EXCLUDED-MINORS FOR THE CLASS OF GRAPHIC SPLITTING MATROIDS

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Abstract

This paper is based on the splitting operation for binary matroids that was introduced by Raghunathan, Shikare and Waphare [Discrete Math. 184 (1998), p.267-271] as a natural generalization of the corresponding operation in graphs. Here, we consider the problem of determining precisely which graphs G have the property that the splitting operation, by every pair of edges, on the cycle matroid $M(G)$ yields a graphic matroid. This problem is solved by proving that there are exactly four minor-minimal graphs that do not have this property.

Mathematics Subject Classifications: 05B35, 05C50 and 05C83.

Key words: Binary matroid, minor, splitting operation, graphic matroid.

1. Introduction

The splitting operation for a graph with respect to a pair of adjacent edges is defined as follows (see [1]): Let G be a connected graph and let v be a vertex of degree at least three in G. If $x = vv_1$ and $y = vv_2$ are two edges incident at v, then splitting away the pair x, y from v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y, and adding a new vertex $v_{x,y}$ adjacent to v_1 and v_2. The transition from G to $G_{x,y}$ is called the splitting operation on G. For practical purposes, we also denote the new edges $v_{x,y}v_1$ and $v_{x,y}v_2$ in $G_{x,y}$ by x and y, respectively. The graph G can be retrieved from $G_{x,y}$ by identifying the vertices v and $v_{x,y}$. We say that $G_{x,y}$ arises from G by the splitting operation.

The following figure illustrates this construction explicitly.

The splitting operation has important applications in graph theory. For example, Fleischner [1] characterized Eulerian graphs and developed an algorithm to find all distinct Eulerian trails in an Eulerian graph using the splitting operation. Tutte [10] characterized 3-connected graphs, and Slater [9] classified 4-connected graphs using a slight modification of this operation. Raghunathan et al. [5] have extended the notion of splitting operation

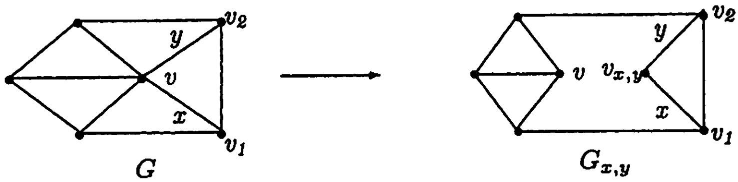

Figure 1

from graphs to binary matroids and used this operation to characterize Eulerian binary matroids.

The splitting operation for binary matroids is defined in the following way: Let M be a binary matroid on a set S and A be a matrix over $GF(2)$ that represents the matroid M. Consider elements x and y of M. Let $A_{x,y}$ be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to x and y where it takes the value 1. Let $M_{x,y}$ be the matroid represented by the matrix $A_{x,y}$. We say that $M_{x,y}$ has been obtained from M by splitting the pair of elements x and y. The two elements x and y in the matroid $M_{x,y}$ are now in series.

Alternatively, the splitting operation can be defined in terms of circuits of binary matroids. Let $M = (S, \mathcal{C})$ be a binary matroid on a set S together with the set $\mathcal{C}$ of circuits. Then $M_{x,y} = (S, \mathcal{C}')$ with $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1$, where
$\mathcal{C}_0 = \{C \in \mathcal{C} : x, y \in C \text{ or } x \notin C, y \notin C\}$; and
$\mathcal{C}_1 = \{C_1 \cup C_2 : C_1,\ C_2 \in \mathcal{C},\ x \in C_1,\ y \in C_2,\ C_1 \cap C_2 = \phi$ and $C_1 \cup C_2$ contains no member of $\mathcal{C}_0$ $\}$.

In particular, if M is graphic, G is a corresponding graph and x, y are adjacent edges of G, then the above notion of splitting operation coincides with the splitting operation in graphs. The following theorem is proved in [5].

Theorem 1.1. Let G be a graph, $M(G)$ be the circuit matroid of G and let x, y be a pair of adjacent edges in G. Then $M(G_{x,y}) = M(G)_{x,y}$. □

The splitting operation on a graphic matroid, in general, need not yield a graphic matroid. Shikare [7] provided some examples in his Ph.D. Thesis to justify this fact. Luis Goddyn in his evaluation report on the Thesis mentioned that the most important and interesting problem in this context is the problem of determining precisely which graphs G have the property that, for every pair x, y of edges, the matroid $M(G)_{x,y}$ is graphic. In this paper, we provide an excluded-minor characterization of the graphic matroids M such that for all $\{x, y\} \in E(M)$, the splitting matroid $M_{x,y}$ is also graphic.

The main result in this paper is the following theorem.

Theorem 1.2. The splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if the circuit matroid of the corresponding graph has no minor isomorphic to the circuit matroid of any of the following four graphs.

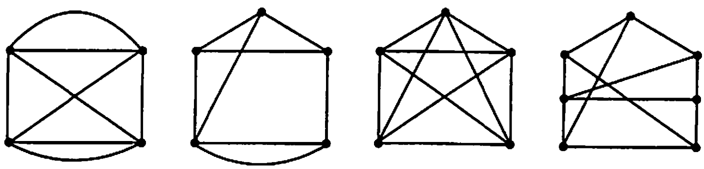

Figure 2

The following theorem is well known.

