A computer search for minimal blocking sets in $PG(2,q)$

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Abstract

We made a computer search for minimal blocking sets in the projective geometry $PG(2,11)$, and found 30000, of which only two nontrivial blocking sets had the possibility of being isomorphic.

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Introduction

In a geometry consisting of sets of lines and points, a *blocking set* is a subset B of points such that every line has at least one point in the subset. These sets are said to be *minimal* if removing any point leaves us with a set of points not blocking all lines. The size $|B|$ of a minimal and non-trivial blocking set in the projective geometry $PG(2,p)$, where p is a prime, is bounded by

$$3(p+1)/2 \leq |B| \leq p\sqrt{p}+1.$$

The lower of these bounds was finally shown by Blokhuis in 1994 [2], after being unproven for about 20 years. It is now known as the Blokhuis bound. The upper bound was shown by Bruen and Thas in 1977 [4], and is valid also when p is not a prime.

Though minimal blocking sets seems to have been studied since 1969, [3], and many contributions have been made, see e.g. [1] or [6] for references, not many computer searches for minimal blocking sets have been done. We made a program in Matlab, in which we used a random method to generate minimal blocking sets in the projective geometries $PG(2,q)$, where $q=11$, $q=13$, $q=17$ and $q=19$. In $PG(2,11)$ we found 30000 minimal blocking sets, of which two trivial sets had the size 12, and the rest were of sizes between 21 and 30, and the most common size being 25. Of these only two non-trivial blockcing sets had the possibility of being isomorphic, as explained in the next section.

Construction and results

To generate minimal blocking sets in $PG(2,q)$ we need the incidence matrix A for the points and lines. To construct this matrix A, we considered the two matrices with all points and all lines in homogeneous coordinates, [7]. The point matrix was then multiplied with the transponate of the line matrix, and the resulting matrix was taken modulo q. We call this matrix $\widetilde{A}$. If $\widetilde{A}(i,j) = 0$ point i is on line j, and if $\widetilde{A}(i,j) \neq 0$ point i is not on line j. By replacing all zeros with ones, and all non-zero elements with zero we get the incidence matrix A of that set of points and lines. Hence, $A(i,j) = 1$ if and only if line j contains point i.

We had a few alternative approaches when trying to generate minimal blocking sets in $PG(2,q)$; we removed a random point, and we used two kinds of greedy algorithms, in which we added the best point to, and removed the worst point from the blocking set, and also their inverses, in which we added the worst and removed the best point. The method we found most appealing was the method in which we removed a random point. This was also the fastest one. Generating 30000 minimal blocking sets in $PG(2,11)$ with the random method took us about 50 minutes.

In the random method we started with the trivial blocking set containing all points, and then randomly chose a point P. If all lines containing this point P had other points in the blocking set the point was removed. To see if it was possible to remove P, we took the sum over every column in the incidence matrix with a one in the row corresponding to the chosen point. If the sum was larger than one in all of them, there was no 1-secant through that point. Hence, the point was not needed in the blocking set, and it was removed. We continued in this way until every point was checked and no more points could be removed.

For an isomorphism, structures are preserved and elements can be linked to each other. We will perform a computer search for blocking sets, and ask whether the sets we generate will be the same, down to isomorphism. Isomorphic sets have the same number of elements, and points will map to points in such a way that lines map to lines. Especially, having two isomorphic blocking sets, a blocking point in one will map to a blocking point in the other, such that the number of i-secants through the two points are the same.

To determine whether there existed any isomorphic blocking sets, we looked at the number of i-secants through every point in every blocking set. We then created a matrix for every blocking set, where each row represented a blocking point. These rows consisted of the vectors $(l_1, l_2, \ldots, l_r)$, where l_i is the number of i-secants the corresponding point is on. Note that l_i theoretically could be zero for $i = 2, \ldots, r-1$, and will in some cases definitely be zero for l_r, since the blocking sets are sorted by size and thus can have different secants. We call this vector the *type of the point*. If two blocking sets are isomorphic, there must for every point in the first set exist a point in the second set which is of the same type. Sorting the rows in

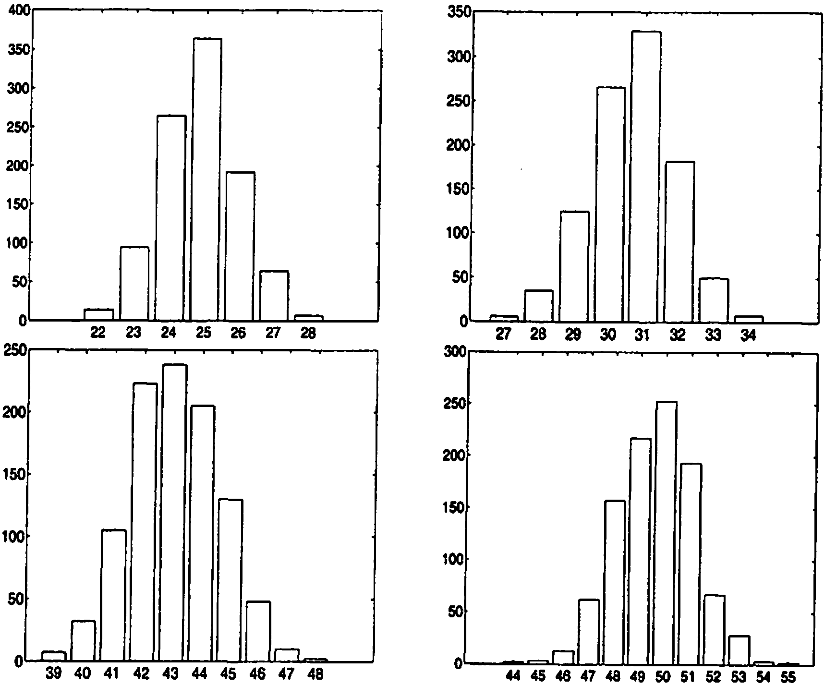

Figure 1: The size distribution of a thousand sets each for $q = 11$ (top left), $q = 13$ (top right), $q = 17$ (bottom left) and $q = 19$ (bottom right). These sets were generated with the random method.

the matrices lexicographically and pair-wise subtracting all matrices representing blocking sets of the same size we can immediatley tell which blocking sets that have the possibility of being isomorphic; the resulting matrix must consist only of zeros.

In $PG(2, 11)$ we generated 30000 minimal and non-trivial blocking sets. Of these, there were only two non-trivial blocking sets for which the matrices constructed as above were the same. Hence, most of the sets were non-isomorphic.

When randomly picking points we got for $q = 11$, $q = 13$ and $q = 17$ the most common size of the sets B being $|B| = 3q - 8$. This is not always the case. For $q = 19$ the most common size was $|B| = 3q-7$. Hence, for $q = 11$ the most common size was 25. In figure 1 we present diagrams of the distribution of sizes among a thousand blocking sets, for $q = 11$, $q = 13$, $q = 17$ and $q = 19$, generated with the random method.

When generating 30000 minimal blocking sets in $PG(2,11)$ ve got the size distribution

size	12	21	22	23	24	25	26	27	28	29	30
number	2	16	446	2827	8062	10461	6239	1743	200	3	1

Let us finally remark that further information, e.g. the programming code, can be found in [5].

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