Enumeration of isomorphism classes of self-orthogonal Latin squares

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Abstract

The numbers of distinct self-orthogonal Latin squares (SOLS) and idempotent SOLS have been enumerated for orders up to and including 9. The isomorphism classes of idempotent SOLS have also been enumerated for these orders. However, the enumeration of the isomorphism classes of non-idempotent SOLS is still an open problem. By utilising the automorphism groups of class representatives from the already enumerated isomorphism classes of idempotent SOLS, we enumerate the isomorphism classes of non-idempotent SOLS implicitly (*i.e.* without generating them). New symmetry classes of SOLS are also introduced, based on the number of allowable transformations that may be applied to a SOLS whithout destroying the property of self-orthogonality, and these classes are also enumerated.

Keywords: Latin square, self-orthogonal Latin square, enumeration, isomorphism classes.

AMS Classification: 05A15, 05B15.

1 Introduction

Given a set of n distinct symbols, a *Latin square* of order n is an $n \times n$ array containing each symbol exactly once in every row and every column. In this paper we denote the entry in row i and column j of a Latin square L by $L(i,j)$ and take the n symbols from the set $\mathbb{Z}_n = \{0, \ldots, n-1\}$. We also use $\mathbb{Z}_n$ as index set for the rows and columns of a Latin square. The *transpose* of a Latin square L, denoted by L^T, is a Latin square such that $L^T(i,j) = L(j,i)$ for all $i, j \in \mathbb{Z}_n$. Two Latin squares L and L' are said

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to be *orthogonal* if each ordered pair $(L(i,j), L'(i,j))$ is unique as i and j vary over $\mathbb{Z}_n$. If a Latin square L is orthogonal to its transpose, then L is said to be a *self-orthogonal Latin square (SOLS)*. Finally, a Latin square L is *idempotent* if $L(i,i) = i$ for all $i \in \mathbb{Z}_n$.

The existence of a SOLS is guaranteed for every order $n \in \mathbb{N}$, except for $n = 2, 3, 6$ [1]. For orders 2 and 3 the non-existence of SOLS is easily verifiable, and it is well known that there exists no pair of *mutually orthogonal Latin squares (MOLS)* of order 6 [9]. Graham and Roberts [4] have recently enumerated SOLS and idempotent SOLS of order n for all $1 \leq n \leq 9$, and they have also classified idempotent SOLS into isomorphism classes. Hence the isomorphism classes of idempotent SOLS have been enumerated. However, the enumeration of the isomorphism classes of non-idempotent SOLS is still an open problem.

In this paper we complete the isomorphic classification of SOLS by enumerating, in addition to the isomorphism classes of idempotent SOLS, the isomorphism classes of non-idempotent SOLS. In §2 a number of well known classes of Latin squares are introduced, and a number of new symmetry classes are additionally defined for the purpose of classifying Latin squares that are either orthogonal or equal to their conjugates in a minimum number of classes. This includes SOLS, and these newly defined classes (of SOLS) are enumerated in §3 together with the isomorphism classes of SOLS for orders $4 \leq n \leq 9$. The paper closes with a summary of contributions and possibilities for future work in §4.

2 Classes of SOLS

Two Latin squares L and L' are *isotopic* if $L(i,j) = s^{-1}(L'(r(i), c(j)))$ for all $i, j, k \in \mathbb{Z}_n$ and some triple $(r, c, s) \in S_n^3$ (where S_n denotes the symmetric group on n elements). The notation $L' = L^{(r,c,s)}$ is henceforth used to denote the fact that $(r, c, s) \in S_n^3$ maps L to L'. In simpler terms, r is a permutation applied to the rows of L, c is a permutation applied to the columns of L and s is a permutation applied to the symbols of L. If $L' = L^{(p,p,p)}$ then L and L' are *isomorphic*, and the notation $L' = L^p$ is used instead. A Latin square L may be represented by the set of triples $T(L) = \{(i,j,k) \mid L(i,j) = k \text{ for } i,j,k \in \mathbb{Z}_n\}$, and these triples may be permuted in six different ways, resulting in six Latin squares (not necessarily distinct from L) known as the *conjugates* or *parastrophes* of L. Two Latin squares L and L' are *paratopic* and in the same *main class* if L is isotopic to a conjugate of L'. For more details on the classification of Latin squares the reader may consult Dénes and Keedwell [3, §4].