Theorem 1.3 [11]. A binary matroid is graphic if and only if it has no minor isomorphic to F_7, F_7^*, $M^*(K_5)$, or $M^*(K_{3,3})$. □

Notation. For convenience, let $\mathcal{F} = \{F_7, F_7^*, M^*(K_5), M^*(K_{3,3})\}$.

For undefined notation and terminology in graphs and matroids, we refer the reader to [1, 3, 4, 6, 13].

2. The splitting operation and minors

In this section, we explore the relationship between the splitting operation and the operations of deletion and contraction.

Proposition 2.1. Let M be a binary matroid on a set S containing T as a subset and let $x, y \in S \setminus T$. Then
(i) $(M \setminus T)_{x,y} = (M_{x,y})\backslash T$; and
(ii) $(M/T)_{x,y} = (M_{x,y})/T$.

Proof. Let A be a matrix over $GF(2)$ that represents the matroid M.

(i) This follows from the fact that a matrix representation for $M \setminus T$ is obtained from A by deleting the columns corresponding to the elements of T.
(ii) Let z be an arbitrary element of T. We show that $(M/z)_{x,y} = (M_{x,y})/z$. If z is a loop, then by (i), $(M/z)_{x,y} = (M \setminus z)_{x,y} = (M_{x,y}) \setminus z = (M_{x,y})/z$. Suppose that z is a non-loop element of T. Let c be the column of A corresponding to z. By pivoting on a nonzero element of c, we can transform A into a matrix A' in which c is a unit vector. Let A'/z be the matrix obtained from A' by deleting the column c and the row containing the unique non-zero entry of c. Then A'/z represents the matroid M/z. The matrix B that represents $(M/z)_{x,y}$ over $GF(2)$ is obtained from A'/z by adding an extra row with zero everywhere except in the columns corresponding to x and y, where it is 1. But B also represents the matroid $(M_{x,y})/z$ over $GF(2)$. We conclude that $(M/z)_{x,y} = (M_{x,y})/z$. Now it follows from this fact that by contracting successively the elements of T in M and then applying splitting operation with respect to the pair x, y, we obtain the desired result. □

In the next proposition, we state some basic properties concerning the splitting operation. Recall that elements e and f of a matroid M are in series if $\{e, f\}$ is a cocircuit of M.

Proposition 2.2. Let x and y be elements of a binary matroid M and let $r(M)$ denote the rank of M. Then
(i) $r(M_{x,y}) = r(M) + 1$ if and only if x and y are not in series;
(ii) $M_{x,y} = M$ if and only if x and y are in series;
(iii) if x_1, x_2 are in series in M, then they are in series in $M_{x,y}$;
(iv) $(M_{x,y})/\{x\} \setminus \{y\} \cong (M_{x,y})/\{y\} \setminus \{x\} \cong (M_{x,y}) \setminus \{x, y\} \cong M \setminus \{x, y\}$;
(v) y is a coloop in $(M_{x,y}) \setminus \{x\}$ while x is a coloop in $(M_{x,y}) \setminus \{y\}$.
The proofs are straightforward. □

Theorem 2.3. Let G be a graph and let x and y be edges of G. Suppose $M(G)_{x,y}$ is not graphic. Then there is a graph G' in which no pair of edges is in series and $M(G')$ is a minor of $M(G)$ such that $M(G')_{x,y}/\{x\} \in \mathcal{F}$ or $M(G')_{x,y}/\{x, y\} \in \mathcal{F}$.

Proof. By Theorem 1.3, the matroid $M(G)_{x,y}$ has a minor $F \in \mathcal{F}$. So there exist subsets T_1 and T_2 of $E(G)$ such that $M(G)_{x,y} \setminus T_1/T_2 \cong F$. Let $T_i' = T_i - \{x, y\}$ for $i = 1, 2$. By Proposition 2.1, $M(G)_{x,y} \setminus T_1'/T_2' =$

$(M(G)\backslash T_1'/T_2')_{x,y}$. Let G_1 denote the graph $G\backslash T_1'/T_2'$. Then F is a minor of $M(G_1)_{x,y}$. We have subsets $T_i" = T_i - T_i', i = 1,2$ of $\{x,y\}$ such that $M(G_1)_{x,y} \setminus T_1"/T_2" \cong F$. F does not use both x and y. If F uses neither x nor y then, by Proposition 2.2(iv), $M(G_1)_{x,y}/\{x,y\} \cong F$. Suppose the minor F uses exactly one of x and y, say y. Then, by Proposition 2.2(v), $M(G_1)_{x,y}/\{x\} \cong F$.

