Graph operations and cordiality

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Abstract

In this paper, we show that the disjoint union of two cordial graphs one of them is of even size is cordial and the join of two cordial graphs both are of even size or one of them is of even size and one of them is of even order is cordial. We also show that $C_m \cup C_n$ is cordial if and only if $m+n \not\equiv 2 \pmod 4$ and mC_n is cordial if and only if $mn \not\equiv 2 \pmod 4$ and for $m, n \geq 3$, $C_m + C_n$ is cordial if and only if $(m, n) \neq (3, 3)$ and $\{m,n\} \not\equiv \{0, 2\} \pmod 4$. Finally, we discuss the cordiality of P_n^k.

1-Introduction

All graphs in this paper are finite, simple and undirected. We follow the basic notations and terminology of graph theory as in [5].

Let G be a graph with vertex set V(G) and edge set E(G). A labeling $f : V(G) \rightarrow \{0, 1$induces an edge labeling $f^*:E(G) \rightarrow \{0, 1\}$, defined by $f^*(xy) = |f(x) - f(y)|$, for each edge $xy \in E(G)$. For $i \in \{0, 1\}$, let $n_i(f) =| \{v \in V(G) : f(v) = i\}|$ and $m_i(f) = |\{e \in E(G) : f^*(e) = i\}|$.A labeling f of a graph G is cordial if $| n_0(f) - n_1(f) | \leq 1$ and $| m_0(f) - m_1(f) | \leq 1$. Note that interchanging the vertex labels 0 and 1 will produce a new cordial labeling of G. So that we may always assume that there is a cordial labeling f with the additional property

$$0 \leq n_0(f) - n_1(f) \leq 1 \quad \text{or} \quad -1 \leq n_0(f) - n_1(f) \leq 0 .$$

A graph G is cordial if it admits a cordial labeling. The notion of cordial labeling of graphs was first introduced by Cahit

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[3], who proved the following : every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$; C_n is cordial if and only if $n \not\equiv 2 \pmod 4$ and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang and Kwong [9] determined all m and n for which mK_n is cordial. Shee and Ho [16] determined the cordiality of $C_m^{(n)}$, the one-point union of n copies of C_m. Diab [6] proved the following graphs are cordial : $C_m + P_n$ if and only if $(m,n) \neq (3,1),(3,2)$ or $(3,1)$; $P_m + K_{1,n}$ if and only if $(m,n) \neq (1,2)$ and $P_m \cup K_{1,n}$ if and only if $(m,n) \neq (1,2)$.Seoud and Abdel Maqsoud [14] proved the following : for $m \geq 2, C_n \times P_m$ is cordial except for the case $C_{4k+2} \times P_2$; $P^2{}_n$ is cordial for all n ; $P^3{}_n$ is cordial if and only if $n \neq 4$; $P^4{}_n$ is cordial if and only if $n \neq 4,5$ or 6. Seoud, Diab and Elsakhawai [15] have proved that the following graphs are cordial $P_m + P_n$ for all m and n except for $(m,n) = (2,2)$; $C_m + C_n$ if $m \not\equiv 0 \pmod 4$, $n \not\equiv 2 \pmod 4$; $C_n + K_{1,m}$ for $n \not\equiv 3 \pmod 4$ and odd m except for $(n,m) = (3,1)$ and $C_n + \bar{K}_m$ when n is odd and when n is even and m is odd. For more details of known results of cordial labeling see [8].

We recall the definitions of graceful, Skolem-graceful, sequential and strongly k-index able labelings of graphs.

Let G be a graph of vertex set V(G) and edge set E(G) with $p = |V(G)|$ and $q = |E(G)|$.

A graph G is called graceful [13] if there exists an injective function f, called a graceful labeling of G,

$f : V(G) \rightarrow \{0, 1, \ldots, q\}$ such that the induced function

$f^* : E(G) \rightarrow \{1, 2, \ldots, q\}$

defined by $f^*(xy) = |f(x) - f(y)|$, for all $xy \in E(G)$ is a bijection.

A graph G is called Skolem-graceful [12] if there exists a bijective function f, called a Skolem- graceful labeling of G,

$f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the induced function

$f^* : E(G) \rightarrow \{1, 2, \ldots, q\}$

defined by $f^*(xy) = |f(x) - f(y)|$, for all $xy \in E(G)$ is a bijection.

