Supereulerian index is stable under contractions and closures

Liming Xiong[*]

Department of Mathematics, Beijing Institute of Technology
Beijing 100081, P.R. China
Department of Mathematics, Jiangxi Normal University
Nanchang 330027, P.R. China

Mingchu Li[†]

School of Software, Dalian University of Technology
Dalian 116024, P.R. China

Abstract

The supereulerian index of a graph G is the smallest integer k such that the k-th iterated line graph of G is supereulerian. We first show that adding an edge between two vertices with degree sums at least three in a graph cannot increase its supereulerian index. We use this result to prove that the supereulerian index of a graph G will not be changed after either of contracting an $A_G(F)$-contractible subgraph F of a graph G and performing the closure operation on G (if G is claw-free). Our results extend a Catlin's remarkable theorem [4] relating that the supereulericity of a graph is stable under the contraction of a collapsible subgraph.

Keywords: supereulerian index, stable property, closure of a graph, contractible graph, collapsible graph, claw-free graph

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1 Introduction

Throughout this article we consider only finite undirected loopless graphs. However, except for Section 4, we admit G to have multiple edges. We use Bondy & Murty [1] for terminology and notation not defined here. For a graph G, set $V_i(G) = \{v \in V(G) : d_G(v) = i\}$ (where i is an integer and $d_G(v)$ is the degree of a vertex v in G) and $W(G) = V(G) \setminus V_2(G)$. A *branch* in G is a nontrivial path with end vertices in $W(G)$ and with internal vertices, if any, of degree 2 in G. If a branch has length 1, then it has no internal vertex. Let $\mathcal{B}(G)$ denote the set of branches of G, and $\mathcal{B}_1(G)$ the subset of $\mathcal{B}(G)$ in which every branch has an end in $V_1(G)$. If $P = x_1, \cdots, x_k$ is a path in a graph G and $S, T \subset G$ are subgraphs of G, then we say that P is an *(S,T)-path* if $x_1 \in V(S)$ and $x_k \in V(T)$. The *distance* of two subgraphs $S, T \subset G$ (denoted by $d_G(S,T)$) is the minimum length of an (S,T)-path. For a subgraph H of G let $\mathcal{B}_H(G)$ be the set of those branches of G which have all edges in H.

If G is a graph and $k \geq 0$ an integer, then $SU_k(G)$ denotes the set of all subgraphs H of G satisfying the following five conditions:

(I) $d_H(x) \equiv 0 \pmod 2$ for every $x \in V(H)$;

(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(III) $d_G(H_1, H - H_1) \leq k$ for every subgraph H_1 of H;

(IV) $|E(b)| \leq k+1$ for every branch $b \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$;

(V) $|E(b)| \leq k$ for every branch $b \in \mathcal{B}_1(G)$.

The line graph $L(G)$ of a graph $G = (V(G), E(G))$ is the graph with the vertex set $E(G)$ and two vertices are adjacent in $L(G)$ if and only if the corresponding edges of G have a vertex in common. $L^k(G)$ is defined recursively by $L^0(G) = G$ and $L^k(G) = L(L^{k-1}(G))$. A graph is called a *supereulerian* if it contains a spanning eulerian subgraph. If $d_G(H_1, H -$

$H_1) = 0$ for every subgraph H_1 of H, then H is a connected graph. Thus a graph G is supereulerian if and only if $SU_0(G) \neq \emptyset$. The following theorem generalizes this result to the k-th iterated line graph $L^k(G)$ of a graph G.

Theorem 1.1. [16] Let G be a connected graph with at least three edges and let $k \geq 0$ be an integer. Then $L^k(G)$ is supereulerian if and only if $SU_k(G) \neq \emptyset$.

The *supereulerian index* of a graph G, denoted by $s(G)$, is the smallest integer k such that $L^k(G)$ is supereulerian. Obviously, the line graph $L(G)$ of a supereulerian graph is supereulerian and is hamiltonian. Thus, Theorem 1.1 equivalently states that $s(G) \leq k$ if and only if $SU_k(G) \neq \emptyset$ for an integer $k \geq 0$ and for any graph G.