If G_1 does not have a pair of edges in series then we take $G' = G_1$. Suppose that the edges x_1 and x_2 are in series in G_1. Then, by Proposition 2.2(iii), x_1 and x_2 are in series in $M(G_1)_{x,y}$. If $\{x_1,x_2\}$ is disjoint from $\{x,y\}$ then x_1 and x_2 are in series in F, a contradiction. By Proposition 2.2(ii), the sets $\{x_1,x_2\}$ and $\{x,y\}$ cannot be equal. Therefore, we may assume that x and x_1 are one and the same but y and x_2 are distinct. Then x, x_2 and y form the series class in $M(G_1)_{x,y}$. If $M(G_1)_{x,y}/\{x\} \cong F$ then x_2 and y are in series in F, a contradiction. We conclude that $M(G_1)_{x,y}/\{x,y\} \cong F$. But then $M(G_1)_{x,y}/\{x,y\} \cong M(G_1)_{x,y}/\{x,x_2\}$. By Proposition 2.1, $M(G_1)_{x,y}/\{x,x_2\} \cong (M(G_1)/\{x_2\})_{x,y}/\{x\}$. By setting $G' = G_1/\{x_2\}$, we have $M(G')_{x,y}/\{x\} \cong F$. This completes the proof of the theorem. □

Definition 2.4. Let G be a graph in which no pair of edges is in series and let $F \in \mathcal{F}$. We say that G is minimal with respect to F if there exist two edges x and y of G such that $M(G)_{x,y}/\{x\} \cong F$ or $M(G)_{x,y}/\{x,y\} \cong F$.

We deduce the following result as a corollary to Theorem 2.3.

Corollary 2.5. Let M be a graphic matroid and G be a corresponding graph. Then the splitting operation, by any pair of elements, on M yields a graphic matroid if and only if G has no minor isomorphic to a minimal graph with respect to some $F \in \mathcal{F}$.

Proof. If $M(G_{x,y})$ is not graphic for some pair x, y of edges then, by Theorem 2.3, G has a minor isomorphic to a minimal graph corresponding to some member of $\mathcal{F}$.

Conversely, suppose G contains a minor, say G_1, isomorphic to a minimal graph that corresponds to $F \in \mathcal{F}$. Then G has edges x and y such that $M(G_1)_{x,y}/\{x\} \cong F$ or $M(G_1)_{x,y}/\{x,y\} \cong F$, where $G_1 = G \setminus T_1/T_2$ and $T_1, T_2 \subseteq E(G) - \{x,y\}$. By Proposition 2.1, $M(G)_{x,y} \setminus T_1/(T_2 \cup \{x\}) \cong F$ or $M(G)_{x,y} \setminus T_1/(T_2 \cup \{x,y\}) \cong F$. Thus $M(G)_{x,y}$ is not graphic, a contradiction. □

In the next theorem, we state some properties of minimal graphs.

Theorem 2.6. Let G be a minimal graph with respect to F where $F \in \mathcal{F}$ and let x, y be two edges of G such that either $M(G)_{x,y}/\{x\} \cong F$ or $M(G)_{x,y}/\{x,y\} \cong F$. Then
(i) G has no loops;
(ii) x and y are non-adjacent edges;
(iii) G is 2-connected;
(iv) if x_1 and x_2 are parallel edges of G then one of them must be either x or y; and
(v) if $M(G)_{x,y}/\{x,y\} \cong F$, then G has at most one pair of parallel edges and there is no 4-circuit in G containing both x and y.

Proof. (i) On the contrary, suppose G has a loop, say z. If z is different from x and y, then it is a loop in $M(G)_{x,y}$ and also a loop in $M(G)_{x,y}/\{x,y\}$. Consequently, it is a loop in F, a contradiction. If z is one of the two elements x and y, say x, then $M(G)_{x,y}/\{x\} \cong M(G)\backslash\{x\}$ and $M(G)_{x,y}/\{x,y\} = M(G)\backslash\{x\}/\{y\}$. We conclude that F is a minor of $M(G)$, a contradiction. Therefore, G can not have loops.
(ii) This follows from Theorem 1.1.
(iii) Suppose G is not 2-connected. By (i), (ii) and the fact that F is 2-connected, it follows that G cannot have more than two blocks and x, y are contained in different blocks. Then there is a graph H with E(H) = E(G), x and y adjacent in H and $M(H) = M(G)$. By Theorem 1.1, $M(G)_{x,y} = M(H_{x,y})$. Hence, the matroid $M(G)_{x,y}$ is graphic, a contradiction.
(iv) If x_1 and x_2 are in a parallel class of G that does not contain x or y, then x_1 and x_2 remain in parallel in each of matroids $M(G)_{x,y}/\{x\}$ and $M(G)_{x,y}/\{x,y\}$. This is a contradiction, since no member of $\mathcal{F}$ contains a parallel class. If x_1 and x_2 are in a parallel class containing x or y, then $M(G)_{x,y}/\{x\}$ and $M(G)_{x,y}/\{x,y\}$ contain a loop; a contradiction.
(v) Suppose that $M(G)_{x,y}/\{x,y\} \cong F$ for $F \in \mathcal{F}$ and G has two pairs of parallel edges. By the properties (ii) and (iv), these pairs must be disjoint, one should contain x and the other should contain y. Then $M(G)_{x,y}$ has a 4-circuit containing x and y. This circuit results in a 2−circuit of $M(G)_{x,y}/\{x,y\}$ which is not possible. Similarly, if G has a 4-circuit containing x and y then it is preserved in $M(G)_{x,y}$ and will give rise to a 2−circuit of F which is impossible. □

A matroid is said to be Eulerian if its ground set can be expressed as a union of disjoint circuits of the matroid (see [12]). A matroid is called bipartite if every circuit of it has an even number of elements. One can check that each of the matroids F_7 and $M^*(K_{3,3})$ is Eulerian while each of F_7^* and $M^*(K_5)$ is bipartite. Welsh [12] showed that a binary matroid is Eulerian if and only if its dual is bipartite. If M is a Eulerian binary matroid and A is a matrix over $GF(2)$ that represents M then the sum of columns of A is zero. It follows that a binary matroid M is Eulerian if and only if $M_{x,y}$ is Eulerian for every pair of elements x and y.

