Diagonally switchable λ-fold 4-cycle systems, $\lambda > 1$[1]

Giovanni Lo Faro and Antoinette Tripodi
Department of Mathematics
University of Messina
Contrada Papardo, 31 - 98166, Sant'Agata
Messina, Italy
lofaro@unime.it
tripodi@dipmat.unime.it

Abstract

A diagonally switchable λ-fold 4-cycle system of order n, briefly DS4CS(n, λ), is a λ-fold 4-cycle system in which by replacing each 4-cycle (a, b, c, d) covering pairs ab, bc, cd, da by either of the 4-cycles (a, c, b, d) or (a, b, d, c) another λ-fold 4-cycle system is obtained. In [3]Adams, Bryant, Grannell, and Griggs proved that a DS4CS$(n, 1)$ exists if and only if $n \equiv 1 \pmod 8$, $n \geq 17$ with the possible exception of $n = 17$. In this paper we prove that for $\lambda \geq 2$ the necessary conditions for the existence of a λ-fold 4-cycle system of order n are also sufficient for the existence of a DS4CS(n, λ) except for $(n, \lambda) = (5, 2)$.

Key words: λ-fold 4-cycle system, complete multipartite graph, group divisible design, diagonally switchable property

1 Introduction

Let G and H be simple finite graphs, and let λH denotes the graph H with each of its edges replicated λ times. The graph K_n denotes the complete graph with n vertices. The graph $K_{n_1,n_2,\ldots,n_t}$ denotes the complete multipartite graph with t partite sets of size $n_1, n_2, \ldots, n_t$ respectively. For convenience, we use $K_n \setminus K_m$ to denote the graph $K_{1,\ldots,1,m}$ with $n - m$ 1s. Note that $K_n \setminus K_m$ is sometimes referred

[1]Supported by P.R.A. and I.N.D.A.M. (G.N.S.A.G.A.).

to as a complete graph of order n with a *hole* of size m. A *λ-fold G-design* of λH, $(\lambda H, G)$-design, is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is a collection of isomorphic copies (called *blocks*) of the graph G whose edges partition the edges of λH. If H is the complete graph K_n, we refer to such a λ-fold G-design as having order n. If $\lambda = 1$, we drop the term "1-fold"and simply say "G-design". Let $\mathcal{H} = \{X_1, X_2, \ldots, X_t\}$, $|X_i| = n_i$, be a partition of the set X into subsets called holes. Let $K_{n_1,n_2,\ldots,n_t}$ be defined on X with parts X_i. A *λ-fold holely G-design*, (G, λ)-HD, is a triple $(X, \mathcal{H}, \mathcal{B})$ where $(X, \mathcal{B})$ is a $(\lambda K_{n_1,n_2,\ldots,n_t}, G)$-design. The *hole-type* of the HD is $\{n_1, n_2, \ldots, n_t\}$. We usually use an "exponential" notation to describe hole-types: the hole-type $1^i 2^j 3^k \ldots$ denotes i occurrences of 1, j occurrences of 2, etc.

Let G be a simple finite graph; and let $V(G)$ and $E(G)$ denote the vertex-set and edge-set of G respectively. For a subgraph Γ_G of G, $G - \Gamma_G$ denotes the graph obtained from G by deleting the edges of Γ_G. Γ_G is said to be *admissible* if there exists a simple graph Γ'_G such that $E(\Gamma'_G) \cap E(G) = \emptyset$, $V(\Gamma'_G) \subseteq V(G)$, and the graph $(G - \Gamma_G) + \Gamma'_G$ obtained from $G - \Gamma_G$ by adding the edges of Γ'_G is isomorphic to G.
if Γ_G is an admissible subgraph of G, then using the vertices of G a further copy of G may be constructed by replacing $E(\Gamma_G)$ by $E(\Gamma_G{}')$.Let $(X, \mathcal{B})$ be a $(\lambda H, G)$-design and Γ_G a fixed admissible subgraph of G. For each block $B \in \mathcal{B}$, $(B - \Gamma_B) + \Gamma'_B$ is called a *Γ_G-transformation* of B. Let $\mathcal{B}' = \{(B - \Gamma_B) + \Gamma'_B : B \in \mathcal{B}\}$. If $(X, \mathcal{B}')$ is also a $(\lambda H, G)$-design, then it is said to be a *Γ_G-transformed design* of $(X, \mathcal{B})$ and we will say that $(X, \mathcal{B})$ is *Γ_G-switchable*. When $E(\Gamma_G) = \{\epsilon\}$, we will simply say ϵ-transformation and ϵ-switchable (see [6, 7]). The motivation for the concept of Γ_G-switchable $(\lambda H, G)$-design can be found in [3], where Adams, Bryant, Grannell, and Griggs studied the diagonally switchable property for the class of 4CSs (4-cycle systems), i.e. designs whose blocks are copies of the graph (a, b, c, d), covering edges ab, bc, cd, da, and with the Γ-switchable property where Γ is the pair of non-adjacent edges of the 4-cycle. More precisely, $(a, b, \;\; c, d)$ may be transformed into (a, b, d, c) or (a, c, b, d) by replacing, respectively, each pair of non-adjacent edges of the original 4-cycle by the diagonals ac and bd. In [3] it is proved