A graph G is collapsible if for any subset $X \subset V(G)$ of even size, there is a connected spanning subgraph $H \subset G$ such that the set of vertices v with odd degree $d_H(v)$ is precisely X. Catlin [4] obtained the following theorem which is very important in supereulerian graph theory.

Theorem 1.2. [4] Let G be a graph and H a collapsible subgraph of G. Then G is supereulerian if and only if G/H is supereulerian, i.e., $s(G) = 0$ if and only if $s(G/H) = 0$, where G/H is the graph obtained from G by identifying the vertices of H as a (new) vertex, by discarding the resulting loops but keeping all multiple edges.

If F is a subgraph of a graph G, then a vertex x is said to be a *attachment vertex* of F in G if $x \in V(F)$ and x has a neighbor in $V(G) \setminus V(F)$. $A_G(F)$ denotes the set of all attachment vertices of a subgraph F in G, and $G|_F$ denotes the graph obtained from G by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges (i.e. edges with one vertex of degree 1) attached to v_F, and we say that $G|_F$ is obtained from G by *contracting* the subgraph F (observe that $|E(G)| = |E(G|_F)|$). Let X be a subset of $V(G)$ and $\mathcal{A}$ a partition of X, Then $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in $E(G)$) such that a_1, a_2 are in the same element of $\mathcal{A}$, and $G^{\mathcal{A}}$ denotes the graph with vertex set $V(G^{\mathcal{A}}) = V(G)$ and edge set $E(G^{\mathcal{A}}) = E(G) \cup E(\mathcal{A})$. Note that $E(G)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e., if $e_1 = a_1a_2 \in E(G)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $G^{\mathcal{A}}$. A graph G is *X-contractible* if, for every even subset $Y \subset X$ and for every partition $\mathcal{A}$ of Y into two-element subsets, the graph $G^{\mathcal{A}}$ has a dominating closed trail (abbreviated DCT) containing all vertices of X and all edges of $E(\mathcal{A})$. Note that this definition allows A to be empty, in which case $G^{\mathcal{A}} = G$. Also, if G is X-contractible, then G is X'-contractible for any $X' \subset X$, since every subset Y of X' is a subset of X. Let

$$d_T(G) = \max\{\ |S| : S \subset E(G) \text{ and there is a closed trail } T \subset G \text{ such that every edge } e \in S \text{ has at least one vertex on } T\}.$$

Theorem 1.3. [14] Let F be a connected graph and let $A \subset V(F)$. Then F is A-contractible if and only if $d_T(G) = d_T(G|_F)$ for every graph G with $F \subset G$ and $A_G(F) = A$.

Note that $G|_F$ may contain multiple edges even if G is simple. However, it is easy to observe that a multiple edge is a contractible subgraph and hence, by a series of subsequent contractions, it is always possible to reduce $G|_F$ to a certain simple graph G' with $d_T(G') = d_T(G|_F) = d_T(G)$.

A simple graph G is *claw-free* if G does not contain a copy of $K_{1,3}$ as an induced subgraph. A subgraph is called a *factor* if it contains all vertices of G. A [2,4]-factor is a connected factor in which every vertex has degree either two or four. It is well-known that every line graph is claw-free. The following result explore the relation between supereulerian claw-free graphs and connected [2,4]-factors in claw-free graphs.

Theorem 1.4. [6] [9] Let G be a claw-free graph. Then G is supereulerian if and only if G has an [2,4]-factor.

Let G be a claw-free graph. A vertex $x \in V(G)$ is *locally connected* if $G[N(x)]$ is a connected graph. For $x \in V(G)$, the graph G'_x with vertex set $V(G'_x) = V(G)$ and edge set $E(G'_x) = E(G) \cup \{uv \mid u, v \in N(x)\}$ is called the *local completion of G at x*. It was shown in [12] that the local completion of a claw-free graph G at x is again claw-free, and if x is a locally connected vertex, then $c(G'_x) = c(G)$ (where $c(G)$ denotes the circumference of G, i.e. the length of a longest cycle in G). Let $\mathrm{cl}(G)$ be a graph obtained from G by recursively performing the local completion operation at locally connected vertices with noncomplete neighborhood, as long as this is possible. The graph $\mathrm{cl}(G)$ is called the *closure* of the graph G. The following theorem summarizes basic properties of the closure operation.