It is easy to verify that a main or isotopy class may consist of SOLS and Latin squares which are not self-orthogonal, while an isomorphism class either contains only SOLS or no SOLS at all (*i.e.* self-orthogonality is an isomorphism class invariant). Hence of these three classes it only makes sense to enumerate the isomorphism classes of SOLS. Graham and Roberts [4] have enumerated the isomorphism classes of idempotent SOLS of orders $4 \leq n \leq 9$ (it is also easy to verify that idempotency is an isomorphism class invariant), and their results are reproduced in Table 2.1. They have also enumerated distinct SOLS and idempotent SOLS of these orders.

n	Distinct SOLS	Idempotent SOLS	Isomorphism classes of idempotent SOLS
4	48	2	1
5	1 440	12	2
6	0	0	0
7	19 353 600	3 840	8
8	4 180 377 600	103 680	8
9	25 070 769 561 600	69 088 320	283

Table 2.1: Enumeration of various classes of SOLS [4].

In order to enumerate the isomorphism classes of non-idempotent SOLS, it is necessary to define another class of Latin squares. Two Latin squares L and L' are *RC-paratopic* if either $L(i,j) = q^{-1}(L'(p(i),p(j)))$ or $L(i,j)^T = q^{-1}(L'(p(i),p(j)))$ for all $i,j,k \in \mathbb{Z}_n$ and some $(p,q) \in S_n^2$. Hence the permutations applied to the rows and columns are restricted to be equal, and the operation of transposition is allowed. It may be noted that the transpose of a Latin square L is the conjugate of L resulting when the first two elements of the triples in $T(L)$ are swapped, *i.e.* the roles of rows and columns are reversed. It is easy to verify that self-orthogonality is an RC-paratopism class invariant, and that this is the largest class of Latin squares under which self-orthogonality is preserved. It is interesting to note that symmetry (a Latin square is *symmetric* if $L = L^T$) is also an RC-paratopism invariant and that this is the largest class of Latin squares under which symmetry is preserved.

Denote by L^{-1} the conjugate of L obtained by reversing the roles of columns and symbols. Furthermore, denote by ${}^{-1}L$ the conjugate obtained by reversing the roles of rows and symbols (Dénes and Keedwell [3] introduced this notation). In view of RC-paratopism classes of Latin squares, two other classes may also be defined, namely *CS-paratopism classes*, where the permutations applied to the columns and symbols are restricted to be equal and the transformation from L to L^{-1} is allowed, and *RS-paratopism classes*, where the permutations applied to the rows and symbols are restricted to be equal and the transformation from L to ${}^{-1}L$ is allowed. In the same way self-orthogonality and symmetry are preserved under an RC-paratopism, the properties of orthogonality and equality between L and L^{-1} are CS-paratopism invariant and the properties of orthogonality and equality between L and ${}^{-1}L$ are CR-paratopism invariant. These are also the largest classes over which these properties are preserved. These classes therefore play an important role in the classification of Latin squares that are either orthogonal or equal to some of their conjugates.

The RC-paratopism classes of SOLS may actually be enumerated directly from the results of Graham and Roberts. The following lemma provides a means for achieving this. Let two Latin squares L and L' be *transpose-isomorphic* if L' is isomorphic to either L or L^T. Hence a transpose-isomorphism is a special case of an RC-paratopism where the permutations applied to the rows, columns and symbols are equal.

Lemma 2.1 *If two idempotent Latin squares are RC-paratopic, then they are transpose-isomorphic.*

Proof: For any two idempotent Latin squares L and L', let either $L(i,j) = q^{-1}(L'(p(i),p(j)))$ or $L(i,j)^T = q^{-1}(L'(p(i),p(j)))$ for all $i,j,k \in \mathbb{Z}_n$ and some $(p,q) \in S_n^2$. Then either $L(i,i) = i = q^{-1}(L'(p(i),p(i))) = q^{-1}(p(i))$ or $L^T(i,i) = i = q^{-1}(L'(p(i),p(i))) = q^{-1}(p(i))$, and therefore $p = q$. ■

