# Lower and Upper Orientable Strong Radius and Strong Diameter of the Cartesian Product of Complete Graphs

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**Abstract**

For two vertices $u$ and $v$ in a strong digraph $D$, the strong distance $sd(u,v)$ between $u$ and $v$ is the minimum size (the number of arcs) of a strong sub-digraph of $D$ containing $u$ and $v$. The strong eccentricity $se(v)$ of a vertex $v$ of $D$ is the strong distance between $v$ and a vertex farthest from $v$. The strong radius $srad(D)$ (resp. strong diameter $sdiam(D)$) of $D$ is the minimum (resp. maximum) strong eccentricity among all vertices of $D$. The lower (resp. upper) orientable strong radius $srad(G)$ (resp. $SRAD(G)$) of a graph $G$ is the minimum (resp. maximum) strong radius over all strong orientations of $G$. The lower (resp. upper) orientable strong diameter $sdiam(G)$ (resp. $SDIAM(G)$) of a graph $G$ is the minimum (resp. maximum) strong diameter over all strong orientations of $G$. In this paper, we determine the lower orientable strong radius and strong diameter of the Cartesian product of complete graphs, and give the upper orientable strong diameter and the bounds on the upper orientable strong radius of the Cartesian product of complete graphs.

**Keywords:** Strong distance, Lower and upper orientable strong radius and strong diameter.

# 1 Introduction

In [1], Chartrand *et al.* defined the strong distance $sd(u, v)$ between two vertices $u$ and $v$ in a strong digraph $D$ as the minimum size (the number of arcs) of a strong sub-digraph of $D$ containing $u$ and $v$. A $(u, v)$-geodesic is a strong sub-digraph of $D$ of size $sd(u, v)$ containing $u$ and $v$. Here we consider only strong oriented graphs of simple graphs. Clearly, if $u \neq v$ then $sd(u, v) \geq 3$. And $sd(u, v) = 3$ if and only if $u$ and $v$ belong to a directed 3-cycle in $D$. Fig. 1 shows a strong digraph with $sd(w, v) = 3$, $sd(u, w) = 5$, $sd(u, x) = 6$.

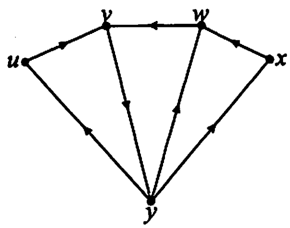


Figure 1: Strong distance in a strong digraph.

The strong eccentricity $se(v)$ of a vertex $v$ in a strong digraph $D$ is

$$se(v) = max\{sd(v, x)|\ x \in V(D)\}.$$

The strong radius $srad(D)$ of $D$ is $srad(D) = min\{se(v)|\ v \in (D)\}$, while the strong diameter $sdiam(D)$ of $D$ is $sdiam(D) = max\{se(v)|\ v \in V(D)\}$.

The strong radius and strong diameter of a strong digraph satisfy the following inequality.

**Theorem 1** [1]. For every strong digraph $D$, $srad(D) \leq sdiam(D) \leq 2srad(D)$.

In [1], Chartrand *et al.* showed that, for any integers $r, d$ with $3 \leq r \leq d \leq 2r$, there exists a strong oriented graph $D$ such that $srad(D) = r$ and $sdiam(D) = d$, and gave an upper bound on strong diameter of a strong oriented graph $D$.

**Theorem 2** [1]. If $D$ is a strong oriented graph of order $n \geq 3$, then $sdiam(D) \leq \lfloor \frac{5(n-1)}{3} \rfloor$.

In [4], Dankelmann *et al.* observed the structure of a $(u, v)$-geodesic for any two vertices $u$ and $v$ in a strong digraph $D$, and gave the following result.

**Theorem 3** [4]. Let $D$ be a strong digraph. For $u, v \in V(D)$, let $D_{uv}$ be a $(u, v)$-geodesic. Then $D_{uv} = P \cup Q$, where $P$ and $Q$ are a directed $(u, v)$-path and a directed $(v, u)$-path, respectively, in $D_{uv}$. There exist directed cycles $C_1, C_2, \ldots, C_k \subset D_{uv}$ such that
(i) $u \in V(C_1)$, $v \in V(C_k)$;
(ii) $\bigcup_{i=1}^{k} C_i = D_{uv}$;
(iii) each $C_i$ contains at least one arc that is in $P$ but not in $Q$, and at least one arc that is in $Q$ but not in $P$;
(iv) $C_i \cap C_{i+1}$ is a directed path for $i = 1, 2, \ldots, k-1$;
(v) $V(C_i) \cap V(C_j) = \emptyset$ for $1 \le i < j-1 \le k-1$.

