7: $m = 6$.

In this case, Lemma 3.4 implies that $e = (7\lambda - 4q + 1)/9$, $r = 2(28q - 7\lambda + 8)/9$, and $r^* = (7q + 14\lambda + 2)/9$. Also, equation (6) implies that $\tau_A = \lambda - [(q+2\lambda-1)/11]\sigma_A$ for all A. Then the inequalities $0 \leq \tau_A \leq e$ imply that $1 \leq \sigma_A \leq 9\lambda/(q+2\lambda-1)$ for all A. Also, $r \geq 8$ gives us $q \geq (\lambda+4)/4$. Combining the last two inequalities, we obtain that $\sigma_A = 1, 2, 3$, or 4 for all A. Now, if $E \in \mathcal{B}$ or $E^* \in \mathcal{B}$, then D is type-1 by Lemma 3.6 (i) and (iv). Therefore, we shall henceforth assume that $E, E^* \notin \mathcal{B}$. So, if there exists a block A with $\sigma_A = 4$, then Lemma 3.6 (ii) implies that $g(A) = 7$ and $m(A) = 14$, so $m^*(A) = 7$ and $D(A)$ is type-1 by case 6. But, then D is also type-1 by Lemma 3.6 (iv). Hence, we may assume that $a_i = 0$ for all i except possibly 1, 2, and 3. Then equations (17), (18), and (19) become

$$a_1 + a_2 + a_3 = 7q + 1, \tag{32}$$

$$a_1 + 2a_2 + 3a_3 = \frac{56q + 49\lambda + 16}{9}, \tag{33}$$

and

$$\sum_{i=1}^{3} \frac{9a_i}{9\lambda - i(4\lambda - 7q - 2)} = \frac{81q^2}{(1 - q - 2\lambda)(2\lambda - 8q - 1)} - \frac{1}{\lambda}. \tag{34}$$

Solving equations (32), (33), and (34) yields

$$a_1 = \frac{a_{13}q^3 + a_{12}q^2 + a_{11}q + a_{10}}{18\lambda(2\lambda - 8q - 1)(2\lambda + q - 1)},$$

where $a_{13} = -1330\lambda + 336$, $a_{12} = 471\lambda^2 - 1365\lambda - 198$, $a_{11} = 672\lambda^3 - 231\lambda^2 - 651\lambda - 126$, and $a_{10} = -245\lambda^4 + 72\lambda^3 + 3\lambda^2 - 56\lambda - 12$,

$$a_2 = \frac{a_{23}q^3 + a_{22}q^2 + a_{21}q + a_{20}}{9\lambda(2\lambda - 8q - 1)(2\lambda + q - 1)},$$

where $a_{23} = 266\lambda - 336$, $a_{22} = -1941\lambda^2 + 2208\lambda + 198$, $a_{21} = 546\lambda^3 - 798\lambda^2 + 861\lambda + 126$, and $a_{20} = 49\lambda^4 + 168\lambda^3 - 96\lambda^2 + 67\lambda + 12$, and

$$a_3 = \frac{a_{33}q^3 + a_{32}q^2 + a_{31}q + a_{30}}{6\lambda(2\lambda - 8q - 1)(2\lambda + q - 1)},$$

where $a_{33} = -70\lambda + 112$, $a_{32} = 549\lambda^2 - 771\lambda - 66$, $a_{31} = -420\lambda^3 + 357\lambda^2 - 273\lambda - 42$, and $a_{30} = 49\lambda^4 - 112\lambda^3 + 39\lambda^2 - 20\lambda - 4$.

Replacing q by a variable x in the above expressions for a_1, a_2, and a_3, we obtain three functions, $a_1(x)$, $a_2(x)$, and $a_3(x)$. Now, we already know $q \geq (\lambda + 4)/4$ and $e \geq 1$ implies that $q \leq 7\lambda - 8$. This implies $(2\lambda - 8q - 1)(2\lambda + q - 1) < 0$. Therefore, $a_1(x)$, $a_2(x)$, and $a_3(x)$ are continuous on the interval $[(\lambda + 4)/4, 7\lambda - 8]$.