Proposition 2.7 Let G be a loopless graph and $x, y \in E(G)$. Then
(i) $M(G)_{x,y}/\{x\}$ is Eulerian if and only if G is Eulerian.
(ii) If x and y are nonadjacent and $M(G)_{x,y}/\{x, y\}$ is Eulerian then either G is Eulerian or the end vertices of x and y are precisely of odd degree.

Proof. (i) This follows from the fact that if G is Eulerian then $E(G)$, the edge set of G can be partitioned into edge-disjoint cycles of G and that $M(G)_{x,y}$ is binary.
(ii) Consider the incidence matrix A of the graph G. The number of ones in a row of A gives the degree of the corresponding vertex. We know that A represents $M(G)$ over $GF(2)$. Let $A_{x,y}$ be the matrix representation of $M(G)_{x,y}$ and $A_{x,y}/\{x, y\}$ represents $M(G)_{x,y}/\{x, y\}$. Suppose $M(G)_{x,y}/\{x, y\}$ is Eulerian and the graph G is not Eulerian. Then the number of ones in each row of $A_{x,y}/\{x, y\}$ is even and there is a row of A consisting of odd number of 1s. In fact, there is a nonzero even number of such rows. Note that the number of 1s in a row of A which corresponds to a vertex other than an endvertex of x or y remains the same as the number of 1s in the corresponding row of $A_{x,y}/\{x, y\}$. Therefore each of the rows of A corresponding to a vertex other than an endvertex of x or y contains even number of 1s. Obtain $A_{x,y}/\{x, y\}$ from $A_{x,y}$ firstly contracting x by pivoting on the nonzero entry that lies in the newly added row (to A)and then contracting y by pivoting on the nonzero entry of a row that has an odd number of 1s. It follows that each row of A corresponding to the end vertices of x and y consists of odd number of 1s. Thus the odd vertices of G are precisely the end vertices of x and y. □

3. The splitting of graphic matroids

In this section, we investigate the minimal graphs corresponding to the four matroids F_7, F_7^*, $M^*(K_{3,3})$ and $M^*(K_5)$, the excluded minors for the class of binary matroids, and give a proof of Theorem 1.2. In particular, the graphs to be examined have at most two more edges than each of the above matroids. We found that there are in all nine minimal graphs, as shown in Figure 3; the graphs G_1 and G_2 correspond to F_7, G_3 corresponds to F_7^*, the four graphs G_4, G_5, G_6 and G_7 correspond to $M^*(K_{3,3})$, and the graphs G_8 and G_9 correspond to $M^*(K_5)$.

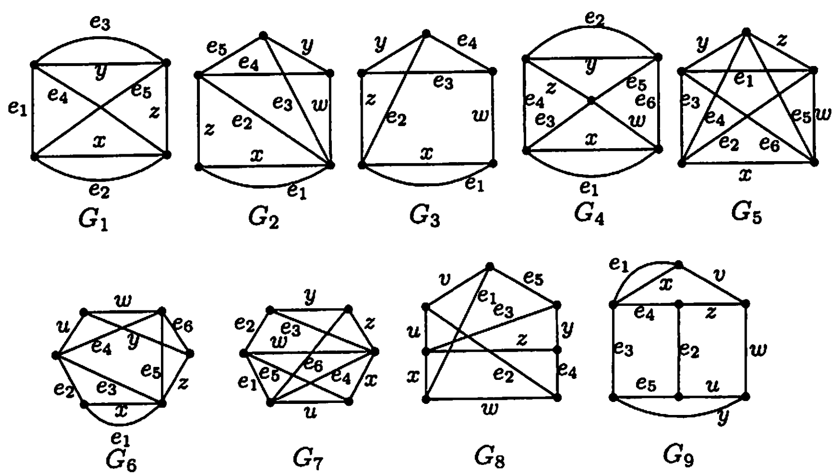

Figure 3

In the following lemma, we characterize minimal graphs corresponding to the Fano matroid F_7.

Lemma 3.1. A graph is minimal with respect to the matroid F_7 if and only if it is isomorphic to one of the two graphs G_1 and G_2 of Figure 3.

Proof. Observe that $M(G_1)_{x,y}/\{x\} \cong F_7$ and $M(G_2)_{x,y}/\{x,y\} \cong F_7$. Therefore G_1 and G_2 are minimal with respect to F_7.