A graph G is called sequential [9] (or strongly c-harmonious [4]) if there exists an injective function f, called a sequential labeling of G,

$f : V(G) \to \{0, 1, \ldots, q-1\}$ and there exists a positive integer c such that the induced function

$f^* : E(G) \to \{c, c+1, \ldots, c+q-1\}$

defined by $f^*(xy) = f(x) + f(y)$, for all $xy \in E(G)$ is a bijection.

A graph G is called strongly k-indexable [2]) if there exists a bijective function f, called a strongly k-indexable labeling of G,

$f : V(G) \to \{0, 1, \ldots, p-1\}$ and there exists a positive integer k such that the induced function

$f^* : E(G) \to \{k, k+1, \ldots, k+q-1\}$

defined by $f^*(xy) = f(x) + f(y)$, for all $xy \in E(G)$ is a bijection.

The following proposition gives the basic relationship between these labelings and the cordial labeling.

Proposition 1

Let G be a graph of vertex set $V(G)$ and edge set $E(G)$ with $p = |V(G)|$ and $q = |E(G)|$, then

(i) G is Skolem-graceful $\Rightarrow$ G is cordial

(ii) G is strongly k-indexable $\Rightarrow$ G is cordial

(iii) G is graceful with $p = q+1$ or $p = q = 2t+1$ $\Rightarrow$ G is cordial.

(iv) G is sequential with $p = q$ or $p = q-1 = 2t+1 \Rightarrow$ G is cordial.

Proof

Let f be a Skolem-graceful (resp. strongly k-indexable, resp. graceful, resp. sequential) labeling of G and define

$$g : V(G) \to \{0, 1\}$$

by $$g(v) = f(v) \pmod 2, \text{ for each } v \in V(G).$$

Note that if G is Skolem-graceful or strongly k-indexable or graceful with $p = q+1$ or sequential with $p = q$, then clearly

$|n_0(g) - n_1(g)| \le 1$ and if G is graceful with $p = q = 2t+1$ or sequential with $p = q-1 = 2t+1$, then since there are exactly t+1 odd integers in the set $\{0, 1, 2, \ldots, 2t+1\}$ we also have $|n_0(g) - n_1(g)| \le 1$ in this case as well. Finally observe that since $g^*(xy) \equiv f^*(xy) \pmod 2$ and f^* is a bijection onto an interval of positive integers, then $|m_0(g) - m_1(g)| \le 1$ in all the stated cases. □

The paper is divided into two sections. In the next section of this paper, we establish the cordiality of the disjoint union and the join of two cordial graphs under special conditions. In Section 3, we determine the cordiality of $C_m \cup C_n$, $m\,C_n$ and $C_m + C_n$. Finally, we discuss the cordiality of P_n^k.

2-Graph operations and cordiality

In this section, we prove the cordiality of the disjoint union and the join of two cordial graphs under special condition.

Theorem 2.1

If G_1 and G_2 are two cordial graphs and G_1 is of even size, then $G_1 \cup G_2$ is cordial.

Proof

Let f, g be cordial labeling of G_1 and G_2 respectively such that

$0 \le n_0(f) - n_1(f) \le 1$ and $-1 \le n_0(g) - n_1(g) \le 0$. Note that since G_1 is of even size, then $m_0(f) = m_1(f)$. Define the labeling

$$h : V(G_1 \cup G_2) \to \{0, 1\}$$

as :

$$h\,|_{V(G_1)} = f$$
$$h\,|_{V(G_2)} = g.$$

We have $|\,n_0(h) - n_1(h)\,| = |\,n_0(f) - n_1(f) + n_0(g) - n_1(g)\,| \le 1$. Also $|\,m_0(h) - m_1(h)\,| = |\,m_0(f) + m_0(g) - m_1(f) - m_1(g)\,| = |m_0(g) - m_1(g)| \le 1$. Hence h is a cordial labeling of $G_1 \cup G_2$. □

Corollary 2.2

If G is cordial of even size, then mG is cordial for all $m \geq 1$.

Theorem 2.3

If G_1 and G_2 are two cordial graphs both are of even size, then $G_1 + G_2$ is cordial.