Now, $a_2(x)$ has zeros only at $-(\lambda + 4)/14$, $7\lambda + 1$, and

$$z_{21} = \frac{7\lambda^2 - 5\lambda + 3}{19\lambda - 24}.$$

Clearly, $-(\lambda + 4)/14, 7\lambda + 1 \notin [(\lambda + 4)/4, 7\lambda - 8]$, so $a_2(x)$ has at most one zero on this interval. However,

$$a_2(\frac{\lambda + 4}{4}) = \frac{-3(\lambda^3 - 4\lambda^2 - 20\lambda + 48)}{8\lambda} < 0$$

and

$$a_2(7\lambda - 8) = \frac{7(22\lambda^2 - 57\lambda + 36)}{\lambda(6\lambda - 7)} > 0.$$

Therefore, $a_2(x)$ has exactly one zero on the interval $[(\lambda + 4)/4, 7\lambda - 8]$. Then the above inequalities imply $a_2(x) < 0$ on the interval $[(\lambda + 4)/4, z_{21})$. Hence, we must have $q \in [z_{21}, 7\lambda - 8]$.

Next, $a_3(x)$ has zeros only at $(\lambda - 2)/7$, $7\lambda + 1$, and

$$z_{31} = \frac{7\lambda^2 - 3\lambda + 2}{2(5\lambda - 8)}.$$

Clearly, $(\lambda - 2)/7, 7\lambda + 1 \notin [z_{21}, 7\lambda - 8]$. Therefore, $a_3(x)$ has at most one zero on this interval. However,

$$a_3(z_{21}) = \frac{7(14\lambda^2 - 16\lambda - 3)}{2(19\lambda - 24)} > 0$$

and

$$a_3(7\lambda - 8) = \frac{-7(\lambda - 2)(8\lambda - 9)}{\lambda(6\lambda - 7)} < 0.$$

Therefore, $a_3(x)$ has exactly one zero on the interval $[z_{11}, z_{12}]$. Then the above inequalities imply $a_3(x) < 0$ on the interval $(z_{31}, 7\lambda - 8)$. Hence, we must have $q \in [z_{21}, z_{31}]$.

Now, $a_1(x)$ has zeros only at $-(5\lambda+2)/7$ and

$$z_{11}, z_{12} = \frac{203\lambda^2 - 175\lambda - 42 \mp 9\sqrt{49\lambda^4 - 442\lambda^3 - 35\lambda^2 + 156\lambda + 36}}{4(95\lambda - 24)}.$$

Clearly, $-(5\lambda+2)/7 \notin [z_{21}, z_{31}]$. Therefore, $a_1(x)$ has at most two zeros on this interval. However,

$$a_1(z_{21}) = \frac{5(16\lambda+3)}{2(19\lambda-24)} > 0,$$

$$a_1(\frac{\lambda}{2}) = \frac{-85\lambda^3 + 728\lambda^2 + 380\lambda + 48}{36\lambda(5\lambda-2)} < 0,$$

and

$$a_1(z_{31}) = \frac{3(11\lambda+2)}{5\lambda-8} > 0,$$

where $z_{21} < \lambda/2 < z_{31}$. Therefore, $a_1(x)$ has exactly two zeros on the interval $[z_{21}, z_{31}]$. Then the above inequalities imply $a_1(x) < 0$ on the interval (z_{11}, z_{12}). Hence, we must have $q \in [z_{21}, z_{11}] \cup [z_{12}, z_{31}]$.