Theorem 1.5. [12] Let G be a claw-free graph. Then

- $\mathrm{cl}(G)$ is uniquely determined,
- $c(\mathrm{cl}(G)) = c(G)$,
- $\mathrm{cl}(G)$ is the line graph of a triangle-free graph,
- G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian.

If $\mathcal{C}$ is a class of graphs, Γ is a graph operation on $\mathcal{C}$ and $\mathcal{P}$ is a graph property, then $\mathcal{P}$ is said to be *stable under* Γ if, for any $G \in \mathcal{C}$, G has $\mathcal{P}$ if and only if $\Gamma(G)$ has $\mathcal{P}$. Similarly, a graph invariant π is said to be *stable under* Γ if for any $G \in \mathcal{C}$ we have $\pi(G) = \pi(\Gamma(G))$. In this terminology, Theorem 1.2 say that the existence of a spanning closed eulerian subgraph is stable under the operation of contraction of a collapsible subgraph H, and Theorem 1.5 say that the circumference and hamiltonicity are stable under the closure operation on claw-free graphs. Very recently, [17] extended the result and obtained that the hamiltonian index (i.e., the smallest intger k such that $L^k(G)$ is hamiltonian) is stable under the closure operation on claw-free graphs. Stability of some further graph properties and invariants under the closure operation was studied e.g. in [2], [13], [8] or [11] (also see the survey paper [3]).

In this paper, our motivation is to extend Theorem 1.2. We investigate the stability of the supereulerian index under the operation of contraction of an $A_G(F)$-contractible subgraph F and under the closure operation on claw-free graphs.

2 Supereulerian index of a subgraph

In this section, we will show that adding an edge between two vertices with degree sums at least three in a graph cannot increase its supereulerian index.

Theorem 2.1. Let G be a connected graph with at least three edges other than a path. Then $s(G) \geq s(G+ab)$ for any two vertices a, b of $V(G)$ with $d_G(a) + d_G(b) \geq 3$.

Proof. Let $G' = G + ab$. Then $s(G') \geq 0$ since $s(G) \geq 0$. Suppose that $d_G(a) + d_G(b) \geq 3$. By Theorem 1.1, there is a subgraph $H \in SU_{s(G)}(G)$. Let H' be the subgraph of G' with the vertex set $V(H') = V(H) \cup \{v \in \{a,b\} : d_{G'}(v) \geq 3\}$ and the edge set $E(H') = E(H)$.

We will show that $H' \in SU_{s(G)}(G')$, i.e., H' satisfies the conditions (I) - (V) of the definition of $SU_{s(G)}(G')$ (for the graph G' and $k = s(G)$). Obviously, H' satisfies conditions (I) and (II).

If one of a, b has degree 1 in G, say, $d_G(a) = 1$, then $d_G(b) \geq 2$ since $d_G(a) + d_G(b) \geq 3$. The branch (denoted by P) of $\mathcal{B}_1(G)$ containing a will become a new branch $P' = Pb$ in $\mathcal{B}(G') \setminus (\mathcal{B}_{H'}(G') \cup \mathcal{B}_1(G'))$ of length $|E(P)| + 1 \leq s(G) + 1$. The other branches of $\mathcal{B}(G') \setminus \mathcal{B}_{H'}(G)$ are the same as those of $\mathcal{B}(G) \setminus \mathcal{B}_H(G)$ except the only case that $d_G(b) = 2$ and b is not in $V(H)$; in this exceptional case, the branch containing b turns into two shorter branches in $\mathcal{B}(G') \setminus \mathcal{B}_{H'}(G')$. This shows that H' satisfies (IV) and (V). If both a and b have degree at least 2 in G, then the branches in $\mathcal{B}(G') \setminus \mathcal{B}_{H'}(G')$ are the same as those in $\mathcal{B}(G) \setminus \mathcal{B}_H(G)$ except the case that

a or b (or both) have degree exactly 2 in G and they are not in $V(H)$; in this exceptional case, the branches in $\mathcal{B}(G') \setminus \mathcal{B}_{H'}(G')$ will be shorter than those in $\mathcal{B}(G) \setminus \mathcal{B}_H(G)$. This shows that H' satisfies (IV) and (V).