Proof

Let f (resp. g) be a cordial labeling of G_1 (resp. G_2) such that
$0 \leq n_0(f) - n_1(f) \leq 1$ and $-1 \leq n_0(g) - n_1(g) \leq 0$. Note that $m_0(f) = m_1(f)$ and $m_0(g) = m_1(g)$ since both G_1 and G_2 are of even size. Define the labeling

$$h : V(G_1 + G_2) \rightarrow \{0, 1\}$$

as :

$$h|_{V(G_1)} = f$$
$$h|_{V(G_2)} = g.$$

Hence $|n_0(h) - n_1(h)| \leq 1$ as in Theorem 2.1 and
$|m_0(h) - m_1(h)| = |m_0(f) + m_0(g) + n_0(f)\, n_0(g) + n_1(f)\, n_1(g) - m_1(f) - m_1(g) - n_0(f)\, n_1(g) - n_1(f)\, n_0(g)|$
$= |(n_0(f) - n_1(f))\,(n_0(g) - n_1(g))| \leq 1$ and $G_1 + G_2$ is cordial as desired. □

Theorem 2.4

If G_1 and G_2 are two cordial graphs and G_1 is of even size and one of them is of even order, then $G_1 + G_2$ is cordial.

Proof

Let f (resp. g) be a cordial labeling of G_1 (resp. G_2). Note that since G_1 is of even size, we have $m_0(f) = m_1(f)$ and since G_1 or G_2 is of even order, we have either $n_0(f) = n_1(f)$ or $n_0(g) = n_1(g)$. Define the labeling

$$h : V(G_1 + G_2) \rightarrow \{0, 1\}$$

as :

$$h|_{V(G_1)} = f$$
$$h|_{V(G_2)} = g\,.$$

We have $|\, n_0(h) - n_1(h)\,| = |\, n_0(f) - n_1(f) + n_0(g) - n_1(g)\,| \le 1$ and
$|\, m_0(h) - m_1(h)\,| = |(m_0(g) - m_1(g)) + (n_0(f) - n_1(f))\,(n_0(g) - n_1(g))\,|$

$= |\, m_0(g) - m_1(g)\,| \le 1$. Hence $G_1 + G_2$ is cordial. □

Corollary 2.5

If $G_1, G_2, \ldots, G_k$ are cordial graphs each of them is of even size and order, then $G_1 + G_2 + \ldots + G_k$ is cordial.

Corollary 2.6

(i) $K_{m,n}$ is cordial for all m and n [3]

(ii) $K_{m,n,p}$ is cordial for all m,n and p [11]

(iii) $P_m + P_n$ is cordial if and only if $(m,n) \neq (2,2)$ [15]

Proof

(i) Follows directly from Theorem 2.3, since $K_{m,n} = \overline{K}_m + \overline{K}_n$.

(ii) $K_{m,n,p} = \overline{K}_m + K_{n,p}$, since $\overline{K}_m$ is cordial graph of even size and either the size or the order of $K_{n,p}$ is even, then $K_{m,n,p}$ is cordial by Theorem 2.3 or Theorem 2.4.

(iii) If m and n are odd , then $P_m + P_n$ is cordial by Theorem 2.3 and if $m + n \equiv 1 \pmod 2$, then $P_m + P_n$ is cordial by Theorem 2.4. If m and n are even with $(m,n) \neq (2,2)$, then $P_m + P_n$ is cordial by the following labeling to P_m and P_n respectively :

(0011)(0011)(0011)....(0011) in case of $m \equiv 0 \pmod 4$
(0011)(0011)(0011)....(0011)(10) in case of $m \equiv 2 \pmod 4$

(0110)(0110)(0110)....(0110) in case of $n \equiv 0 (\mod 4)$

(0110)(0110)(0110)....(0110)(01) in case of $n \equiv 2 (\mod 4)$

Finally if $(m,n) = (2,2)$, then $P_2 + P_2 = K_4$ which is not cordial by Cahit [3]. □

3- New families of cordial graphs

In this section, we determine the cordiality of $C_m \cup C_n$, mC_n and $C_m + C_n$ and we discuss the cordiality of P_n^k.

Theorem 3.1

For m, n ≥ 3, $C_m \cup C_n$ is cordial if and only if $m+n \not\equiv 2$ (mod 4).

Proof

If $m+n \equiv 2$ (mod 4), then $C_m \cup C_n$ is not cordial by [3]

If $m+n \equiv 3$ (mod 4), then $C_m \cup C_n$ is graceful of odd size by [1]. Therefore, $C_m \cup C_n$ is cordial by Proposition 1.

If $m+n \equiv 1$ (mod 4), then we may assume that either

(i) $m \equiv 0$ (mod 4) and $n \equiv 1$ (mod 4)

In this case $C_m \cup C_n$ is cordial by [3] and Theorem 2.1

or (ii) $m \equiv 2$ (mod 4) and $n \equiv 3$ (mod 4)

In this case $C_m \cup C_n$ is cordial by the following labeling : Label the cycle C_m, $m \equiv 2$ (mod 4) by the successive labels:

(0011) (0011) . . . (0011) (10)

and label the cycle C_n, $n \equiv 3$ (mod 4) by the successive labels: (0011) (0011) . . . (0011) (001).