that a diagonally switchable 4-cycle system of order n exists if and only if $n \equiv 1 \pmod 7$, $n \geq 17$ with the possible exception of $n = 17$.

It is well known that the spectrum for λ-fold 4CSs is the set of all integers n such that $\lambda n(n-1) \equiv 0 \pmod 8$ and $\lambda(n-1) \equiv 0 \pmod 2$. In this paper we determine the necessary and sufficient for the existence of a diagonally switchable λ-fold 4-cycle system of order n (briefly, DS4CS(n, λ)), for $\lambda \geq 2$, proving the following theorem as the main result.

Main Theorem There exists a diagonally switchable $4CS(n, \lambda)$ for all $\lambda \geq 2$ and admissible n, except for $(n, \lambda) = (5, 2)$.

2 Working lemmas

A *group divisible design* (or GDD) with index λ is a triple $(X, \mathcal{H}, \mathcal{A})$, which satisfies the following properties:

(1) $\mathcal{H}$ is a partition of X into subsets called *groups.*

(2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point.

(3) Every pair of points from distinct groups occurs in exactly λ blocks.

The group-type of a GDD $(X, \mathcal{H}, \mathcal{A})$ is the multiset $\{|H| : H \in \mathcal{H}\}$. We will use an "exponential" notation to describe group-types: the group-type $1^i 2^j 3^k$... denotes i occurrences of 1, j occurrences of 2, etc. We call the GDD $(X, \mathcal{H}, \mathcal{A})$ a (K, λ)-GDD if $|A| \in K$ for every $A \in \mathcal{A}$. A $(\{k\}, \lambda)$-GDD is briefly written as (k, λ)-GDD. A (v, K, λ)-PBD is a (K, λ)-GDD of type 1^v, and a $(v, \{k\}, \lambda)$-PBD is called a (v, k, λ)-BIBD and is denoted by $S_\lambda(2, k, v)$ ($S(2, k, v)$ when $\lambda = 1$).

A *transversal design* (TD) TD(k, n) is a GDD of group type n^k and block size k. It is well known that a TD(k, n) is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order n. If q is a prime power, then there exist $q - 1$ MOLS(q)s and therefore a TD$(q + 1, q)$. For later use, we recall the following existence results for TDs:

Lemma 2.1 ([1, 2])

1) *A TD*$(4,n)$ *exists for all integers* $n \neq 2, 6$.

2) *A TD*(k,q) *exists for any prime power* q *and any* $k \leq q+1$.

The following Constructions 2.2-2.5 are well described in [6]. The first one is a variation of Wilson's Fundamental Construction in [9].