Conversely, let G be a minimal graph with respect to F_7 and let x and y be edges of G such that either $M(G)_{x,y}/\{x\} \cong F_7$ or $M(G)_{x,y}/\{x,y\} \cong F_7$.

Case (i). Let $M(G)_{x,y}/\{x\} \cong F_7$. Since the rank of F_7 is 3, $r(M(G)_{x,y}) = 4$ and $|E(M(G)_{x,y})| = 8 = |E(M(G))|$. By Proposition 2.2 (i), $r(M(G)_{x,y}) = r(M(G)) + 1$ so, $r(M(G)) = 3$. Further, by Proposition 2.6, G is 2-connected and contains at most two pairs of parallel edges. We conclude that $|V(G)| = 4$ and $|E(G)| = 8$. Thus the simplification of G is the complete graph on four vertices. Moreover, G is obtained by adding two nonadjacent parallel edges to its simplification. Hence G is isomorphic to the graph G_1 of Figure 2.

Case (ii). Assume that $M(G)_{x,y}/\{x,y\} \cong F_7$. In this case, $r(M(G)_{x,y}) = 5$ and $|E(M(G)_{x,y})| = 9 = |E(M(G)|$. By Proposition 2.2 (i), $r(M(G)) = 4$. Further, by Theorem 2.6(iii), G is 2-connected. Accordingly, $|V(G)| = 5$ and $|E(G)| = 9$. Now, by Theorem 2.6(v), x and y must be edges in a cycle of G with five edges. Two of the possible edges from endvertices of x to endvertices of y can't be edges of G or a four cycle containing x and y is obtained. Thus the simplification of G is as given by G_2 deletes a parallel edge. Put the parallel edge back in the only possible place to get G_2. This completes the proof of the lemma. □

Lemma 3.2. A graph G is minimal with respect to the matroid F_7^* if and only if it is isomorphic to the graph G_3.

Proof. Observe that $M(G_3)_{x,y}/\{x\} \cong F_7^*$. Therefore G_3 is minimal with respect to F_7^*.

Conversely, let G be a minimal graph with respect to F_7^* and let x and y be edges of G such that either $M(G)_{x,y}/\{x\} \cong F_7^*$ or $M(G)_{x,y}/\{x,y\} \cong F_7^*$. We note that the rank of F_7^* is four and every circuit of it has four elements.

Case (i). Suppose that $M(G)_{x,y}/\{x\} \cong F_7^*$. Then $r(M(G)_{x,y}) = 5$ and $|E(M(G)_{x,y})| = 8 = |E(M(G))|$. By Proposition 2.2(i), $r(M(G)_{x,y}) = r(M(G)) + 1$ so, $r(M(G)) = 4$. Further, by Theorem 2.6, G is 2-connected and has no edges in series. Thus $|V(G)| = 5$, $|E(G)| = 8$ and the degree sequence of G is (4, 3, 3, 3, 3). If G is simple then it must be isomorphic to $G_5 \setminus \{e_2, e_6\}$ (see [3], p. 217). However, every pair of non-adjacent edges in this graph is contained in a 4-circuit which will reduce to a 3-circuit of $M(G)_{x,y}/\{x\}$, that is, of F_7^*, a contradiction since every circuit of F_7^* has size four. Hence G cannot be simple. Suppose G is a multigraph. Then it must be obtained from a simple graph with 5 vertices and 7 edges by putting an edge in parallel. There is just one simple graph of this type (see [3], p. 217) and that is isomorphic to $G_5 \setminus \{e_2, e_4, e_6\}$. This graph has two edges in series. The graph G is obtained from this graph by putting an edge parallel to one of the two edges which are in series. Then G is isomorphic to the graph G_3 of Figure 3.

Case (ii). Let $M(G)_{x,y}/\{x,y\} \cong F_7^*$. Then we have $r(M(G)_{x,y}) = 6$, $|E(M(G_{x,y}))| = 9 = |E(M(G))|$. Moreover, by Proposition 2.2(i),

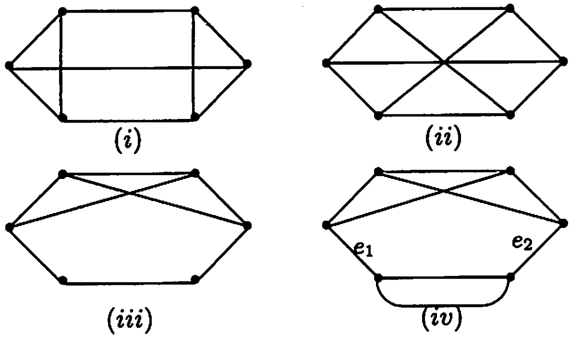

Figure 4

$r(M(G)) = 5$ and, by Theorem 2.6(i), G is 2-connected. Consequently, G has 6 vertices, 9 edges and there are no edges in series. Thus the degree sequence for G is (3, 3, 3, 3, 3, 3). If G is simple, then it must be one of the two graphs (i) and (ii) of Figure 4 (see [3], p. 222). But every pair of non-adjacent edges in each of the two graphs is contained in either a 4-circuit or a 5- circuit which will give rise to a 2-circuit or a 3-circuit of $M(G)_{x,y}/\{x,y\}$ which is impossible. Hence G cannot be simple. If G is a multigraph then, by Theorem 2.6(v), it has at most one pair of parallel edges. G can be obtained from a simple graph, say G', with 6 vertices and 8 edges, by adding an edge parallel to the edge having endvertices of degree 2. The graph G' has the degree sequence (3, 3, 3, 3, 2, 2) and the vertices of degree 2 must be adjacent. Indeed, the graph G' is isomorphic to the graph (iii) of Figure 4 (see [3], p.221). The corresponding graph G is isomorphic to the graph (iv) of Figure 4. However, the edges e_1 and e_2 of this graph are in series and hence it is not minimal with respect to F_7^*.