If $m+n \equiv 0$ (mod 4), then we may assume that either

(i) $m \equiv 0$ (mod 4) and $n \equiv 0$ (mod 4).

In this case $C_m \cup C_n$ is cordial by [3] and Theorem 2.1.

or (ii) $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$.

In this case $C_m \cup C_n$ is cordial by the following labeling : Label the cycle C_m, $m \equiv 1 \pmod 4$ by the successive labels:

(0011) (0011) . . . (0011) (0)

and the cycle C_n, $n \equiv 3 \pmod 4$ by the successive labels:

(0011) (0011) . . . (0011) (011).

or (iii) $m \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$.

In this case $C_m \cup C_n$ is cordial by the labeling

(0011) (0011) . . . (0011) (10) for C_m and

(0011) (0011) . . . (0011) (01) for C_n. □

Theorem 3.2

For $m \geq 1$, $n \geq 3$, mC_n is cordial if and only if $mn \not\equiv 2 \pmod 4$.

Proof

If $mn \equiv 2 \pmod 4$, then mC_n is not cordial by [3]

If m and n are odd, mC_n is sequential [17], hence is cordial by Proposition 1.

If $n \equiv 0 \pmod 4$, then mC_n is cordial for all $m \geq 1$ by [3] and Corollary 2.2

If $n \equiv 2 \pmod 4$ and m is even, then mC_n is cordial by the labeling : $(\underbrace{11\ldots1}_{\frac{n}{2}\text{-ones}})(\underbrace{00\ldots0}_{\frac{n}{2}\text{-zeros}})$ for $\frac{m}{2} C_n$ and

(0011) (0101) (0101) . . . (0101) (01) for the other $\frac{m}{2} C_n$.

If $n \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 4$

In this case mC_n is cordial by the labeling :

(0011) (0011) . . . (0011) (0) for $\frac{m}{2} C_n$ and

(0011) (0011) . . . (0011) (1) for $\frac{m}{4} C_n$ and

(1010) (1100) (1100) . . . (1100) (1) for $\frac{m}{4} C_n$.

If $n \equiv 3 \pmod 4$ and $m \equiv 0 \pmod 4$
In this case mC_n is cordial by the labeling :

(0011) (0011) . . . (0011) (001) for $\frac{3m}{4} C_n$ and

(0011) (0011) . . . (0011) (111) for $\frac{m}{4} C_n$. □

Theorem 3.3

For $m, n \geq 3$, $C_m + C_n$ is cordial if and only if $(m, n) \neq (3, 3)$ and $\{m,n\} \not\equiv \{0, 2\} \pmod 4$.

Proof

If $(m, n) = (3, 3)$, then $C_3 + C_3 = K_6$, which is not cordial by [3].

If $\{m, n\} \equiv \{0, 2\} \pmod 4$, then $C_m + C_n$ is not cordial by [3], since $|E(C_m + C_n)| = mn + m+n \equiv 2 \pmod 4$.

If $m \equiv 0 \pmod 4$ and $n \not\equiv 2 \pmod 4$, then $C_m + C_n$ is cordial by [3] and Theorem 2.4.

If $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$
$C_m + C_n$ is cordial by the following labeling :
Label the cycle C_m by the successive labels:
(0011) (0011) . . . (0011) (0)
Label the cycle C_n by the successive labels
(0011) (0011) . . . (0011) (1)

If $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$
$C_m + C_n$ is cordial by the labeling :
(0011) (0011) . . . (0011) (1) for C_m
and (0011) (0011) . . . (0011) (00) for C_n

If $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$
$C_m + C_n$ is cordial by the labeling :

(0011) (0011) . . . (0011) (1) for C_m
and (0011) (0011) . . . (0011) (001) for C_n

If $m \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$
$C_m + C_n$ is cordial by the labeling :
(0011) (0011) . . . (0011) (00) for C_m
and (1100) (1100) . . . (1100) (11) for C_n

If $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$
$C_m + C_n$ is cordial by the labeling :
(0011) (0011) . . . (0011) (00) for C_m
and (1100) (1100) . . . (1100) (110) for C_n

If $m \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$, $m \geq 3$ and $n \geq 7$
$C_m + C_n$ is cordial by the labeling :
(0011) (0011) . . . (0011) (001) for C_m
and (0011) (0011) . . . (0011) (110) for C_n. □

Recall that the k^{th} power G^k of a graph G, $2 \leq k \leq |V(G)| - 1$, is the graph having $V(G^k) = V(G)$ with the vertices u and v adjacent in G^k wherever $d(u, v) \leq k$ in G.

