PSEUDO-COMPLEMENTS IN FINITE PROJECTIVE PLANES

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Abstract

We show that a finite linear space with $b = n^2 + n + 1$ lines, $n \geq 2$, constant point-degree $n+1$ and containing a sufficient number of lines of size n can be embedded in a projective plane of order n. Using this fact, we also give characterizations of some pseudo-complements, which are the complements of certain subsets of finite projective planes.

Key Words: Linear space, projective plane, affine plane, pseudo-complement.

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1 Introduction

Let us first recall some definitions and results. For more details, (see [1], [2]).

A *finite linear space* is a pair $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a finite set of *points* and $\mathcal{L}$ is a family of proper subsets of $\mathcal{P}$, which are called *lines*, such that

(L1) Any two distinct points lie on exactly one line,

(L2) Any line contains at least two points,

(L3) There exist at least two lines.

It is clear that (L3) could be replaced by an axiom (L3)$'$: There are three lines of $\mathcal{S}$ not incident with a common point. In any case, (L3) and (L3)$'$ are 'non-triviality' conditions. Systems satisfying (L1) and (L2) but not (L3) are called *trivial linear spaces.*

In a finite linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, v and b denote the total number of points and lines, respectively. The degree $b(p)$ of a point p is the total number of lines through p, and the size $v(l)$ of a line l is the total number of points on l. Thus; if $v(l) = k$ then l is called a k−line. The total number of k−lines is denoted by b_k.

The integer n defined by $n + 1 = \max\{b(p) : p \in \mathcal{P}\}$ is the *order* of a linear space. It is clear that any line of size $n + 1$ meets every other line in a linear space of order n.

The numbers v, b, $v(l)$ and $b(p)$ will be called the *parameters* of $\mathcal{S}$.

A *projective plane* π is a linear space in which all lines meet and in which all points are on $n+1$ lines, $n \geq 2$. The number n is called the order of π.

An *affine plane* $\mathbb{A}$ is a linear space in which, for any point p not on a line l, there is a unique line on p missing l, and in which all points are on $n + 1$ lines, $n \geq 2$.

For any line l of a linear space $\mathcal{S}$ of order n, the difference $n + 1 - v(l)$ is called a *deficiency* of l, denoted $d(l)$. Since the size of any line cannot exceed $n + 1$, the deficiency of any line is non-negative.

Let μ and λ be the respective minimum and maximum deficiencies among those lines of $\mathcal{S}$ which have size less than n.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a linear space and let $\mathcal{X}$ be a subset of $\mathcal{P}$ containing three non-collinear points. Then we can define the linear space $\mathcal{S}' = (\mathcal{X}, \{l \cap \mathcal{X} : l \in \mathcal{L}, |l \cap \mathcal{X}| \geq 2\})$. If $\mathcal{C} = \mathcal{P} - \mathcal{X}$, then $\mathcal{S}'$ is called the *complement* of $\mathcal{C}$ in $\mathcal{S}$ and we say that $\mathcal{S}'$ is obtained by removing $\mathcal{C}$ from $\mathcal{S}$. We denote the complement of $\mathcal{C}$ in $\mathcal{S}$ by $\mathcal{S} - \mathcal{C}$.

Let $\mathcal{X}$ be a set of points in a projective plane π of order n. Suppose that we remove $\mathcal{X}$ from π. We obtain a linear space $\pi - \mathcal{X}$ having certain parameters (i.e., the number of points, the number of lines, the point-degrees and line-degrees) (see [1]).

We call any linear space, which has the same parameters as $\pi - \mathcal{X}$, a *pseudo-complement* of $\mathcal{X}$ in π.

We have already encountered the notation of a pseudo-complement, namely the *pseudo-complement of one line.* This is a linear space with n^2 points, $n^2 + n$ lines in which any point has degree $n + 1$ and any line has degree n. We know that this is an affine plane, which is a structure embeddable in a projective plane of order n.

A linear space with n^2+n-m^2-m points, $b = n^2+n+1$ lines, constant point-degree $n + 1$ and containing at least $m^2 + m + 1$ lines of size $n - m$ will be called the *pseudo-complement of a projective subplane of order* m in a projective plane of order n. It is clear that $m < n$.

A linear space with $n^2+n+1-m^2$ points, $b = n^2+n+1$ lines, constant point-degree $n + 1$ and containing at least $m^2 + m$ lines of size $n + 1 - m$ will be called the *pseudo-complement of an affine subplane order* m in a projective plane of order n. It is clear that $m < n$.

Two lines l and l' are *parallel* if $l = l'$ or $l \cap l' = \phi$. Two lines l and l' are *disjoint* if $l \cap l' = \phi$.