Each RC-paratopism class of SOLS contains exactly one transpose-isomorphism class of idempotent SOLS (it is easy to see that every RC-paratopism class of SOLS contains an idempotent SOLS), and therefore the number of RC-paratopism classes of SOLS is equal to the number of transpose-isomorphism classes of idempotent SOLS. Since two SOLS L and L' are transpose-isomorphic if L' is isomorphic to either L or L^T, and since Graham and Roberts [4] provide information regarding which idempotent SOLS are isomorphic to their transposes, the number of transpose-isomorphism classes of idempotent SOLS (*i.e.* the number of RC-paratopism classes of SOLS) may be derived from the results in [4]. If one isomorphism class contains the transpose of a SOLS from another isomorphism class, then those two classes are counted as one transpose-isomorphism class. For instance, one of the two isomorphism classes of idempotent SOLS of order 5 contains the transpose of an idempotent SOLS found in the other class [4], and therefore there is only one transpose-isomorphism class of idempotent SOLS of order 5. The number of RC-paratopism classes of SOLS found in this way is shown in Table 2.2.

The above results were also verified independently via computer generation of RC-paratopism class representatives of SOLS in [2]. The problem of enumerating the transpose-isomorphism and isomorphism classes of all SOLS (not only idempotent SOLS) has so far not yet been addressed, and this is done in the next section.

n	RC-paratopism classes
4	1
5	1
6	0
7	4
8	4
9	175

Table 2.2: Enumeration of RC-paratopism classes of SOLS.

3 Transpose-isomorphism classes and isomorphism classes of SOLS

In order to enumerate the transpose-isomorphism and isomorphism classes of SOLS, we utilise a theorem of McKay *et al.* [8] which counts the number of isomorphism classes within a given isotopism class of Latin squares by using a class representative from that isotopy class. We adapt the theorem here in order to count the number of transpose-isomorphism and isomorphism classes within a given RC-paratopism class. In what follows, an RC-paratopism is denoted by an ordered triple $(p, q, t) \in S_n^2 \times S_2$, where p is the permutation applied to the rows and columns of a SOLS, q is the permutation applied to the symbols of a SOLS and t permutes the roles of rows and columns.

An *RC-autoparatopism* is an RC-paratopism that maps a SOLS $\boldsymbol{L}$ to itself and a *transpose-automorphism* is a transpose-isomorphism that maps $\boldsymbol{L}$ to itself. For any two RC-paratopisms $\alpha = (p, q, t)$ and $\beta = (p', q', t')$ the notation $\alpha\beta = (p \circ p', q \circ q', t \circ t')$ is used to denote the action of first applying β and then α. Furthermore, $\alpha^{-1} = (p^{-1}, q^{-1}, t^{-1})$. It should be noted that each RC-autoparatopism of an idempotent SOLS is in fact, by Lemma 2.1, a transpose-automorphism. Denote by $A(\boldsymbol{L})$ the set of all RC-autoparatopisms (*i.e.* transpose-automorphisms) admitted by an idempotent SOLS $\boldsymbol{L}$. The elements of $A(\boldsymbol{L})$ are therefore either of the form $(p, p, t) \in S_n^2 \times S_2$ or of the form $(p, t) \in S_n \times S_2$, determined by the context in which it is used.

All the transpose-isomorphism classes in a single RC-paratopism class are orbits of the group $S_n \times S_2$ and we may find the number of orbits by the Cauchy-Frobenius lemma [5] stating that if $F(\sigma)$ is the number of SOLS (in a single RC-paratopism class) fixed by the transpose-isomorphism $\sigma \in S_n \times S_2$, then the number of transpose-isomorphism classes of SOLS in that class is

$$\sum_{\sigma \in S_n \times S_2} \frac{F(\sigma)}{|S_n \times S_2|}.$$

In other words, the number of transpose-isomorphism classes in a single RC-paratopism class of SOLS is the sum of the number of transpose-automorphisms of all SOLS in the class divided by the total number of possible transpose-isomorphisms.

A permutation p of order n is of *type* $(a_1, a_2, \ldots, a_n)$ if it has a_i cycles of length i for $1 \leq i \leq n$. If two permutations p_1 and p_2 are of the same type $(a_1, a_2, \ldots, a_n)$, then they are called *conjugates* and there exist $\prod_{i=1}^{n} a_i! i^{a_i}$ permutations q such that $q \circ p_1 \circ q^{-1} = p_2$. For any $\alpha = (p, p, t) \in A(L)$, let $\psi(\alpha) = \prod_{i=1}^{n} a_i! i^{a_i}$ if p is of type $(a_1, a_2, \ldots, a_n)$.