The following proposition gives a necessary condition for the cordiality of P_n^k, $2 \leq k \leq n-1$.

Proposition 3.4

Let $n \geq 4$. If P_n^k is cordial, then $n \geq k+1 + \left\lceil \sqrt{k-2} \right\rceil$.

Proof

We have $|E(P_n^k)| = kn - \frac{k}{2}(k+1)$ .Let that P_n^k is cordial, then,

$|E(P_n^k)| \leq$ maximum number of edges of a cordial graph of order n

$$= \begin{cases} \frac{n(n-2)}{2} + 1 & , \text{ if } n \text{ is even} \\ \frac{(n-1)^2}{2} + 1 & , \text{ if } n \text{ is odd} \end{cases}$$

by Du [7]

If n is even, then we must have

$$nk - \frac{k}{2}(k+1) \le 4\frac{n}{4}(\frac{n}{2}-1)+1$$

or $\quad (n-(k+1))^2 \ge k-1$

or $\quad n \ge k + 1 + \sqrt{k-1}$. Hence $n \ge k + 1 + \left\lceil \sqrt{k-2} \right\rceil$ in this case.

If n is odd, then we must have

$$nk - \frac{k}{2}(k+1) \le \frac{(n-1)}{4}\frac{(n-3)}{2} + \frac{(n+1)}{2}\frac{(n-1)}{2} + 1$$

or $\quad (n-(k+1))^2 \ge k-2$

or $\quad n \ge k + 1 + \left\lceil \sqrt{k-2} \right\rceil$. □

We conjectured that this necessary condition is in fact sufficient. For k = 2, 3 and 4, the results of Seoud and Abdel Maqsoud [14] confirm our conjecture for these k's. The following theorem confirms our conjecture for k = 5, 6, 7, 8 and 9.

Theorem 3.5

Let $n \ge 4$, then

(a) P_n^5 is cordial if and only if $n \ge 8$

(b) P_n^6 is cordial if and only if $n \ge 9$

(c) P_n^7 is cordial if and only if $n \ge 11$

(d) P_n^8 is cordial if and only if $n \ge 12$

(e) P_n^9 is cordial if and only if $n \ge 13$

Proof

The necessity of the stated condition in each case follow from Proposition 3.4. We prove its sufficiency.

(a) if $n \geq 8$, then for n even, P_n^5 is cordial by the labeling :

$$(00001111)\,(01)\prod_{i=1}^{k}(10)^3\,(01)^3$$

and for n odd, P_n^5 is cordial by the labeling :

$$(001001111)\,(01)\prod_{i=1}^{k}(10)^3\,(01)^3 .$$

(b) if $n \geq 9$, then for n odd, P_n^6 is cordial by the labeling :

$$(000011111)\,(01)\prod_{i=1}^{k}(10)^2\,(01) ,$$

and for n even, P_n^6 is cordial by the labeling :

$$(0010001111)\,(01)\prod_{i=1}^{k}(10)^2\,(01) .$$

(c) if $n \geq 11$, then for n odd, P_n^7 is cordial by the labeling :

$$(00000111111)\,(01)\prod_{i=1}^{k}(10)^4\,(01)^4 ,$$

and for n even, P_n^7 is cordial by the labeling :

$$(001000011111)\,(01)\prod_{i=1}^{k}(10)^4\,(01)^4 .$$

(d) if $n \geq 12$, then for n even, P_n^8 is cordial by the labeling
:

$$(000000111111)\,(01)\prod_{i=1}^{k}(10)^3\,(01) ,$$

and for n odd, P_n^8 is cordial by the labeling :

$$(0010000111111)\,(01)\prod_{i=1}^{k}(10)^3\,(01) .$$

(e) if $n \geq 13$, then for odd n, P_n^9 is cordial by the labeling :

$$(0000001111111)\,(01)\prod_{i=1}^{k}(10)^5\,(01)^5 ,$$

and for even n, P_n^9 is cordial by the labeling :

$$(00100000111111)\,(01)\prod_{i=1}^{k}(10)^5\,(01)^5 . \quad \square$$

One may extend the labeling of P_n^k for $k \geq 10$ in a similar way as in Theorem 3.5.

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