Construction 2.2 (Weighting Construction) *Suppose* $(X, \mathcal{H}, \mathcal{A})$ *is a* (K,λ)*-GDD. Let* $\omega : X \longmapsto Z^{+} \cup \{0\}$ *be any function (we refer to* ω *as a weighting) and, for every* $x \in X$*, let* $S(x)$ *be a set of* $\omega(x)$ *"copies" of* x*. For every* $A \in \mathcal{A}$*, suppose that*

$$(\cup_{x \in A} S(x),\ \{S(x) : x \in A\},\ \mathcal{B}_A)$$

is a diagonally switchable (C_4, μ)*-HD with hole-type* $\{\omega(x) :\ x \in A\}$*. Then*

$$(\cup_{x \in X} S(x),\ \{\cup_{x \in H} S(x) :\ H \in \mathcal{H}\},\ \cup_{A \in \mathcal{A}} \mathcal{B}_A)$$

is a diagonally switchable $(C_4, \lambda\mu)$*-HD with hole-type* $\{\sum_{x \in H} \omega(x) : H \in \mathcal{H}\}$.

Construction 2.3 (PBD-construction) *Suppose that there exists a* (v, L, μ)*-PBD. If, for each* $l \in L$*, there is a diagonally switchable* λ*-fold* C_4*-design of order* l*, then there exists a diagonally switchable* $\mu\lambda$*-fold* C_4*-design of order* v*.*

Construction 2.4 (Filling subdesigns) *Let* a *be a nonnegative integer. Suppose that there exists exists a diagonally switchable* (C_4, λ)*-HD of hole-type* $\{n_1, n_2, \ldots, n_t\}$*. If there exists a diagonally switchable* $(\lambda(K_{n_i+a} \backslash K_a), C_4)$*-design for each* $1 \leq i \leq t$*, then there exists a diagonally switchable* $(\lambda(K_{v+a} \setminus K_a), C_4)$*-design, where* $v = \sum_{i=1}^{t} n_i$*. If further there exists a diagonally switchable* $4CS(a, \lambda)$ *then there exists a diagonally switchable* $4CS(v + a, \lambda)$

Construction 2.5 (Repeating blocks) *If there exists a diagonally switchable* $(\lambda_1 H, C_4)$*-design and a diagonally switchable* $(\lambda_2 H, C_4)$*-design, then there exists a diagonally switchable* $((\mu_1\lambda_1 + \mu_1\lambda_1)H, C_4)$*-design for any positive integers* μ_1 *and* μ_2*.*

We quote the following known results for later use.

Lemma 2.6 ([10]) *Necessary and sufficient conditions for the existence of a* $(4,\lambda)$*-GDD of type* m^u *are* $u \geq 4$, $\lambda(u-1)m \equiv 0 \pmod 3$ *and* $\lambda u(u-1)m^2 \equiv 0 \pmod{12}$ *with the exception of* $(m,u,\lambda) \in \{(2,4,1),(6,4,1)\}$, *in which case no such GDD exists.*

Lemma 2.7 ([8]) *There exists a* $(4,1)$*-GDD of type* $4^u m^1$ *for each* $u \geq 6$, $u \equiv 0 \pmod 3$ *and* $m \equiv 1 \pmod 3$ *with* $1 \leq m \leq 2(u-1)$.

Lemma 2.8 ([5]) *A* $(4,1)$*-GDD of type* $2^u 5^1$ *exists if and only if* $u \equiv 0 \pmod 3$, $u \geq 9$.

Lemma 2.9 ([4]) *There exists a* $(v,\{4,5,6,7\},1)$*-PBD for any integer* $v \geq 4$ *and* $v \neq \{8-12, 14, 15, 18, 19, 23\}$.

3 The case of $\lambda = 2$

In this section we deal with the case where $\lambda = 2$. To start with, we note that the necessary and sufficient condition for the existence of a 4CS$(n,2)$ is $n \equiv 0,1 \pmod 4$ $(n \geq 4)$. We want to prove that conditions is also sufficient for the existence of a DS4CS$(n,2)$, except for $n = 5$.

3.1 DS4Cs of small order

Here we give a proof of the nonexistence of a DS4CS$(5,2)$ as well as some examples which are necessary for the main constructions. We point out that throughout the paper, the corresponding transformation of the block $B = (a,b,c,d)$ is (a,b,d,c).

Lemma 3.1 *There does not exist a* $DS4CS(5,2)$.