A *parallel class* in the linear space $(\mathcal{P}, \mathcal{L})$ is a subset of $\mathcal{L}$ with the property that each point of $\mathcal{P}$ is on a unique element of this subset.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{L}')$ be two finite linear spaces. We say that $\mathcal{S}$ can be *embedded* in $\mathcal{S}'$ if $\mathcal{P} \subseteq \mathcal{P}'$ and $\mathcal{L} = \{l' \cap \mathcal{P} : l' \in \mathcal{L}' \text{ and } |l' \cap \mathcal{P}| \geq 2\}$. Hall proved in [10] that every finite linear space can be embedded in an infinite projective plane.

The complementation problem with respect to a projective plane is the following:
Remove a certain subset of points and lines from the projective plane. Determine the parameters of the resulting space. Now assume that you are starting with a space having these parameters. Does this somehow force this subset to reappear, thus giving an embedding in the original projective plane? A number of people have considered complementation problems ([1], [2], [3], ..., [13]). In 1970, Dickey solved the problem for the case where the configuration removed was a unital [7]. Batten [2] gave characterizations of linear spaces which are the complement of affine or projective subplanes of finite projective planes.

In this article, we show that a linear space of order n whose parameters are those of the complement of a point subset in a finite projective plane π of order n such that no line is removed and a sufficient number of lines lose only one point, is embeddable in π. Using this fact, we also give a new characterization of pseudo-complements which are complements of affine or projective subplanes of finite projective planes.

In this paper, for any two disjoint lines l and l', which have size less than n in a finite linear space, we will use $m(l, l')$ and $m_n(l, l')$ to denote the total number of lines and $n-$lines, respectively, meeting l or l' excluding l and l' themselves.

2 Main Results

Lemma 2.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a finite linear space with b lines, v points, constant point-degree $n+1$ and containing at least one $n-$line , $n \geq 2$. Then the following statements hold for any two disjoint lines l and l' which have size less than n in S:*

(i) $m(l, l') = n^2 - (d(l) - 1) \cdot (d(l') - 1)$,

(ii) $m_n(l, l') \leq n^2 - (\mu - 1)^2 - b_{n+1}$,

(iii) *if $b - v - \lambda \geq 0$ and each point on any line k which has size less than n is on at most $b - v - d(k)$ lines of size n,* $m_n(l, l') \leq 2(n + 1 - \mu)(b - v - \mu)$.

Proof: Assume that $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a finite linear space with constant point-degree $n+1$ and contains at least one n−line , $n \geq 2$. Let l and l' be two disjoint lines which have size less than n in S. Therefore, $b(p) = n+1$ for all points p, $d(l) \geq \mu$ and $d(l') \geq \mu$.

Let x be the number of lines meeting l (excluding l itself); let y be the number of lines meeting l' (excluding l' itself); and let z be the number of lines meeting both l and l'. The following three equations are obtained by a simple counting method :

$$x = n(n+1-d(l)),$$

$$y = n(n+1-d(l)),$$

and

$$z = (n+1-d(l))(n+1-d(l')).$$

Therefore, $x + y - z = n^2 - (d(l)-1)(d(l')-1)$.
Since, $m(l, l') = x + y - z$, we have

$$m(l, l') = n^2 - (d(l)-1)(d(l')-1) \leq n^2 - (\mu - 1)^2,$$

which proves (i). Since any line of size $n+1$ meets every other line, all the lines of size $n+1$ meet both l and l'. Therefore, since $b_{n+1} \geq 0$, $d(l) \geq \mu$ and $d(l') \geq \mu$, we have

$$m_n(l, l') \leq m(l, l') - b_{n+1} \leq n^2 - (\mu - 1)^2 - b_{n+1},$$

which proves (ii).

If $b - v - \lambda \geq 0$ and each point on any line k which has size less than n is on at most $b - v - d(k)$ lines of size n, then there are at most $(n+1-d(l))(b-v-d(l))$ lines of size n meeting l, and there are at most $(n+1-d(l'))(b-v-d(l'))$ lines of size n meeting l'. Since $d(l) \geq \mu$ and $d(l') \geq \mu$, we have $m_n(l, l') \leq 2(n+1-\mu)(b-v-\mu)$, which proves (iii).

Theorem 2.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a finite linear space with $b = n^2 + n + 1$ lines, $n \geq 2$, constant point-degree $n+1$. If $b_n > n^2 - (\mu - 1)^2 - b_{n+1} \geq 0$, then $\mathcal{S}$ can be embedded in a projective plane of order n.*

Proof: Assume that $\mathcal{S}$ is a finite linear space with $b = n^2 + n + 1$ lines, $n \geq 2$, constant point-degree $n+1$ and $b_n > n^2 - (\mu - 1)^2 - b_{n+1} \geq 0$. That is,

(1) $b = n^2 + n + 1$,

(2) $b(p) = n + 1$, for all points p,

(3) $b_n > n^2 - (\mu - 1)^2 - b_{n+1} \geq 0$.