Theorem 3.1 *Let $\mathcal{I}(n)$ be a set of class representatives, one from each RC-paratopism class of SOLS of order n. Furthermore, let each element of $\mathcal{I}(n)$ be idempotent. Then the number of transpose-isomorphism classes of SOLS of order n is*

$$\sum_{L \in \mathcal{I}(n)} \frac{1}{|A(L)|} \sum_{\alpha \in A(L)} \psi(\alpha). \tag{1}$$

Proof: Let σ be an RC-paratopism and let $\alpha \in A(L)$ for some $L \in \mathcal{I}(n)$. Then the RC-paratopism $\sigma\alpha\sigma^{-1}$ is an RC-autoparatopism of some SOLS in the RC-paratopism class of L, and any RC-autoparatopism of a SOLS in the RC-paratopism class of L may be written in this form for some RC-paratopism α and some $\sigma \in A(L)$.

Let $\sigma = (p, q, r)$ and $\alpha = (s, s, t)$. For $\sigma\alpha\sigma^{-1}$ to be a transpose-automorphism it must hold that $p \circ s \circ p^{-1} = q \circ s \circ q^{-1}$. The total number of RC-paratopisms σ for which $\sigma\alpha\sigma^{-1}$ is a transpose-automorphism is $2n!\psi(\alpha)$. This number may be found by noting that p may be chosen in $n!$ ways, q may be chosen in $\prod_{i=1}^{n} a_i! i^{a_i}$ ways, given that s is of type $(a_1, a_2, \ldots, a_n)$, and finally r may be chosen in two ways. We thus count $\sum_{\alpha \in A(L)} 2n!\psi(\alpha)$ transpose-automorphisms over all SOLS in the RC-paratopism class of L, although we may have counted some transpose-automorphisms more than once.

It is therefore necessary to find the number of equivalence classes of pairs (σ, α), where σ is an RC-paratopism for which $\sigma\alpha\sigma^{-1}$ is a transpose-automorphism and $\alpha \in A(L)$. Here two pairs (σ, α) and (σ', α') are equivalent, denoted by $(\sigma, \alpha) \sim (\sigma', \alpha')$, if $\sigma\alpha\sigma^{-1} = \sigma'\alpha'\sigma'^{-1}$ and σ and σ' both map L to the same SOLS. If we let $\beta \in A(L)$, $\sigma' = \sigma\beta$ and $\alpha' = \beta^{-1}\alpha\beta$, then $\sigma\alpha\sigma^{-1} = \sigma'\alpha'\sigma'^{-1}$. Furthermore, $\alpha' \in A(L)$ and σ' and σ both map L to the same SOLS. We may therefore find, for every $\beta \in A(L)$, a pair (σ', α') such that $(\sigma, \alpha) \sim (\sigma', \alpha')$, and these equivalence classes have cardinality at least $|A(L)|$.

Conversely, let $\sigma\alpha\sigma^{-1} = \sigma'\alpha'\sigma'^{-1}$ be transpose-automorphisms such that σ' and σ both map L to the same SOLS and $\alpha, \alpha' \in A(L)$. Furthermore, let $\beta = \sigma^{-1}\sigma' \in A(L)$. Then $\sigma\alpha\sigma^{-1} = \sigma'\alpha'\sigma'^{-1} = \sigma\beta\alpha'\beta^{-1}\sigma^{-1}$ and $\alpha' = \beta^{-1}\alpha\beta$. We now have the equivalence found above, and therefore the equivalence classes have size exactly $|A(L)|$.