Proof Suppose $(I_5, \mathcal{B})$ is a DS4CS$(5,2)$, where $I_5 = \{1,2,3,4,5\}$. Every vertex of I_5 occurs in four blocks, thus for every $x \in I_5$

there exists exactly one block whose vertex set is $I_5 \setminus \{x\}$. If we fix a pair, say $\{1,2\}$, the blocks containing both 1 and 2 are of the form : *i*) $(1,2,*,*)$, $(1,2,*,*)$; or *ii*) $(1,*,*,2)$, $(1,*,*,2)$, $(1,*,2,*)$, $(1,*,2,*)$; or *iii*) $(1,*,*,2)$, $(1,2,*,*)$, $(1,*,2,*)$. To start with, note that the second configuration is impossible; otherwise, the fifth block is constructed on three verteces. If $\{1,2\}$ appears in configuration *i*), then the four blocks involving the vertex 1 are in the form of $(1,2,a,*)$, $(1,2,b,*)$, $(1,*,c,*)$, $(1,*,d,*)$, with $\{a,b,c,d\} \subseteq \{3,4,5\}$, and so there exists a vertex α such that the pair $\{1,\alpha\}$ appears in configuration *ii*) which is impossible. Therefore, all pairs of vertices appear in configuration *iii*). For each $(a,b,c,d) \in \mathcal{B}$ if we put the quadruple $\{a,b,c,d\}$ in $\mathcal{B}'$; the resulting design $(I_5, \mathcal{B}')$ is an $S_2(2,4,5)$. Let $\mathcal{B}' = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}$. Each block of $\mathcal{B}'$ contains three pairs of parallel edges: one pair of horizontal edges, one of vertical edges, and one of diagonal edges. Without loss of generality assume that in the first quadruple 12, 34 are horizontal, 14, 23 vertical, and 13, 24 diagonal (i.e. the corresponding 4-cycle of $\mathcal{B}$ is $(1,2,3,4)$). Then in the second one 12, 35 are vertical or diagonal. If 12, 35 are vertical, in the second quadruple 13, 25 are horizontal and so 15, 23 diagonal. It follows that in the third one 12, 45 are necessarily diagonal and so 14, 25 can be neither vertical nor horizontal. If in the second quadruple 12, 35 are diagonal, then 15, 23 are horizontal and so 13, 25 vertical. It follows that in the fifth one 23, 45 are necessarily diagonal and so 34, 25 can be neither vertical nor horizontal. That is a contradiction. □

All of the remaining examples are defined on the set $X = Z_n$ (or $X = Z_{n-1} \cup \{\infty\}$) and are cyclic (or 1-rotational, respectively). They are obtained under the group action $i \to i+1$ $(\infty + 1 = \infty)$ on the sets of starters listed below.

n=4
$(\infty,0,1,2)$

n=8
$(\infty,6,1,0)$, $(2,6,0,4)$

n=9
$(1,-1,2,-2)$, $(1,-1,-2,2)$

n=12
$(3,2,0,\infty)$, $(1,5,-3,3)$, $(1,0,3,-3)$

n=17
$(6,-6,7,-7)$, $(6,-6,-7,7)$, $(3,-3,5,-5)$, $(3,-3,-5,5)$

n=20
$(6,0,3,\infty)$, $(2,-7,5,-3)$, $(2,-7,-3,5)$, $(1,3,7,8)$, $(1,3,8,7)$

n=21
$(1,15,2,12)$, $(5,18,3,6)$, $(4,20,11,7)$, $(8,17,19,13)$, $(14,16,9,10)$

n=53
$(0,5g_i,g_i,4g_i)$, for $i=1\ldots,(q-1)/4$, where $g_1,\ldots,g_{(q-1)/4}$ are all representatives in the quotient group $C_0/\{1,-1\}$ and C_0 is the group of non-zero squares in F_{53}.

3.2 Main constructions

In this section we give some constructions which collectively, along with the examples in Section 3.3 and a few other examples that we will give later, allow us to settle the case of $\lambda = 2$. They are all direct constructions, except for the first one which is a recursive construction. Before giving those constructions, which we will present as theorems, we need a further lemma.