Using (2) it is easy to see that lines have at most $n+1$ points and that every line meets an $(n+1)-$line. By (3) $b_n \geq 1$. Thus, there exists at least one $n-$line.

For every $n-$line l, we define

$$\Pi_l = \{l\} \cup \{x : x \text{ is a line disjoint to } l\}.$$

Since each point has degree $n + 1 = v(l) + 1$, each point outside l lies on exactly one line which is parallel to l. This shows that Π_l is a partition of the points of $\mathcal{S}$ into disjoint lines, and Π_l induces an equivalence relation among the lines in $\mathcal{S}$ of size n. We will refer to this equivalent relation on the lines of size n as parallelism. Since l meets n^2 other lines, $|\Pi_l| = n+1$. Hence, each n-line induces a partition of the points into $n + 1$ lines which we will refer to as the parallel class associated with that $n-$line.

Suppose that l and l' are two different n-lines which meet. Then l' meets n lines of Π_l, so $|\Pi_l \cap \Pi_{l'}| = 1$.

We let each such parallel class corresponds to a "new point". Consider the structure $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*)$ where $\mathcal{P}^*$ is $\mathcal{P}$ along with the new points, and $\mathcal{L}^*$ consists of the lines of $\mathcal{L}$ "extended" by those parallel classes to which they belong. We first of all prove that $\mathcal{S}^*$ is a linear space. It is clear that two old points (points of $\mathcal{P}$) are on a unique line of $\mathcal{L}^*$. Since any old point is on exactly one line contained in the parallel class corresponding a new point, an old point and a new point are on a unique line of $\mathcal{L}^*$. Let x and y be distinct new points. We show that they determine a unique line of $\mathcal{L}^*$. Let l_x and l_y be $n-$lines which determine the parallel classes corresponding to x and y. If l_x and l_y do not meet, then $x = y$ which is a contradiction. So l_x and l_y meet. By (2) each point of l_y is on a unique line of the parallel class determined by l_x.This leaves precisely one line of the parallel class parallel to both l_x and l_y. This is the required line. It follows from our method of construction that each point of $\mathcal{S}^*$ is on $n + 1$ lines.

Finally we prove that any two lines of $\mathcal{S}^*$ always meet. Let l and l' be lines of $\mathcal{S}^*$ which do not meet in $\mathcal{S}$. Then neither l nor $l'$are $n + 1-$lines. To prove that they meet in $\mathcal{S}^*$, it suffices to find an n-line parallel to both.

If either l or l' is an n-line, then both of them belong to same parallel class, and they meet in $\mathcal{S}^*$.

If neither l nor $l'$are $n-$lines, $v(l) < n$ and $v(l') < n$. Hence by lemma 2.1 (*ii*) and (3), $m_n(l, l') \leq n^2 - (\mu - 1)^2 - b_{n+1} < b_n$. There is at least one $n-$line parallel to both. Therefore $\mathcal{S}^*$ is a projective plane of order n.

Theorem 2.2 *Every pseudo-complement of a projective subplane of order m in a projective plane of order n can be embedded into a projective plane as the complement of a projective subplane order m. Moreover, either $n = m^2$*

(if there is no (n+1)-lines), or $n \geq m^2+m$ *(if there is at least one* $(n+1)$*-line).*

Proof: Let $\mathcal{S}$ be the pseudo-complement of a projective subplane of order m in a projective plane of order n. So, $\mathcal{S}$ is a linear space with n^2+n-m^2-m points, $b=n^2+n+1$ lines, constant point-degree $n+1$ and contains at least m^2+m+1 lines of size $(n-m)$.We show that $\mathcal{S}$ contains m^2+m+1 lines of size $n-m$, $(n-m)(m^2+m+1)$ lines of size n and $n^2-(m^2+m)n+m^3$ lines of size $n+1$.

Since b_i is the number of i-lines in $\mathcal{S}$, by a simple counting method, we have

(i) $\sum_i b_i = n^2+n+1,$

(ii) $\sum_i ib_i = (n+1)v = (n+1)(n^2+n-m^2-m),$

(iii) $\sum_i i(i-1)b_i = v(v-1).$

Hence,

(iv) $\sum_i (n+1-i)(n-i)b_i = (m^2+m)(m^2+m+1).$

However, $\mathcal{S}$ has at least m^2+m+1 $(n-m)$-lines, and each of them contributes m^2+m to the left hand side of the equality (iv) above. Thus $b_i = 0$, for $i \notin \{n-m, n+1, n\}$. Therefore, by (i)-(iv), $\mathcal{S}$ contains m^2+m+1 lines of size $n-m$, $(n-m)(m^2+m+1)$ lines of size n and $n^2-(m^2+m)n+m^3$ lines of size $n+1$.