□.

The following lemma characterizes minimal graphs corresponding to the matroid $M^*(K_{3,3})$.

Lemma 3.3. A graph is minimal with respect to the matroid $M^*(K_{3,3})$ if and only if it is isomorphic to one of the four graphs G_4, G_5, G_6 and G_7 of Figure 3.

Proof. One can check that none of the graphs G_4, G_5, G_6 and G_7 of Figure 3 contains edges in series. Further, each of the matroids $M(G_4)_{x,y}/\{x\}$, $M(G_5)_{x,y}/\{x\}$, $M(G_6)_{x,y}/\{x,y\}$, and $M(G_7)_{x,y}/\{x,y\}$ is isomorphic to

$M^*(K_{3,3})$. Therefore, each of these graphs is minimal with respect to $M^*(K_{3,3})$.

Now, suppose G is a minimal graph with respect to $M^*(K_{3,3})$ and let x and y be the edges of G with the property that $M(G)_{x,y}/\{x\} \cong M^*(K_{3,3})$ or $M(G)_{x,y}/\{x,y\} \cong M^*(K_{3,3})$.

Case (i). $M(G)_{x,y}/\{x\} \cong M^*(K_{3,3})$. Since the rank of $M^*(K_{3,3})$ is four, $M(G)_{x,y}$ is a matroid of rank 5 and $|E(M(G)_{x,y})| = 10$. In the light of Proposition 2.2(i), the matroid $M(G)$ has rank 4 and its ground set has 10 elements. By Theorem 2.6(iii), G is 2-connected. We conclude that G is a graph with 5 vertices, 10 edges and there are no edges in series. Further, by Proposition 2.7(i), G must be Eulerian. If G is simple, then it must be isomorphic to G_5 which is the complete graph on five vertices (see [2], p.217). Suppose G is a multigraph and has one pair of parallel edges. Then it must be obtained from the complete graph on five vertices by removing one of its edges and adding a parallel edge to an edge having each end vertex of degree four. But then such a graph is not Eulerian. Hence G cannot have just one pair of parallel edges. Suppose G has two pairs of parallel edges. In this case, G is obtained from the complete graph on five vertices by removing two nonadjacent edges and then putting two parallel edges in such a way that the resulting graph is Eulerian. The graph G must be isomorphic to the graph G_4 of Figure 3.

Case (ii). Assume that $M(G)_{x,y}/\{x,y\} \cong M^*(K_{3,3})$. Then $M(G)_{x,y}$ is a matroid with 11 elements and has rank 6. In the light of Proposition 2.2(i), the rank of $M(G)$ is 5 and its ground set has 11 elements. In view of Theorem 2.6, G must be a 2-connected graph with $|V(G)| = 6$ and $|E(G)| = 11$. Further, by Proposition 2.7(ii), G is Eulerian or precisely the end vertices of x and y are of odd degree. Figure 5 shows all 2-connected simple graphs on 6 vertices and 11 edges ([3], p.223).

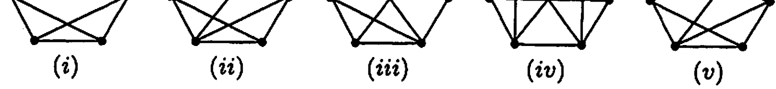

Figure 5

Every pair of non-adjacent edges of the graph (iii) of Figure 5 is contained in a 4-circuit and therefore, by Theorem 2.6(v), it is not minimal.

The graph *(ii)* of Figure 5 is nothing but the graph G_7 of Figure 3. Each of the remaining graphs of Figure 5 does not satisfy either of the two properties stated in Proposition 2.7(ii). Hence none of these graphs is minimal with respect to $M^*(K_{3,3})$.

Now suppose that the graph G is a multigraph. Then, by Theorem 2.6(v), it has at most one pair of parallel edges and, by minimality, has no edges in series. Thus G is obtainable from a connected simple graph with 6 vertices and 10 edges, by putting a parallel edge in a suitable place. The connected simple graphs each with 6 vertices and 10 edges are shown in Figure 6 ([3], p.223).