It remains to show that H' satisfies (III). Suppose there is a subgraph H_1' of H' such that $d_{G'}(H_1', H' - H_1') \geq s(G) + 1 \geq 2$. It is easy to see that $V(H_1') \cap V(H)$ and $V(H' - H_1') \cap V(H)$ cannot be both empty. Suppose first that $V(H_1') \cap V(H) = \emptyset$ and $V(H' - H_1') \cap V(H) \neq \emptyset$ (note that the case that $V(H_1') \cap V(H) \neq \emptyset$ and $V(H' - H_1') \cap V(H) = \emptyset$ is symmetric). Then $V(H_1') \subseteq \{a, b\}$. If $V(H_1') = \{a, b\}$, then $d_G(a), d_G(b) \leq 2$ since $\{a, b\} \cap V(H) = \emptyset$ and H satisfies (II). By the definition of $H', d_{G'}(a), d_{G'}(b) \geq 3$. Hence $d_G(a) = d_G(b) = 2$, implying that both a and b are on some branches of $\mathcal{B}(G) \setminus \mathcal{B}_H(G)$. Since H satisfies (IV) and (V), $d_G(\{a, b\}, H) \leq s(G)$; in this case, any shortest $(\{a, b\}, H)$-path in G is also an $(H_1', H' - H_1')$-path in G'. Hence $d_{G'}(H_1', H' - H_1') \leq d_G(\{a, b\}, H) \leq s(G)$, a contradiction. This implies that H_1' has exactly one vertex, say, $V(H_1') = \{a\}$. Similarly, $d_G(\{a\}, H) \leq s(G)$ and any shortest $(\{a\}, H)$-path in G is an $(H_1', H' - H_1')$-path in G', implying that $d_{G'}(H_1', H' - H_1') \leq d_G(\{a\}, H) \leq s(G)$, a contradiction. Finally, suppose that both $V(H_1') \cap V(H)$ and $V(H' - H_1') \cap V(H)$ are nonempty, and set $H_1 = H_1' \cap H$. In this case, any shortest $(H_1, H - H_1)$-path in G is also an $(H_1', H' - H_1')$-path in G'. Hence $d_{G'}(H_1', H' - H_1') \leq d_G(H_1, H - H_1) \leq s(G)$, a contradiction. This shows that H' satisfies (III). Thus $H' \in SU_{s(G)}(G')$, implying $s(G') \leq s(G)$. Therefore the proof of Theorem 1 is completed. □

Remark 2.2. An example from [17] shows that the assumption $d_G(a) + d_G(b) \geq 3$ in Theorem 2.1 cannot be relaxed.

The following corollary is easily obtained from Theorem 2.1.

Corollary 2.3. Let G be a connected graph with at least three edges other than a path and G' be a graph obtained from G by recursively adding the edges whose ends a and b satisfy $d(u) + d(v) \geq 3$. Then $s(G) \geq s(G')$.

3 Supereulerian index is stable under contraction

We begin this section with the following well-known result which will be used in our proofs. Here a bond of a graph G is a minimal edge cut set. Note that a graph may have different bonds with different values.

Theorem 3.1. [10] A connected graph is eulerian if and only if each bond contains an even number of edges.

The following lemma will be used in our proof, and its proof is easy and omitted here.

Lemma 3.2. Let G be a graph. Then for any $H \in EU_{h(G)}(G)$ and any subgraph H_1 of H, if the distance between H_1 and $H - H_1$ is at least 2, then the shortest path of G between H_1 and $H - H_1$ is a branch of G, whose ends are adjacent in G.

If G is a supereulerian graph (i.e. $s(G) = 0$) and $F \subset G$ is a nontrivial subgraph of G, then $G|_F$ cannot be supereulerian (since it has connectivity 1), and it is easy to observe that any closed spanning eulerian subgraph H in G turns into a DCT in $G|_F$. Hence $s(G) = 0$ implies $s(G|_F) = 1$ for any nontrivial subgraph $F \subset G$. However, the following theorem shows that for $s(G) \geq 1$, i.e. for nonsupereulerian graphs, the supereulerian index is stable under contraction of a contractible subgraph.