In [4], Dankelmann *et al.* presented the upper bounds on strong diameter of $D$ in terms of order $n$, directed girth $g \ge 2$, and strong connectivity $\kappa$ as $sdiam(D) \le \lfloor \frac{(n-1)(g+2)}{g} \rfloor$ and $sdiam(D) \le \frac{5}{3}(1 + \frac{n-2}{\kappa})$. They also gave an upper bound on the strong radius of a strong oriented graph $D$.

**Theorem 4** [4]. For any strong oriented graph $D$ of order $n$, $srad(D) \le n$, and this bound is sharp.

In [6], for a connected graph $G$, Lai *et al.* defined the lower orientable strong radius $srad(G)$ of $G$ as

$$srad(G) = min\{srad(D) |\ D\ is\ a\ strong\ orientation\ of\ G\},$$

while the upper orientable strong radius $SRAD(G)$ of $G$ is

$$SRAD(G) = max\{srad(D) |\ D\ is\ a\ strong\ orientation\ of\ G\},$$

they also defined the lower orientable strong diameter $sdiam(G)$ of $G$ as

$$sdiam(G) = min\{sdiam(D) |\ D\ is\ a\ strong\ orientation\ of\ G\},$$

while the upper orientable strong diameter $SDIAM(G)$ of $G$ is

$$SDIAM(G) = max\{sdiam(D) |\ D\ is\ a\ strong\ orientation\ of\ G\}.$$

In [3], the present authors investigated the lower and upper orientable strong diameter of graphs satisfying the Ore condition.

**Theorem 5** [3]. Let $G = (V, E)$ be a bridgeless simple graph with girth $g$ and order $n$. If $\sigma_2(G) = min\{deg(x) + deg(y) |\ \forall\ xy \notin E(G)\} \ge n$, then $sdiam(G) \ge g$, $n \le SDIAM(G) \le n+1$, and the bounds are sharp.

The lower orientable strong radius and diameter, and the upper orientable strong diameter as well as bounds for the upper orientable strong radius of complete $k$-partite graphs were given in [7].

**Theorem 6** [7]. Let $k \geq 3$ and $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$. Then

$$srad(K(m_1, m_2, \ldots, m_k)) = \begin{cases} 3, & \text{if } 1 = m_1 \leq m_2 \leq \cdots \leq m_k; \\ 4, & \text{if } 2 \leq m_1 \leq m_2 \leq \cdots \leq m_k. \end{cases}$$

**Theorem 7** [7]. Let $k \geq 3$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, where $m_k \geq 2$, and let $m = m_1 + m_2 + \cdots + m_{k-1}$. Then

$$sdiam(K(m_1, m_2, \ldots, m_k)) = \begin{cases} 4, & \text{if } \binom{m}{\lfloor \frac{m}{2} \rfloor} \geq m_k; \\ 5, & \text{if } \binom{m}{\lfloor \frac{m}{2} \rfloor} < m_k. \end{cases}$$

**Theorem 8** [7]. Let $k \geq 3$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, and let $m = m_1 + m_2 + \cdots + m_{k-1}$. Then

$$SDIAM(K(m_1, m_2, \ldots, m_k)) = \begin{cases} 2m + 2, & \text{if } m < m_k; \\ m + m_k + 1, & \text{if } m \geq m_k. \end{cases}$$

**Theorem 9** [7]. Let $k \geq 3$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$ and $m = m_1 + m_2 + \cdots + m_{k-1}$, then

$$SRAD(K(m_1, m_2, \ldots, m_k)) \geq \begin{cases} m + 1, & \text{if } m < m_k; \\ \lfloor \frac{m+m_k}{2} \rfloor + 1, & \text{if } m \geq m_k. \end{cases}$$

$$SRAD(K(m_1, m_2, \ldots, m_k)) \leq \begin{cases} min\{2m + 2, m + m_k\}, & \text{if } m < m_k; \\ m + m_k, & \text{if } m \geq m_k. \end{cases}$$

For a graph $G$, let $\mathcal{D}(G)$ be the family of strong orientations of $G$, and define $\vec{d}(G) = min\{d(D) |\ D \in \mathcal{D}(G)\}$, where $d(D)$ denotes the diameter of the digraph $D$. In [5], Koh *et al.* obtained the exact values of $\vec{d}(K_m \times K_n)$, for $m \geq 2$ and $n \geq 2$.

**Theorem 10** [5]. For $m \geq 2$ and $n \geq 2$,

$$\vec{d}(K_m \times K_n) = \begin{cases} 4, & \text{if } (m, n) = (3, 2); \\ 3, & \text{otherwise}. \end{cases}$$

Some other results about strong distance can be found in [2, 8].

In this paper, we determine the lower orientable strong radius and strong diameter of the Cartesian product of complete graphs, and give the upper orientable strong diameter and the bounds on the upper orientable strong radius of the Cartesian product of complete graphs.