To prove the embedding, we must show that all conditions of Theorem 2.1 hold.

Since $\lambda = \mu = m+1$,

$$b_n = (n-m)(m^2+m+1) > n^2-(\mu-1)^2-b_{n+1}.$$

Thus, condition of Theorem 2.1 valid in $\mathcal{S}$. By theorem 2.1, $\mathcal{S}$ can be embedded in a projective plane π of order n.

Let $\pi-\mathcal{S}$ be the complement of $\mathcal{S}$ in π. $\pi-\mathcal{S}$ has $b-v=m^2+m+1$ points. Since lines of $\pi-\mathcal{S}$ are the complements of $(n-m)$ –lines of $\mathcal{S}$ in π, lines of $\pi-\mathcal{S}$ have $m+1$ points. So, there are m^2+m+1 such lines in $\pi-\mathcal{S}$, as noted above.

Let p be a point of $\pi-\mathcal{S}$. Since there are no $(n+1)-$ lines in the parallel class corresponding to p, there are exactly $n-m$ lines of size n in this parallel class. So, there are exactly $m+1$ lines of size $n-m$ in this parallel class. Therefore, there are $m+1$ lines on p in $\pi-\mathcal{S}$. So all lines meet, and $\pi-\mathcal{S}$ is a projective plane of order m.

The fact that $n=m^2$ or $n \geq m^2+m$ follows from Bruck's theorem [4].

Theorem 2.3 *Every pseudo-complement of affine subplane of order m in a projective plane of order n can be embedded into a projective plane as the complement of a affine subplane order m.*

Proof: Let $\mathcal{S}$ be the pseudo-complement of affine subplane of order m in a projective plane of order n. Then $\mathcal{S}$ is a linear space with $n^2+n+1-m^2$ points, $b = n^2+n+1$ lines, constant point-degree $n+1$ and contains at least m^2+m lines of size $n+1-m$.

We show that $\mathcal{S}$ contains m^2+m lines of size $n+1-m$, $m^2(n-m)$ lines of size n and $n^2-(m^2-1)(n+1-m)$ lines of size $n+1$.

By equations (i), (ii) and (iii) in the proof of Theorem 2.2,

$$\text{(iv)} \ \sum_i (n+1-i)(n-i)b_i = m^4 - m^2.$$

However $\mathcal{S}$ has at least m^2+m lines of size $n+1-m$, and each of them contributes m^2-m to the left hand side of the equality above. Thus $b_i = 0$, $i \neq n+1-m, n, n+1$. Therefore, by equations (i), (ii) and (iii) in the proof of Theorem 2.2 and (iv), $\mathcal{S}$ contains m^2+m lines of size $n+1-m$, $m^2(n-m)$ lines of size n and $n^2-(m^2-1)(n+1-m)$ lines of size $n+1$. Since $\mathcal{S}$ is the linear space with n^2+n+1 lines, constant point-degree $n+1$, Π_l is the parallel class with $n+1$ lines for each n–line l.

To prove the embedding, we must show that $\mathcal{S}$ has exactly m^2 parallel classes. Let a be the number of n-lines in a fixed class. then

$$an + (n+1-a)(n+1-m) = n^2+n+1-m^2, \text{ implies } a = n-m.$$

Since $b_n = (n-m)m^2$, the number of distinct parallel classes is m^2.

Consider the structure $\pi = (\mathcal{P}^*, \mathcal{L}^*)$ where $\mathcal{P}^*$ is $\mathcal{P}$ along with the new points, and $\mathcal{L}^*$ consists of the lines of $\mathcal{L}$ "extended" by those parallel classes to which they belong. Thus π is a linear space with n^2+n+1 points and n^2+n+1 lines. By [4], π is a projective plane of order n.

Let $\pi - \mathcal{S}$ be the complement of $\mathcal{S}$ in π. The structure $\pi - \mathcal{S}$ has $b - v = m^2$ points. Since lines of $\pi - \mathcal{S}$ are the complements of $(n+1-m)$ –lines of $\mathcal{S}$ in π, lines of $\pi - \mathcal{S}$ have m points. So, there are m^2+m such lines in $\pi - \mathcal{S}$, as noted above.

Let p be a point of $\pi - \mathcal{S}$ and let a be total number of n–lines in the parallel class corresponding to p. Since there are no $(n+1)$ – lines in this class, $a = n-m$, as noted above. So, there are exactly $n+1-a = m+1$ lines of size $n+1-m$ in this class. Therefore there are $m+1$ lines on p in $\pi - \mathcal{S}$. So, $\pi - \mathcal{S}$ is a linear space in which for any point p not on a line l, there is a unique line on p missing l, and in which all points are on $m+1$ lines $n \geq 2$. Therefore, $\pi - \mathcal{S}$ is an affine plane of order m.

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