Theorem 3.3. Let G be a nonsupereulerian graph other than a path and F be an $A_G(F)$-contractible subgraph of G. Then $s(G) = s(G|_F)$.

Proof. Let $G' = G|_F$. Obviously, $s(G) \geq 1$ if and only if $s(G') \geq 1$. It is sufficient to consider the case that $s(G) \geq 1$. Hence $s(G') \geq 1$. We first prove that $s(G') \leq s(G)$. By Theorem 1.1, we can take a subgraph H in

$SU_{s(G)}(G)$. Let H' be the graph obtained from $H|_F$ by deleting the new pendant edges. We prove that H' is in $SU_{s(G)}(G')$, i.e., H' satisfies the conditions of the definition of $SU_{s(G)}(G')$ for the graph G' and $k = s(G)$. By Theorem 3.1, H' satisfies (I) and (II) in the definition of $SU_{s(G)}(G')$. From the definitions of $A_G(F)$ and A-contractible graph, every vertex in $A_G(F)$ has degree at least 3 in G. Using Lemma 3.2, we easily obtain that H' satisfies also the other conditions in the definition of $SU_G(G')$, and hence $s(G') \leq s(G)$.

We now prove that $s(G) \leq s(G')$. Since $s(G') \geq 1$, by Theorem 1.1, we can take a subgraph H' in $SU_{s(G')}(G')$. Let $r(x)$ denote the number of branches of $\mathcal{B}_{H'}(G)$ with an end-vertex x. Let

$$V_b(H') = \{x \in F : x \text{ is an endvertex of a branch of } \mathcal{B}_{H'}(G)\}$$

$$V_b^j = \{x \in V_b(H') : r(x) \equiv j \pmod 2\}.$$

Since H' satisfies (I), $\sum_{x\in V_b^1} r(x) + \sum_{x\in V_b^2} r(x) = \sum_{x\in V_b} r(x) = d_{H'}(v_F)$ is even. Since $\sum_{x\in V_b^2} r(x)$ is even, it follows that $\sum_{x\in V_b^1} r(x)$ is also even. Hence $|V_b^1|$ is even. Let $X = V_b^1$. Take one partition $\mathcal{A}$ of X into two-element subsets. Since F is $A_G(F)$-contractible, $F^{\mathcal{A}}$ has a DCT T containing all vertices of $A_G(F)$ and all edges of $E(\mathcal{A})$. Let H be the graph with vertex set $V(H) = V(H') \cup (\bigcup_{i=3}^{\Delta(G)} V_i(G)) \cup V(T)$ and the edge set $E(H) = E(H') \cup (E(T) \setminus E(\mathcal{A}))$.

We prove that $H \in SU_{s(G')}(G)$. Obviously, H satisfies the conditions (I) and (II) in the definition of $SU_{s(G')}(G)$. Since T is a DCT which contains all vertices of $A_G(F)$ and all edges of $E(\mathcal{A})$, by Claim 1, H satisfies (IV) and (V). By Lemma 3.2, H satisfies (III). Hence $H \in SU_{s(G')}(G)$, implying $s(G) \leq s(G')$. This completes the proof of Theorem 3.3. □

The following corollary extends Theorem 1.2, which follows from Theorem 1.2 and Theorem 3.3.

Corollary 3.4. Let G be a graph and H be a collapsible subgraph of G. Then $s(G) = s(G/H)$.

4 Supereulerian index of a claw-free graph is stable under the closure

In this section, assume that all graphs are simple (i.e. without multiple edges). In order to prove our results, we start with the following basic result.

Theorem 4.1. Let G be a claw-free graph and v a locally connected vertex of G such that $G[N(v)]$ is not complete and let $N' = \{xy \notin E(G)\colon x, y \in N(v)\}$ and G' be a graph with vertex set $V(G)$ and the edge set $E(G') = E(G) \cup N'$. Then G has a [2,4]-factor if and only if G' has a [2,4]-factor.