## 2 The lower orientable strong radius and diameter of the Cartesian product of complete graphs

In this section, we consider the lower orientable strong radius and diameter of the Cartesian product of complete graphs.

The Cartesian product $G = G_1 \times G_2$ of two vertex-disjoint graphs $G_1$ and $G_2$ has vertex set $V(G) = V(G_1) \times V(G_2)$ and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $G$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

For the Cartesian product $K_m \times K_n$ of complete graphs $K_m$ and $K_n$ with $m \geq 2$, $n \geq 2$, let $V(K_m \times K_n) = \{(i,j)|\ 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Thus, two vertices $(i,j)$ and $(i',j')$ are adjacent in $K_m \times K_n$ if and only if $i = i'$ or $j = j'$. In the following, without loss of generality, we assume that $2 \leq m \leq n$.

For any strong orientation $D$ of $K_2 \times K_2$, every vertex of $D$ has strong eccentricity 4, so $srad(K_2 \times K_2) = SRAD(K_2 \times K_2) = sdiam(K_2 \times K_2) = SDIAM(K_2 \times K_2) = 4$. Hence we only need to consider the case of $n \geq m \geq 2$ and $n > 2$.

Let $D$ be any strong orientation of $K_m \times K_n$. For any vertex $(i,j)$ of $K_m \times K_n$, there is a vertex $(i',j')$ with $i \neq i'$, $j \neq j'$. Clearly, $sd_D((i,j),(i',j')) \geq 4$, so $se_D((i,j)) \geq 4$ and $srad(D) \geq 4$, implying that $srad(K_m \times K_n) \geq 4$. In the following, we will prove that $srad(K_m \times K_n) = 4$ for $2 \leq m \leq n$.

**Theorem 11.** $srad(K_m \times K_n) = 4$ for $2 \leq m \leq n$, $sdiam(K_m \times K_n) = 5$ for $2 = m < n$, and $5 \leq sdiam(K_m \times K_n) \leq 6$ for $3 \leq m \leq n$.

**Proof.** Since $srad(K_m \times K_n) \geq 4$, in order to prove $srad(K_m \times K_n) = 4$, we only need to give a strong orientation $D$ of $K_m \times K_n$ such that $srad(D) = 4$.

Let $D$ be a strong orientation of $K_m \times K_n$ such that $A(D) = \{((1,i),(1,j))|1 \leq i < j \leq n\} \cup \{((s,j),(s,i))|1 \leq i < j \leq n,\ 2 \leq s \leq m\} \cup \{((j,1),(i,1))|1 \leq i < j \leq m\} \cup \{((i,s),(j,s))|1 \leq i < j \leq m,\ 2 \leq s \leq n\}$.

Consider the vertex $(1,1) \in V(K_m \times K_n)$. For any vertex $(i,j) \in$

$V(K_m \times K_n)$ with $i \geq 2$, $j \geq 2$, $(1,1)$ and $(i,j)$ are contained in the directed 4-cycle $(1,1)(1,j)(i,j)(i,1)(1,1)$. Clearly, this directed 4-cycle also contains $(i,1)$ and $(1,j)$. So, $se_D((1,1)) = 4$, and $srad(D) = 4$. Hence $srad(K_m \times K_n) = 4$ for $2 \leq m \leq n$.

If $n > m = 2$, it is not difficult to see that $se_D((1,1)) = se_D((2,1)) = 4$, $se_D((i,j)) = 5$ for any $(i,j) \in V(K_2 \times K_n) \setminus \{(1,1),(2,1)\}$. So $sdiam(D) = 5$.

To prove $sdiam(K_2 \times K_n) = 5$, it suffices to prove $sdiam(D) \geq 5$ for any strong orientation $D$ of $K_2 \times K_n$, where $n > 2$. Assume there is a strong orientation $D$ of $K_2 \times K_n$ such that $sdiam(D) = 4$. Then $se_D((i,j)) = 4$ for any $(i,j) \in V(K_m \times K_n)$. So $sd_D((1,1),(2,2)) = sd_D((1,1),(2,3)) = 4$, $(1,1)$ and $(2,2)$ are contained in a directed 4-cycle in $D$, say $(1,1)(1,2)(2,2)(2,1)(1,1)$. Then $(1,1)$ and $(2,3)$ must be in the directed 4-cycle $(1,1)(1,3)(2,3)(2,1)(1,1)$. However, in this orientation, $(1,3)$ and $(2,2)$ can't be contained in any directed 4-cycle, a contradiction.