Lemma 3.2 *There exists a diagonally switchable $(C_4, 2)$-HD of hole-type n^4 for any integer n.*

Proof For $n \neq 2, 6$, take a TD$(4,n)$ from Lemma 2.1 and replace every quadruple $\{a,b,c,d\}$ by the three 4-cycles (a,b,c,d), (a,c,d,b), (a,d,b,c) to obtain the required design. For $n=2$, the assertion follows by Construction 2.5 with a diagonally switchable $(C_4, 1)$-HD of hole-type 2^4, whose existence is proved in [3]. For $n = 6$, take a TD$(4, 3)$ from Lemma 2.1, give each point a weight of 2, and apply Construction 2.2 with diagonally switchable $(C_4, 2)$-HDs of hole-type 2^4. □

Theorem 3.3 *If there exists a $DS4CS(n, 2)$, then there exists a $DS4CS(4n, 2)$ and a $DS4CS(4n - 3, 2)$.*

Proof Take a diagonally switchable $(C_4, 2)$-HD of hole-type n^4 (or $(n-1)^4$) from Lemma 3.2 and apply Construction 2.4 with $a = 0$ (or $a = 1$, respectively) and DS4CS$(n, 2)$s to obtain the required design. □

Theorem 3.4 *There exists a $DS4CS(n, 2)$ for any $n \equiv 1, 4 \pmod{12}$.*

Proof Take an S$(2, 4, n)$, which exists for any $n \equiv 1, 4 \pmod{12}$, and apply Construction 2.3 with DS4CS$(4, 2)$s, given in Section 3.1, to obtain the required system. □

Theorem 3.5 *There exists a $DS4CS(12k,\ 2)$ for every $k \geq 3$.*

Proof Take a 4-GDD of type 6^k, $k \geq 5$, from Lemma 2.6, give each point a weight of 2 and apply Construction 2.2 to obtain a diagonally switchable $(C_4, 2)$-HD of hole-type 12^k, where the input designs are from Lemma 3.2. Apply again Construction 2.4 with $a = 0$ and with DS4CS$(12, 2)$s, given in Section 3.1, to obtain a DS4CS$(12k, 2)$. For $k = 3, 4$ apply Theorem 3.3 $(n \to 4n)$ with $n = 9, 12$, respectively. □

Theorem 3.6 *There exists a $DS4CS(24k + 20 + a, 2)$, $a = 0, 1$, for every $k \geq 1$, $(a, k) \neq (0, 1)$.*

Proof Take a 4-GDD of type $4^{3k}10^1$, $k \geq 2$, from Lemma 2.7, and give each point a weight of 2. Apply Construction 2.2 to obtain a diagonally switchable $(C_4, 2)$-HD of hole-type $8^{3k}20^1$ and then Construction 2.4 with $a = 0, 1$ and with DS4CS$(8 + a, 2)$s and a DS4CS$(20 + a, 2)$ to obtain the required design. For $(a, k) = (1, 1)$ apply Theorem 3.3 ($n \to 4n - 3$) with $n = 12$. □

Theorem 3.7 *There exists a* $DS4CS(24k + 8, 2)$ *for every* $k \geq 1$.

Proof Take a 4-GDD of type 4^{3k+1}, $k \geq 1$, from Lemma 2.6, give each point a weight of 2. Apply Construction 2.2 to obtain a diagonally switchable $(C_4, 2)$-HD of hole-type 8^{3k+1} and then Construction 2.4 with $a = 0$ and with DS4CS$(8, 2)$s to obtain a $DS4CS(24k+8, 2)$. □

Theorem 3.8 *There exists a* $DS4CS(24k + 5, 2)$ *for every* $k \geq 1$, $k \neq 4, 6$.

Proof First, note that applying Theorem 3.3 ($n \to 4n - 3$) with $n = 8$ gives a DS4CS$(29, 2)$. From Lemma 2.7 take a 4-GDD of type $4^{3h}(3s+1)^1$, where $h \geq 2$ and $0 \leq s \leq 2h-1$. Put $s = 2$and give each point a weight of 4. Apply Construction 2.2 to obtain a diagonally switchable $(C_4, 2)$-HD of hole-type $16^{3h}28^1$ then Construction 2.4 with $a = 1$ and with DS4CS$(17, 2)$s and a DS4CS$(29, 2)$ to obtain a DS4CS$(48h + 29, 2)$. Apply Theorem 3.3 ($n \to 4n - 3$) with $n = 20$ to obtain a DS4CS$(77, 2)$. Put $s = 4$ (clearly $h \geq 3$)and give each point a weight of 4. Apply Construction 2.2 to obtain a diagonally switchable $(C_4, 2)$-HD of hole-type $16^{3h}52^1$ then Construction 2.4 with $a = 1$ and with DS4CS$(17, 2)$s and a DS4CS$(53, 2)$ to obtain a $DS4CS(48h + 53, 2)$. □

Finally, the next lemma deals with the small orders left.