The number of distinct transpose-automorphisms over all SOLS in the RC-paratopism class of L is therefore

$$\sum_{\alpha \in A(L)} \frac{2n!\psi(\alpha)}{|A(L)|}.$$

Since $|S_n \times S_2| = 2n!$, the above expression should, by the Cauchy-Frobenius lemma, be divided by $2n!$ in order to count the number of transpose-isomorphism classes of SOLS in the RC-paratopism class of L. To find the total number of transpose-isomorphism classes of SOLS of order n we sum over all the elements of $\mathcal{I}(n)$. ∎

The following theorem shows how the set $\mathcal{I}(n)$ may be used to count the isomorphism classes of SOLS if it is known which elements of $\mathcal{I}(n)$ are isomorphic to their transposes, and which are not. Let $\mathcal{I}'(n) \subseteq \mathcal{I}(n)$ consist of all elements of $\mathcal{I}(n)$ which are isomorphic to their transposes and let $\mathcal{I}''(n) = \mathcal{I}(n) \backslash \mathcal{I}'(n)$.

Theorem 3.2 *The number of isomorphism classes of SOLS of order n is*

$$\sum_{L \in \mathcal{I}'(n)} \frac{1}{|A'(L)|} \sum_{\alpha \in A'(L)} \psi(\alpha) + \sum_{L \in \mathcal{I}''(n)} \frac{2}{|A(L)|} \sum_{\alpha \in A(L)} \psi(\alpha),$$

where $A'(L) \subset A(L)$ is the set of the transpose-automorphisms of L which do not transpose L.

Proof: Let $L \in \mathcal{I}'(n)$. Then there exists some RC-paratopism $(p, p, \iota) \in S_n \times S_2$ that maps L to L^T, where ι is the identity element of S_2, *i.e.* it does not transpose L. If L' is any SOLS in the RC-paratopism class of L and the RC-paratopism (q, r, τ) maps L to L' (where τ is the operation of transposition), then the RC-paratopism (q, r, ι) maps L^T to L'. Hence the RC-paratopism (qp, rp, ι) maps L to L', and any SOLS in the RC-paratopism class of L may therefore be mapped to L via an RC-paratopism that does not use the operation of transposition.

If the steps of Theorem 3.1 are followed again, this time only considering RC-paratopisms which do not use the operation of transposition, then the number of isomorphism classes of SOLS in the RC-paratopism class of L is

$$\sum_{L \in \mathcal{I}'(n)} \frac{1}{|A'(L)|} \sum_{\alpha \in A'(L)} \psi(\alpha).$$

Next suppose $\boldsymbol{L} \in \mathcal{I}''(n)$. Let some SOLS $\boldsymbol{L}'$ in the RC-paratopism class of $\boldsymbol{L}$ be isomorphic to its transpose via some isomorphism $p \in S_n$, and let the RC-paratopism $(q, r, t) \in S_n \times S_2$ map $\boldsymbol{L}$ to $\boldsymbol{L}'$. Then (q, r, t) also maps $\boldsymbol{L}^T$ to $\boldsymbol{L}'^T$ and hence the RC-paratopism $(q^{-1}pq, r^{-1}pr, \iota)$ maps $\boldsymbol{L}$ to $\boldsymbol{L}^T$. Since both $\boldsymbol{L}$ and $\boldsymbol{L}^T$ are idempotent, it follows that they are isomorphic, contradicting the property of the elements of $\mathcal{I}''(n)$. Hence no SOLS in the RC-paratopism class of $\boldsymbol{L}$ is isomorphic to its transpose, and therefore each transpose-isomorphism class in the RC-paratopism class of $\boldsymbol{L}$ splits into two disjoint isomorphism classes, where one contains the tranposes of the SOLS in the other. Consequently there are twice as many isomorphism classes as transpose-isomorphism classes in the RC-paratopism class of $\boldsymbol{L}$. ■

Idempotent RC-paratopism class representatives of SOLS of orders $4 \leq n \leq 9$ are available online [6], and these may be used as the set $\mathcal{I}(n)$ in (1) for $4 \leq n \leq 9$. Information on whether or not these representatives are isomorphic to their transposes are also provided in [6]. In order to obtain the transpose-automorphism groups of these SOLS, the computer program `nauty` [7] was used. Since `nauty` takes only graphs as input, we construct a graph in such a way that the isomorphisms of the graph correspond to the transpose-isomorphisms of the SOLS under consideration. McKay *et al.* [8] describe the construction of graphs for isomorphism, isotopy and main classes of Latin squares, and we use a similar graph representation approach for the transpose-isomorphism classes of idempotent SOLS.