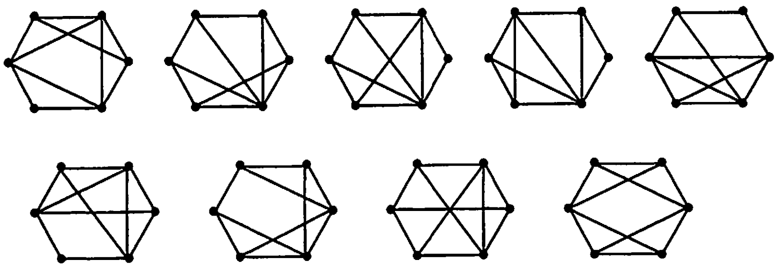

Figure 6

Consider the first graph in the first row of Figure 6. This has a vertex of degree two. If a parallel edge is to be put up in this graph to obtain the graph G the edge must be put up parallel to an edge which has one end vertex of degree two. There are two ways, to put a parallel edge in this graph. In fact, the two ways are symmetric and give rise to two graphs each of which is isomorphic to the graph G_6 of Figure 3. In each of the remaining graphs of Figure 6 if we put a parallel edge in all possible ways, we obtain graphs each of which does not satisfy either of the two conditions stated in Proposition 2.7(ii). Thus none of these graphs is minimal with respect to $M^*(K_{3,3})$. This completes the proof. □

The following lemma characterizes the minimal graphs corresponding the matroid $M^*(K_5)$.

Lemma 3.4. A graph is minimal with respect to $M^*(K_5)$ if and only if it is isomorphic to G_8 or G_9.

Proof. Observe that neither of the graphs G_8 and G_9 of Figure 3 has edges in series. Moreover, each of the matroids $M(G_8)_{x,y}/\{x\}$ and $M(G_9)_{x,y}/\{x\}$ is isomorphic to $M^*(K_5)$. Thus, each of the graphs G_8 and G_9 of Figure 3 is minimal with respect to $M^*(K_5)$.

Conversely, suppose that G is a minimal graph with respect to $M^*(K_5)$ and let x, y be a pair of edges in G such that either $M(G)_{x,y}/\{x\} \cong M^*(K_5)$ or $M(G)_{x,y}/\{x,y\} \cong M^*(K_5)$. We note that $M^*(K_5)$ is a bipartite matroid of rank 6. In fact, it has 5 circuits of size 4 and 10 circuits of size 6.

Case (i). Suppose $M(G)_{x,y}/\{x\} \cong M^*(K_5)$. Then $M(G)_{x,y}$ is a matroid on a 11-element set and has rank 7. By Proposition 2.2(i), $M(G)$ has rank 6 and its ground set has 11 elements. By Theorem 2.6, G is 2-connected, has no edges in series and hence no vertex of degree 2. Consequently, G has 7 vertices, 11 edges and the degree sequence (4, 3, 3, 3, 3, 3, 3). From the nature of circuits of $M^*(K_5)$ and the definition of $M(G)_{x,y}$, it follows that G cannot have (i) two or more edge-disjoint triangles and (ii) two or more pairs of parallel edges. Suppose G is simple. The non-isomorphic simple graphs each of which has degree sequence (4, 3, 3, 3, 3, 3, 3) can be constructed from the simple graphs with degree sequence (3, 3, 2, 2, 2, 2) by adding a vertex adjacent to vertices of degree two. There are precisely four non-isomorphic simple graphs each with degree sequence (3, 3, 2, 2, 2, 2) ([3], p. 220). All of the non-isomorphic simple graphs each with degree sequence (4, 3, 3, 3, 3, 3, 3) are shown in Figure 7. However, each of the graphs (i), (ii) and (iii) of Figure 7 has two or more triangles and therefore none of these graphs is minimal with respect to $M^*(K_5)$. The graph (iv) of Figure 7 is the graph G_8 of Figure 3.

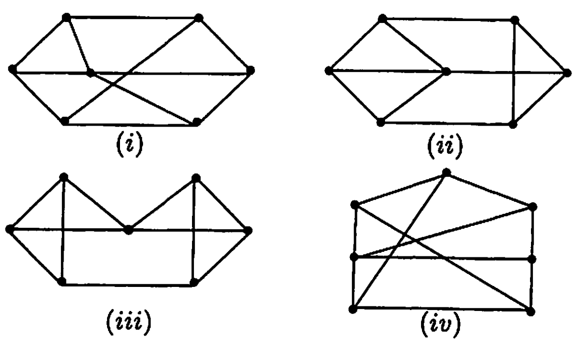

Figure 7

If G is not simple then, as noted above, it has just one pair of parallel edges. Thus G can be obtained from a simple graph whose degree sequence is (3, 3, 3, 3, 3, 3, 2) or (4, 3, 3, 3, 3, 2, 2) by adding an edge in parallel. However, any such graph obtained from a simple graph having degree sequence (4, 3, 3, 3, 3, 2, 2) contains a pair of edges in series. Hence G cannot arise from these type of graphs. Now all non-isomorphic simple graphs with degree sequence (3, 3, 3, 3, 3, 3, 2) can be obtained from the non-isomorphic simple graphs each with degree sequence (3, 3, 2, 2, 2, 2) (see [3], p.220-221). There are in all five non-isomorphic simple graphs each with degree sequence (3, 3, 3, 3, 3, 3, 2). The non-isomorphic multigraphs obtained from each of the five graphs, by adding a parallel edge to an edge having an endvertex of degree 2, are shown in Figure 8.