Lemma 3.9 *There exists a* $DS4CS(n, 2)$ *for* $n = 24, 44, 101, 149$.

Proof $n = 24$: In $Z_{23} \cup \{\infty\}$ take the base blocks $(\infty, 10, 0, 22)$, $(14, 12, 7, 20)$, $(19, 8, 15, 18)$, $(4, 9, 13, 6)$, $(11, 5, 17, 3)$, $(21, 2, 16, 1)$.

$n = 44$: Start from a 5-GDD of type 4^6 by deleting one point from a S(2, 5, 25). In the last group give 2 points a weight of zero and the remaining 2 points a weight of 2. Give all the other points of the 5-GDD a weight of 2 and apply Construction 2.2 to obtain a diagonally switchable $(C_4, 2)$-HD of hole-type $8^5 4^1$; the input designs are from Lemma 3.2 and from [3] where a diagonally switchable $(C_4, 1)$-HD of hole-type 2^5 is given. Apply again Construction 2.4 with $a = 0$ and with DS4CS(8, 2)s and a DS4CS(4, 2) to obtain a DS4CS(44, 2).

$n = 101$: Take a TD(5, 11) from Lemma 2.1. Select a block and give all its point a weight of zero. Give all the other points of the TD a weight of 2 and apply Construction 2.2 to obtain a diagonally switchable $(C_4, 2)$-HD of hole-type 20^5. Apply again Construction 2.4 with $a = 1$ and with DS4CS(21, 2)s to obtain a DS4CS(101, 2).

$n = 149$: Take a TD(5, 16) from Lemma 2.1. In the last group give 6 points a weight of zero and the remaining 10 points a weight of 2. Give all the other points of the TD a weight of 2 and apply Construction 2.2 to obtain a diagonally switchable $(C_4, 2)$-HD of hole-type $32^4 20^1$. Apply again Construction 2.4 with $a = 1$ and with DS4CS(33, 2)s and a DS4CS(21, 2) to obtain a DS4CS(149, 2). □

4 The case of $\lambda = 4$

In this section we prove that a DS4CS$(n, 4)$ exists for any $n \geq 4$.

Lemma 4.1 *Let $q \geq 5$ be an odd prime power and $s \leq (q-1)/2$ be a non-negative integer. Then there exists a diagonally switchable $(4(K_{q+s} \setminus K_s), C_4)$-design.*

Proof Let F_q be a finite field with q elements. Let $g_1, \ldots, g_{(q-1)/2}$ be all representatives in the quotient group $F_q^*/\{1, -1\}$. Choose an element $x \in F_q^*$ such that $x \neq \pm y$. $(F_q \cup \{\infty_1, \ldots, \infty_t\}, \mathcal{B})$, where $\mathcal{B}$ is the collection of 4-cycles obtained by listing the following base blocks in $(F_q, +)$:

$(0, (x+y)g_i, yg_i, \infty_i)$, $(yg_i, xg_i, 0, \infty_i)$, where $i = 1, \ldots, s$,

$(0, (x+y)g_i, yg_i, xg_i)$, where $i = s+1, \ldots, (q-1)/2$,

is the required design. □

Lemma 4.2 *There exists a $DS4CS(n, 4)$ for every $n \geq 4$.*

Proof For $n = 4$, take two copies of a DS4CS(4, 2) from Section 3.

For $n \in \{5, 6, 7\}$, n can be written as $q + s$ by taking $(q, s) = (5, 0)$, $(5, 1)$, and $(7, 0)$, respectively. By Lemma 4.1 there exists a DS4CS(n, 4).

For any integer $n \geq 4$ and $n \notin \{8-12, 14, 15, 18, 19, 23\}$, there is a $(n, \{4, 5, 6, 7\}, 1)$-PBD by Lemma 2.9. Hence there exists a DS4CS(n, 4) by Construction 2.3. The input designs exist as pointed before.