Proof. Obviously, G' has a [2,4]-factor if G has a [2,4]-factor. Now assume that G' has a [2,4]-factor F. We will prove that G has a [2,4]-factor F'. Let $A = \{x \in V(G) : d_F(x) = 4\}$ and $N'' = \{xy \in E(F)\colon xy \in N'\}$. If $N' = \emptyset$, then we are done. Thus $N' \neq \emptyset$. If $A = \emptyset$, then F is a hamiltonian cycle of G'. From [12], G has a hamiltonian cycle and so we are done. Thus $A \neq \emptyset$. Assume that F is a [2,4]-factor with the minimum number $m := |N''|$ and $xy \in N'$. Then $xv, yv \in E(G)$. Furthermore assume that F is a [2,4]-factor with the minimal vertices of degree 4 in F among all [2,4]-factors with m edges of N'. Consider two cases according to the value of $d_F(v)$.

Case 1. $d_F(v) = 2$.

If $xv, yv \notin E(F)$, then removing the edge xy from F and adding xv, yv into F, we obtain a [2,4]-factor F' with $|N''|-1$ edges of N', a contradiction. Thus $xv \in E(F)$ or $yv \in E(F)$ (say $xv \in E(F)$). We can assume that v, x and y are in the same cycle of F since otherwise v and y are in two different cycles of F, respectively, and we have $d_F(x) = 4$. Removing the edges xy, xv and adding the edge vy we obtain a new connected [2,4]-factor F' with $|N''| - 1$ edges of N' and $d_{F'}(x) = 2$, a contradiction. Assume that C is such a cycle of F such that y, x and v is in order on C. Since $A \neq \emptyset$ and $d_F(v) = 2$, there is a vertex w on F such that xvw is a path on C and $w \notin \{x,y,v\}$. Then, from $G[v,w,x,y] \neq K_{1,3}$ and $xy \notin E(G)$, $wx \in E(G)$ or $wy \in E(G)$. Further, we have $wy \in E(G)$ since otherwise $wx \in E(G)$ and then removing the edges xy and wv from F and adding the edges wx and vy into F we obtain a new [2,4]-factor F' with $|N''|-1$ edges of N', a contradiction, which also shows that $wx \notin E(G)$. Since v is locally connected, there is a vertex z such that $xz \in E(G)$ and $z \in N(v)$.

We first have $d_F(z) \neq 2$ since otherwise removing the edges xy from F and adding the new edges vz, vy and xz into F, we obtain a new [2,4]-factor F' with $|N''|-1$ edges of N' such that $d_{F'}(z)=4$ and $d_{F'}(v)=4$, a contradiction. Thus $d_F(z)=4$. Without loss of generality assume that $z \in V(C)$ (the proof of other case $z \notin V(C)$ is similar). Since $wx \notin E(G)$, $w \neq z$. Let $N_F(z)=\{z_1,z_2,z_3,z_4\}$ and the cycle $C=(z_1zz_2...yxvw...z_1)$ and the cycle $(z_3zz_4...z_3)$ meet at the vertex z. Then $z_1z_3, z_1z_4 \notin E(G)$ and $z_2z_3, z_2z_4 \notin E(G)$ since otherwise, e.g., $z_1z_3 \in E(G)$, removing the edges z_1z, z_3z, xy, wv from F and adding z_1z_3, wy, vz, xz into F, we obtain a new [2,4]-factor with $|N''|-1$ edges of N' such that $d_{F'}(z)=4$, a contradiction. It follows that $z_1z_2, z_3z_4 \in E(G)$. Removing the edges xy, zz_1 and zz_2 from F and adding the edges z_1z_2, vz, xz, vy into F, we obtain a new [2,4]-factor with $|N''|-1$ edges of N' such that $d_{F'}(v)=4$ and $d_{F'}(z)=4$ and xvz is a cycle of F', a contradiction. Thus Case 1 is proved.

Case 2. $d_F(v)=4$.

Let $N_F(v)=\{v_1,v_2,v_3,v_4\}$ and $C_1 := (v_1vv_2...v_1)$ and $C_2 := (v_3vv_4...v_3)$ be two cycles of F meeting at the vertex v. If $\{x,y\}=\{v_1,v_2\}$ or $\{x,y\}=\{v_3,v_4\}$ (say $x=v_1$ and $y=v_2$), then, from $G[v,x,y,v_3] \neq K_{1,3}$, $x(=v_1)v_3 \in E(G)$ or $y(=v_2)v_3 \in E(G)$ (say $xv_3 \in E(G)$). Removing v_1v, vv_3 from F and adding v_1v_3 into F we can obtain a new [2,4]-factor F' with m edges of N' such that $d_{F'}(v)=2$, a contradiction. Thus $\{x,y\} \neq \{v_1,v_2\}$ and $\{x,y\} \neq \{v_3,v_4\}$.