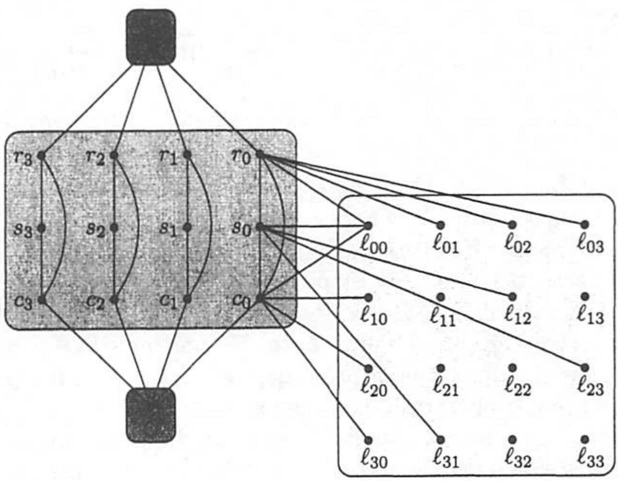

Figure 3.1: The graph $G(\boldsymbol{L}_4)$ with some edges omitted. The automorphisms of this graph correspond to the transpose-automorphisms of $\boldsymbol{L}_4$.

Let L be a SOLS of order n and let $G(L)$ be a vertex-coloured graph with vertex set $V(G) = \{r_i, c_i, s_i, \ell_{ij} \mid i, j \in \mathbb{Z}_n\} \cup \{R, C\}$, where one colour is assigned to $\{r_i, c_i, s_i \mid i \in \mathbb{Z}_n\}$, another to $\{R, C\}$ and a third colour to $\{\ell_{ij} \mid i, j \in \mathbb{Z}_n\}$. Furthermore, $G(L)$ has edge set $E(G) = \{r_i\ell_{ij}, c_j\ell_{ij}, s_k\ell_{ij} \mid L(i, j) = k\} \cup \{Rr_i, Cc_i, r_ic_i, r_is_i, c_is_i \mid i \in \mathbb{Z}_n\}$ and the isomorphisms of $G(L)$ are colour-preserving. For illustrative purposes a part of $G(L_4)$ is shown in Figure 3.1 for the SOLS

$$L_4 = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 \end{bmatrix}.$$

The transpose-automorphism group of L may be derived from the automorphism group of $G(L)$ by the similarity in structure of the two designs. This approach may be used, together with Theorems 3.1 and 3.2, to count the number of transpose-isomorphism classes of SOLS and the number of isomorphism classes of SOLS. It may be noted that these counts include classes containing idempotent SOLS. The results are shown in Table 3.3. The computation of the autotopism groups via **nauty** was immediate for $4 \leq n \leq 8$ and required a total of 0.92 seconds computing time for all 175 cases corresponding to $n = 9$.

n	transpose-isomorphism classes	isomorphism classes
4	5	6
5	11	22
6	0	0
7	1 986	3972
8	52 060	104 120
9	34 564 884	69 112 956

Table 3.3: The number of transpose-isomorphism classes and isomorphism classes of SOLS.

4 Conclusion

In this paper we presented three new classes of Latin squares, namely RC-, RS- and CS-paratopism classes, which may be used to catalogue Latin squares that are either orthogonal or equal to some of their conjugates, and it was shown how the number of RC-paratopism classes of SOLS of orders $4 \leq n \leq 9$ may be derived from the results of Graham and Roberts [4]. Utilising available repositories of class-representatives from the newly defined RC-paratopism classes of SOLS, we enumerated the number of transpose-isomorphism and isomorphism classes of SOLS utilising the Cauchy-Frobenius lemma and a theorem of McKay *et al.* [8].

Due to the lack of class-representatives of RC-paratopism classes of SOLS for orders greater than 9, the number of transpose-isomorphism and isomorphism classes of SOLS of these orders have not been enumerated. An attempt to enumerate and generate class representatives of the RC-paratopism classes of SOLS of order 10 was undertaken in [2], and it was shown that the methods used in [2] are not able to complete the enumeration within a realistic time frame. The results of this paper show, however, that if the difficult problem of generating an exhaustive list of class representatives of RC-paratopism classes of SOLS of orders larger than 9 may be solved, then the transpose-isomorphism and isomorphism classes of SOLS of these orders may be enumerated without much difficulty, largely due to the speed with which `nauty` [7] computes graph-automorphisms.

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