Now, we easily obtain the inequalities $z_{21} > (7\lambda - 3)/19$ and $z_{31} < (7\lambda+10)/10$. Next, $49\lambda^4 - 442\lambda^3 - 35\lambda^2 + 156\lambda + 36 > (7\lambda^2 - 32\lambda - 109)^2$ implies $z_{11} < (140\lambda^2 + 113\lambda + 939)/[4(95\lambda - 24)]$ and $z_{12} > (266\lambda^2 - 463\lambda - 1023)/[4(95\lambda - 24)]$. This implies $z_{11} < (7\lambda+11)/19$ and $z_{12} > (7\lambda - 13)/10$. Therefore, we have $q \in ((7\lambda-3)/19, (7\lambda+11)/19) \cup ((7\lambda-13)/10, (7\lambda+10)/10)$. Thus, we must have $q \in \{(7\lambda+j)/19 | -2 \le j \le 10\} \cup \{(7\lambda+k)/10 | -12 \le k \le 9\}$.

If $q \in \{(7\lambda+j)/19 | -2 \le j \le 10\}$, then $r = (18q + 2j + 16)/9$ for some integer $-2 \le j \le 10$. However, clearly r is an integer only for $j = 1$ and 10. If $q = (7\lambda+1)/19$, then $r^* = 5(7\lambda+1)/19$ so 19 divides $7\lambda+1$. Therefore, 19 divides $19\lambda - 2(7\lambda+1) = 5\lambda - 2$. But, then $r^*(r^*-1)/(v-1) = 5(5\lambda-2)/19$ is an integer and D is type-1 by Theorem 2.6. Thus, we must have $q = (7\lambda+10)/19$. However, then

$$a_1(q) = \frac{-328\lambda^3 + 5266\lambda^2 + 4662\lambda + 396}{19\lambda(10\lambda^2 + 53\lambda - 11)} < 0$$

for $\lambda \ge 17$. So, we must have $\lambda \le 16$, a contradiction. Hence, we must have $q \in \{(7\lambda+k)/10 | -12 \le k \le 9\}$.

Then $r = (36q + 2k + 16)/9$ for some integer $-12 \le k \le 9$. However, clearly r is an integer only for $k = -8$ and 1. If $q = (7\lambda - 8)/10$, then

$r = 2(7\lambda - 8)/5$, so 5 divides $7\lambda - 8$. Therefore, 5 divides $7\lambda - 8 - 5(\lambda - 1) = 2\lambda - 3$. But, then $r(r-1)/(v-1) = 4(2\lambda - 3)/5$ is an integer and D is type-1 by Theorem 2.6. If $q = (7\lambda + 1)/10$, then $r^* = 3(7\lambda + 1)/10$, so 10 divides $7\lambda + 1$. Therefore, 10 divides $10\lambda - (7\lambda + 1) = 3\lambda - 1$. But, then $r^*(r^* - 1)/(v-1) = 3(3\lambda - 1)/10$ is an integer and D is type-1 by Theorem 2.6.

Case 8: $m = 8$.
In this case, Lemma 3.4 implies that $r = 2(21q - 7\lambda + 6)/5$ and $r^* = (14\lambda - 7q - 2)/5$. Now, if there exists a block A with $\sigma_A \leq -1$, then $m(A) \leq 6$ and D is type-1 by previous cases. If there exists a block A with $\sigma_A \geq 3$, then $m(A) \geq 14$, so $m^*(A) \leq 7$ and once again D is type-1 by previous cases. Therefore, we may assume that $\sigma_A = 0, 1$, or 2 for all A. Hence, $a_i = 0$ for all i except possibly 0, 1, and 2. Then equations (17), (18), and (19) become

$$a_0 + a_1 + a_2 = 7q + 1, \tag{35}$$

$$a_1 + 2a_2 = \frac{49\lambda - 42q - 12}{5}, \tag{36}$$

and

$$\sum_{i=0}^{2} \frac{5a_i}{5\lambda - i(4\lambda - 7q - 2)} = \frac{25q^2}{(q - 2\lambda + 1)(2\lambda - 6q - 1)} - \frac{1}{\lambda}. \tag{37}$$