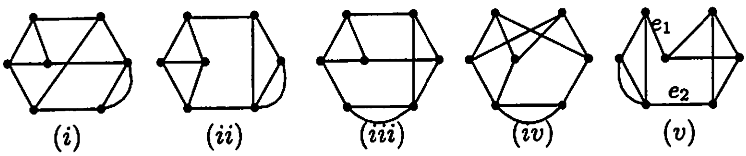

Figure 8

Now, each of the graphs (ii), (iii) and (v) of Figure 8 contains a pair of edge-disjoint triangles and hence none of them is minimal. In the graph (iv) of Figure 8, one of the two edges x and y, say x must be in a 2-circuit and the edge y being non-adjacent with x must be contained in a 4-circuit which is disjoint from the 2-circuit. But then $M(G)_{x,y}/\{x\}$ will contain a circuit whose cardinality is different from 4 and 6, which is impossible. Thus G must be isomorphic to the graph (i) of Figure 8 which is nothing but the graph G_9 of Figure 3.

Case (ii). Assume that $M(G)_{x,y}/\{x,y\} \cong M^*(K_5)$. Since $M^*(K_5)$ has rank 6, $M(G)_{x,y}/\{x,y\}$ is a matroid of rank 6 and its ground set has 10 elements. Consequently, $M(G)_{x,y}$ is a matroid of rank 8 on a set with 12 elements. By Proposition 2.2(i), $M(G)$ has rank 7 and its ground set has 12 elements. By Proposition 2.6(iii), G is 2-connected. Thus the graph G must have 8 vertices, 12 edges and no edges in series. Accordingly, the degree sequence of G is (3, 3, 3, 3, 3, 3, 3, 3). Suppose G is simple. Then G is obtained from a simple graph having degree sequence (2, 2, 2, 2, 2, 2), (3, 2, 2, 2, 2, 1), (3, 3, 2, 2, 1, 1) or (3, 3, 3, 1, 1, 1) by adding two more vertices adjacent to vertices of degree two and one. There are 11 non-isomorphic graphs each of which has one of the above degree sequences (see [3], p. 221, 222). Figure 9 shows all non-isomorphic 2-connected simple graphs each of which has degree sequence (3, 3, 3, 3, 3, 3, 3, 3) and no pair of edges in series.

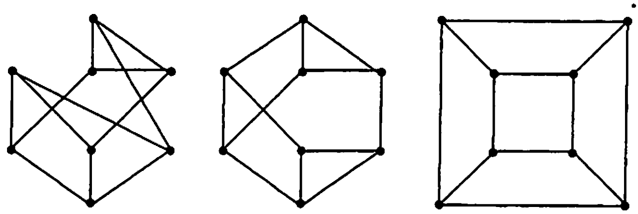

Figure 9

The matroid $M^*(K_5)$ has only even sized circuits and contains 5 circuits each of size four. Therefore the nonadjacent edges x and y satisfy the following properties.

i) If G contains a triangle then precisely one of x and y must be an edge of the triangle.

ii) G has no cycle of odd size which contains both x and y.

iii) The total number of 4-cycles containing neither x nor y and 6-cycles containing both x and y should not exceed 5.

iv) By Proposition 2.6(v), G has no 4-cycle which contains both x and y.

Now one can check that neither of the three graphs of Figure 9 contains a pair of nonadjacent edges x and y which satisfy the above properties. Thus there cannot be a simple graph G with the condition $M(G)_{x,y}/\{x,y\} \cong M^*(K_5)$.

Suppose G is a multigraph. Then, by Theorem 2.6(v), it has at most one pair of parallel edges. Accordingly, G is obtained from a connected simple graph, say H, with 8 vertices and 11 edges by adding an edge in parallel. Further, the graph H must have two adjacent vertices each of which has degree two and the edge to be added must be parallel to the edge joining these two vertices. But the two edges of G adjacent to the pair of parallel edges are in series. We conclude that there is no graph G such that $M(G)_{x,y}/\{x,y\} \cong M^*(K_5)$. □

Now we use Lemmas 3.1, 3.2, 3.3 and 3.4 to prove Theorem 1.2.

Proof of Theorem 1.2. Let M be a graphic matroid and let G be a graph such that $M = M(G)$. On combining Corollary 2.5 and Lemmas

3.1, 3.2, 3.3, and 3.4, it follows that $M(G)_{x,y}$ is graphic for every pair $\{x, y\}$ of edges of G if and only if $M(G)$ has no minor isomorphic to any of the matroids $M(G_i)$, $i = 1, 2, ..., 9$, where the graphs G_i are as shown in Figure 3. However, we have $M(G_3) \cong M(G_2) \setminus e_2 \cong M(G_4) \setminus \{e_2, w\} \cong M(G_6)/e_2 \setminus \{e_3, e_5\} \cong M(G_7)/e_1 \setminus \{w, e_4\} \cong M(G_9)/\{v, z\} \setminus e_1$. Thus, $M(G)_{x,y}$ is graphic if and only if $M(G)$ has no minor isomorphic to any of the matroids $M(G_1)$, $M(G_3)$, $M(G_5)$ and $M(G_8)$. But the graphs G_1, G_3, G_5 and G_8 are precisely the graphs given in the statement of the theorem.

□

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