For $n = 8, 9, 12$, take two copies of a DS4CS(n, 2) from Section 3.

For $n = 10, 14, 18$, by Lemma 4.1 there exists a DS4CS(n, 4) by taking $q = n - 1$ and $s = 1$.

For $n = 11, 19, 23$, by Lemma 4.1 there exists a DS4CS(n, 4) by taking $q = n$ and $s = 0$.

For $n = 15$, by Lemma 4.1 there exists a diagonally switchable $(4(K_{15} \setminus K_4), C_4)$-design. By Construction 2.4 with $a = 4$ and $t = 1$, there exists a DS4CS(15, 4). The needed DS4CS(4, 4) exists as pointed before. □

5 Conclusion

We are in position to present our main result.

Main Theorem There exists a diagonally switchable $4CS(n, \lambda)$ for all $\lambda \geq 2$ and admissible n , except for $(n, \lambda) = (5, 2)$.

Proof $\lambda = 2$: In [3] it has been proved that a DS4CS(n, 1) exists for every $n \equiv 1 \pmod 8$, except for $n = 9$ and possibly $n = 17$.So by Construction 2.5 and combining all of the results in Section 3 we have a complete solution to our problem.

$\lambda \equiv 1 \pmod 2$, $\lambda \geq 3$: The conclusion follows by Construction 2.5 together with the following examples for $(n, \lambda) = (9, 3)$ and $(17, 3)$.

$(n, \lambda) = (9, 3)$: take the base blocks $(0, 2, 5, 1)$, $(0, 4, 3, 2)$, $(0, 6, 1, 3)$ in Z_9.

$(n, \lambda) = (17, 3)$: take the base blocks $(3, -5, -4, 1)$, $(3, -5, 1, -4)$, $(0, 1, 7, 4)$, $(0, 1, 4, 7)$, $(0, 2, 11, 7)$, $(0, 2, 7, 11)$ in Z_{17}.

$\lambda \equiv 0 \pmod 4$: The conclusion follows by Section 4 and Construction 2.5.

$\lambda \equiv 2 \pmod 4$, $\lambda \geq 4$: The conclusion follows by Construction 2.5 together with the following example for $(n, \lambda) = (5, 6)$.

$(n, \lambda) = (5, 6)$: take the base blocks $(4, 0, 2, 1)$, $(0, 2, 4, 1)$, $(0, 1, 2, 4)$ in Z_5. □

References

[1] R. J. R. Abel, C. J. Colbourn and J. H. Dinitz, Mutually Orthogonal Latin Squares, in: *CRC Handbook of Combinatorial Designs* (C. J. Colbourn and J. H. Dinitz, eds.) CRC Press 2006, 160–193.

[2] R. J. R. Abel, C. J. Colbourn and M. Wojtas, *Concerning seven and eight mutually orthogonal Latin squares*, *J. Combin. Des.*, 12 (2004), 123–131.

[3] P. Adams, D. Bryant, M. Grannell, and T. Griggs, *Diagonally switchable 4-cycle systems*, Australasian J. Combin., 34(2006), 145-152.

[4] F. E. Bennett, H. D. O. F. Gronau, A. C. H. Ling and R. C. Mullin, *PBD-closure,* in: C. J. Colbourn, J. H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, 203-213.

[5] A. E. Brouwer, A. Schrijver and H. Hanani, *Group divisible designs with block size 4*, Discrete Math., 20(1977), 1-10.

[6] Y. Chang, G. Lo Faro, and A. Tripodi, *Three types of edge-switchable kite systems,*

[7] Y. Chang, T. Feng, G. Lo Faro, and A. Tripodi, *Two types of switchable* λ*-fold* $(K_4 - e)$*-designs,*

[8] G. Ge and Alan C. H. Ling, *Group divisible designs with block size four and group type* $g^u m^1$ *for small* g, Discrete Math., 285, No. 1-3(2004), 97-120.

[9] R. M. Wilson, *Constructions and uses of pairwise balanced designs, Math. Centre Tracts* 55 (1974), 18–41.

[10] L. Zhu, *Some recent developments on BIBDs and related designs,* Discrete Math., 123(1993), 189-214.