Solving equations (35), (36), and (37) yields

$$a_0 = \frac{a_{02}q^2 + a_{01}q + a_{00}}{2(2\lambda - 6q - 1)(2\lambda - q - 1)},$$

where $a_{02} = -176\lambda - 12$, $a_{01} = 189\lambda^2 - 119\lambda - 14$, and $a_{00} = -49\lambda^3 + 46\lambda^2 - 13\lambda - 2$,

$$a_1 = \frac{a_{13}q^3 + a_{12}q^2 + a_{11}q + a_{10}}{5(2\lambda - 6q - 1)(2\lambda - q - 1)},$$

where $a_{13} = 672$, $a_{12} = -982\lambda + 976$, $a_{11} = 189\lambda^2 - 504\lambda + 336$, and $a_{10} = 49\lambda^3 + 54\lambda^3 - 72\lambda + 32$, and

$$a_2 = \frac{a_{23}q^3 + a_{22}q^2 + a_{21}q + a_{20}}{10(2\lambda - 6q - 1)(2\lambda - q - 1)},$$

where $a_{23} = -924$, $a_{22} = 1864\lambda - 1342$, $a_{21} = -1043\lambda^2 + 1183\lambda - 462$, and $a_{20} = 147\lambda^3 - 298\lambda^2 + 169\lambda - 44$.

Replacing q by a variable x in the above expressions for a_0, a_1, and a_2, we obtain three functions, $a_0(x)$, $a_1(x)$, and $a_2(x)$. Now, the inequalities $r \geq 8$ and $r^* \geq 8$ imply $(\lambda + 2)/3 \leq q \leq 2\lambda - 6$. This implies $(2\lambda - 6q -$

$1)(2\lambda - q - 1) < 0$. Therefore, $a_0(x)$, $a_1(x)$, and $a_2(x)$ are continuous on the interval $[(\lambda+2)/3, 2\lambda - 6]$.

Now, $a_1(x)$ has zeros only at $-(\lambda+2)/7$ and

$$z_{11}, z_{12} = \frac{77\lambda - 56 \mp 5\sqrt{49\lambda^2 - 176\lambda + 64}}{96}.$$

Clearly, $-(\lambda+2)/7 \notin [(\lambda+2)/3, 2\lambda-6]$, so $a_1(x)$ has at most two zeros on this interval. However,

$$a_1(\frac{\lambda+2}{3}) = \frac{-2(\lambda-4)^2(\lambda+2)}{3(\lambda-1)} < 0,$$

$$a_1(\frac{4\lambda}{5}) = \frac{10527\lambda^3 - 34430\lambda^2 - 24600\lambda - 4000}{25(6\lambda-5)(14\lambda+5)} > 0,$$

and

$$a_1(2\lambda - 6) = \frac{-15\lambda^3 + 184\lambda^2 - 720\lambda + 896}{2\lambda - 7} < 0,$$

where $(\lambda+2)/3 < 4\lambda/5 < 2\lambda - 6$. Therefore, $a_1(x)$ has exactly two zeros on the interval $[(\lambda+2)/3, 2\lambda-6]$ at z_{11} and z_{12}. The above inequalities then imply $a_1(x) < 0$ on $[(\lambda+2)/3, z_{11}) \cup (z_{12}, 2\lambda-6]$ Hence, we must have $q \in [z_{11}, z_{12}]$.

Next, $a_2(x)$ has zeros only at $(3\lambda-4)/14$ and

$$z_{21}, z_{22} = \frac{119\lambda - 7 \mp 5\sqrt{49\lambda^2 - 374\lambda + 121}}{132}.$$

Clearly, $(3\lambda-4)/14 \notin [z_{11}, z_{12}]$, so $a_2(x)$ has at most two zeros on this interval. However,

$$a_2(z_{11}) = \frac{49\lambda + 40 + 7\sqrt{49\lambda^2 - 176\lambda + 64}}{32} > 0,$$

$$a_2(\frac{4\lambda}{5}) = \frac{-(41\lambda+20)(99\lambda^2 - 690\lambda - 275)}{50(6\lambda-5)(14\lambda+5)} < 0,$$

and

$$a_2(z_{12}) = \frac{49\lambda + 40 - 7\sqrt{49\lambda^2 - 176\lambda + 64}}{32} > 0,$$

where $z_{11} < 4\lambda/5 < z_{12}$. Therefore, $a_2(x)$ has exactly two zeros on the interval $[z_{11}, z_{12}]$ at z_{21} and z_{22}. The above inequalities then imply $a_2(x) < 0$ on the interval (z_{21}, z_{22}). Hence, we must have $q \in [z_{11}, z_{21}] \cup [z_{22}, z_{12}]$.