Since F is connected, there is a path in F connecting x and one of $\{v_1,v_2,v_3,v_4\}$(say v_1) but not going through v. If $v_1v_2 \in E(G)$, then removing vv_1, vv_2, xy from F and adding the edges v_1v_2, xv, yv into F, we obtain a new [2,4]-factor with $|N''|-1$ edges of N' such that $d_{F'}(v) = 4$, a contradiction. Thus $v_1v_2 \notin E(G)$. Since $G[v, v_1, v_2, v_3] \neq K_{1,3}$, $v_1v_3 \in E(G)$ or $v_2v_3 \in E(G)$. If $v_iv_3 \in E(G)$ for $i = 1, 2$, then removing vv_i, vv_3, xy from F and adding v_iv_3, xv, vy into F we obtain a new [2,4]-factor F' with $|N''| - 1$ edges of N' such that $d_{F'}(v) = 4$, a contradiction. Thus Case 2 is proved. Therefore we complete the proof of Theorem 4.1. $\square$

Theorem 4.1 equivalently states that G is supereulerian if and only if its closure is supereulerian. The following result, which generalizes this result and is the main result of this section, shows that the supereulerian index is stable under the closure operation on claw-free graphs.

Theorem 4.2. Let G be a connected claw-free graph with at least three edges other than a path. Then $s(G) = s(\text{cl}(G))$.

Proof. Since $G \subseteq \text{cl}(G)$ and $V(G) = V(\text{cl}(G))$, we have $s(G) \geq s(\text{cl}(G))$ by the definition of $\text{cl}(G)$ and by Theorem 2.1. It remains to prove that $s(G) \leq s(\text{cl}(G))$. It is sufficient to prove that $s(G) \leq s(G + xy)$ for any pair of vertices x and y with $xy \notin E(G)$ such that they have a common neighbor in G which is a locally connected vertex of G.

Let $G' = G + xy$ and let u be a locally connected common neighbor of x and y. Then there is an (x, y)-path P in $G[N(u)]$ such that $|E(P)| \geq 2$ and $E(P) \subseteq E(G[N(u)])$. It is easy to see that the internal vertices of P have degree at least 3 in G. Thus, by the definition of $\text{cl}(G)$ and by Theorem 2.1, $s(G') \geq s(\text{cl}(G))$. By Theorem 1.1, $SU_{s(G')}(G') \neq \emptyset$. Taking an $H \in SU_{s(G')}(G')$, we construct a subgraph H' of G as follows:

$$V(H') = V(H) \setminus \{v \in \{x, y\} : d_G(v) = 2 \textit{ and } d_{G'}(v) = 3 \textit{ and } d_H(v) = 0\},$$

$$E(H') = \begin{cases} E(H) & \text{if } xy \notin E(H), \\ (E(H) \Delta E(P)) \setminus \{xy\} & \text{if } xy \in E(H). \end{cases}$$

where $E(H)\Delta(E(P)$ denotes the symmetric difference $(E(H) \setminus E(P)) \cup (E(P) \setminus E(H))$.

We show that $H' \in SU_{s(G')}(G)$, i.e., H' satisfies the conditions of the definition of $SU_{s(G')}(G)$ for the graph G and $k = s(G')$. Obviously, H'

satisfies conditions (I) and (II). By the definition of $G + xy$ and Claim 1, all branches of length at least 2 in G are the same as in G' except the case when x or y (or both) have degree 2 in G; in this exceptional case, each of x, y is on a branch in $\mathcal{B}(G) \setminus \mathcal{B}_1(G)$ with adjacent end vertices and length exactly 2. Hence by Claim 1 and Lemma 3.2, H' satisfies the other conditions of the definition of $SU_{s(G')}(G)$, implying $H' \in SU_{s(G')}(G)$. By Theorem 1.1, $s(G) \leq s(G')$, which proves Theorem 4.2. $\square$

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