Suppose first that $q \in [z_{22}, z_{12}]$. Now, we easily obtain the inequalities $49\lambda^2 - 374\lambda + 121 > (7\lambda - 32)^2$ and $49\lambda^2 - 176\lambda + 64 < (7\lambda - 12)^2$. These imply that $z_{22} > (7\lambda - 11)/6$ and $z_{12} < (7\lambda - 7)/6$. Therefore, we have $q \in ((7\lambda-11)/6, (7\lambda-7)/6)$. Thus, we must have $q = (7\lambda-10)/6$, $q = (7\lambda-9)/6$, or $q = (7\lambda-8)/6$. If $q = (7\lambda-10)/6$, then $r = (30q-8)/5$ is not an integer, a contradiction. If $q = (7\lambda-9)/6$, then $r = (30q-6)/5$ is not an integer, a contradiction. If $q = (7\lambda-8)/6$, then $r = (30q-4)/5$ is not an integer, a contradiction. Therefore, we must have $q \in [z_{11}, z_{21}]$.

Now, $a_0(x)$ has zeros only at

$$z_{01}, z_{02} = \frac{189\lambda^2 - 119\lambda - 84 \mp 5\sqrt{49\lambda^4 - 598\lambda^3 + 77\lambda^2 + 52\lambda + 4}}{8(44\lambda+3)}.$$

Also, we have

$$a_0(z_{11}) = \frac{1568\lambda^2 - 3673\lambda + 1304 - (224\lambda - 161)\sqrt{49\lambda^2 - 176\lambda + 64}}{237\lambda - 120 - 21\sqrt{49\lambda^2 - 176\lambda + 64}} > 0,$$

$$a_0(\frac{\lambda}{2}) = \frac{-3\lambda^2 + 36\lambda + 4}{2(3\lambda-2)} < 0,$$

and

$$a_0(z_{21}) = \frac{5929\lambda^2 - 16976\lambda + 4807 - (847\lambda - 427)\sqrt{49\lambda^2 - 374\lambda + 121}}{6(193\lambda - 55 - 7\sqrt{49\lambda^2 - 374\lambda + 121})} > 0,$$

where $z_{11} < \lambda/2 < z_{21}$. Therefore, $a_0(x)$ has exactly two zeros on the interval $[z_{11}, z_{21}]$. The above inequalities then imply $a_0(x) < 0$ on the interval (z_{01}, z_{02}). Hence, we must have $q \in [z_{11}, z_{01}] \cup [z_{02}, z_{21}]$.

Next, we easily obtain the inequalities $z_{11} > 7\lambda/16$ and $z_{21} < (7\lambda+7)/11$. Also, $49\lambda^4 - 598\lambda^3 + 77\lambda^2 + 52\lambda + 4 > (7\lambda^2 - 43\lambda - 263)^2$, which implies $z_{01} < (154\lambda^2 + 96\lambda + 1301)/[8(44\lambda+3)]$ and $z_{02} > (224\lambda^2 - 334\lambda - 1329)/[8(44\lambda+3)]$. This implies $z_{01} < (7\lambda+1)/16$ and $z_{02} > (7\lambda-1)/11$. Therefore, we have $q \in (7\lambda/16, (7\lambda+1)/16) \cup ((7\lambda-1)/11, (7\lambda+7)/11)$. But, there are no integers in the interval $(7\lambda/16, (7\lambda+1)/16)$. Thus, we must have $q \in \{(7\lambda+k)/11 | 0 \le k \le 6\}$.

Then we have $r = (20q + 2k + 12)/5$ for some integer $0 \le k \le 6$. However, clearly r is an integer only for $k = 4$. Therefore, we must have $q = (7\lambda+4)/11$. But, then we have

$$a_2(q) = \frac{(\lambda+21)(13\lambda+20)}{6(\lambda-1)(4\lambda+7)} < 1$$

for $\lambda \ge 27$. So, since a_2 is a non-negative integer and $\lambda \ge 35$, we must have $a_2 = 0$. This means that $(7\lambda+4)/11$ is a zero of $a_2(x)$. Clearly, $(7\lambda+4)/11 > (3\lambda-4)/14$. Also, $49\lambda^2 - 374\lambda + 121 < (7\lambda-26)^2$ implies that $z_{21} > (84\lambda+53)/132$, which is strictly greater than $(7\lambda+4)/11$, a contradiction.

Case 9: $m = 9$.

If there exists a block A with $\sigma_A \leq -1$, then $m(A) \leq 7$ and D is type-1 by previous cases. If there exists a block A with $\sigma_A \geq 2$, then $m(A) \geq 13$, so $m^*(A) \leq 8$ and once again D is type-1 by previous cases. Therefore, we may assume that $\sigma_A = 0$ or 1 for all A. Hence, $a_i = 0$ for all i except possibly 0 and 1. Then equations (17), (18), and (19) become

$$a_0 + a_1 = 7q + 1, \tag{38}$$

$$a_1 = \frac{49\lambda - 70q - 20}{3}, \tag{39}$$

and

$$\sum_{i=0}^{1} \frac{3a_i}{3\lambda - i(4\lambda - 7q - 2)} = \frac{9q^2}{(2q - 2\lambda + 1)(2\lambda - 5q - 1)} - \frac{1}{\lambda}. \tag{40}$$

Solving equations (38) and (39) yields

$$a_0 = \frac{91q - 49\lambda + 23}{3}$$

and

$$a_1 = \frac{49\lambda - 70q - 20}{3}.$$

Inserting the above expressions for a_0 and a_1 into equation (40) and manipulating the result, we arrive at

$$(4\lambda - 7q - 2)^2[130q^2 - (161\lambda - 91)q + 49\lambda^2 - 46\lambda + 13] = 0.$$

Now, $4\lambda - 7q - 2 \neq 0$ since $v \neq 4\lambda - 1$. Therefore, we obtain that

$$130\lambda^2 - (161\lambda - 91)q + 49\lambda^2 - 46\lambda + 13 = 0. \tag{41}$$

Thus, the discriminant of the left-hand-side of equation (41), which is a quadratic in q, must be a perfect square. This implies that

$$49\lambda^2 - 598\lambda + 169 = c^2$$

for some integer c, which can be transformed into

$$(1680\lambda - 7c - 102557)(1680\lambda + 7c - 102557) = 81120. \tag{42}$$

However, by considering all possible ways of factoring 81120 into the product of two integers, it can easily be shown that equation (42) has no solution in integers. We thereby obtain a contradiction.

Case 10: $m = 10$.

If there exists a block A with $\sigma_A \leq -1$, then $m(A) \leq 8$ and D is type-1 by previous cases. If there exists a block A with $\sigma_A \geq 1$, then $m(A) \geq 12$, so $m^*(A) \leq 9$ and once again D is type-1 by previous cases. Therefore, we may assume that $\sigma_A = 0$ for all A, a contradiction. This concludes the proof. □

Remark 4.2. Although we were able to prove the content of Theorem 2.5, unfortunately we were not able to prove that all λ-designs on $v = 7p + 1$ points, p prime, are type-1. We were not even able to prove this for a single conguence class of primes modulo 7. The reason for this is that a proof attempt analogous to those in [5], [6], [9], [10], and [11] breaks down when p divides g.

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