By a similar argument to that given above, we have $sdiam(K_m \times K_n) \geq 5$ for $3 \leq m \leq n$. On the other hand, by Theorem 10, we know that there exists a strong orientation $D'$ of $K_m \times K_n$ such that $d(D') = 3$, where $d(D')$ denotes the diameter of the digraph $D'$. Thus, for any two vertices $(i,j), (i',j') \in V(K_m \times K_n)$, there exists a directed $((i,j),(i',j'))$-path $P$ of length at most 3 and a directed $((i',j'),(i,j))$-path $Q$ of length at most 3, and so $sd_{D'}((i,j),(i',j')) \leq |A(P \cup Q)| \leq 6$. Now it follows that $5 \leq sdiam(K_m \times K_n) \leq sdiam(D') \leq 6$ for $3 \leq m \leq n$. □

We have the following conjecture.

**Conjecture 12.** $sdiam(K_m \times K_n) = 6$ for $3 \leq m \leq n$.

# 3 The upper orientable strong radius and diameter of the Cartesian product of complete graphs

For a vertex $v$ of a graph $G$, let $N_G(v)$ denote the set of the vertices adjacent to $v$ in $G$.

**Lemma 13.** Let $D$ be any strong orientation of $K_m \times K_n$ with $n \geq m \geq 2$ and $n > 2$. Then $sdiam(D) \leq mn + 1$.

**Proof.** For any strong orientation $D = (V(D), A(D))$ of $K_m \times K_n$, let $u,\ v \in V(D)$ be two vertices such that $sd_D(u,v) = sdiam(D)$. We will prove that $sd_D(u,v) \leq mn + 1$.

Assume $uv \in E(K_m \times K_n)$, then without loss of generality, assume that $(v,u) \in A(D)$. Let $P$ be a shortest directed $(u,v)$-path and $C = P + (v,u)$. Clearly, $C$ is a directed cycle containing $u$ and $v$. So $sd_D(u,v) \leq |A(C)| \leq mn$.

Assume $uv \notin E(K_m \times K_n)$. Let $D_{uv}$ be a $(u,v)$-geodesic in $D$. By Theorem 3, we have $D_{uv} = P \cup Q$, where $P$ is a directed $(u,v)$-path and $Q$ is a directed $(v,u)$-path in $D_{uv}$. Furthermore, there exist directed cycles $C_1, C_2, \ldots, C_k$ in $D_{uv}$ such that $D_{uv} = \bigcup_{i=1}^{k} C_i$ satisfying (i)-(v) in Theorem 3.

If $k = 1$, then $sd_D(u,v) = |A(C_1)| = |V(C_1)| \leq mn$.

If $k = 2$, then $sd_D(u,v) = |A(C_1 \cup C_2)| = |V(C_1 \cup C_2)| + 1 \leq mn + 1$.

If $k \geq 3$, by Theorem 3 (iv), let $P_i = C_i \cap C_{i+1}$ be a directed path starting from the vertex $a_i$ of $P_i$, $i = 1, 2, \ldots, k-1$. Take $S = \{u, a_2, a_4, \ldots, a_{2(\lfloor \frac{k-1}{2} \rfloor - 1)}, v\}$. By Theorem 3 (iii) and the minimality of $D_{uv}$, for any $x = (i,j)$ and $y = (i',j')$ in $S$, $xy \notin E(K_m \times K_n)$, and so $i \neq i'$, $j \neq j'$. Then, for any $z \in S - v$, $z$ and $v$ have exactly two common neighbors and $N_{K_m \times K_n}(z) \cap N_{K_m \times K_n}(v) \cap V(D_{uv}) = \emptyset$ by Theorem 3 (iii) and the minimality of $D_{uv}$. Moreover for $k \geq 5$ and any $z, z' \in S - v$, $(N_{K_m \times K_n}(z) \cap N_{K_m \times K_n}(v)) \cap (N_{K_m \times K_n}(z') \cap N_{K_m \times K_n}(v)) = \emptyset$ since $zz', zv, z'v \notin E(K_m \times K_n)$. Thus there are at least $2|S - v| = 2(\lfloor \frac{k-1}{2} \rfloor)$ vertices adjacent to $v$ and not in $D_{uv}$, implying that $mn - |V(D_{uv})| \geq 2(\lfloor \frac{k-1}{2} \rfloor)$, $|V(D_{uv})| \leq mn - 2(\lfloor \frac{k-1}{2} \rfloor)$. Hence, $sd_D(u,v) = |A(D_{uv})| = |V(D_{uv})| + k - 1 \leq mn - 2\lfloor \frac{k-1}{2} \rfloor + k - 1 \leq mn + 1$. The proof is completed. □

**Theorem 14.** $SDIAM(K_m \times K_n) = mn + 1$ for $n \geq m \geq 2$ and $n > 2$.

**Proof.** By Lemma 13, we only need to give a strong orientation $D$ of $K_m \times K_n$ such that $sdiam(D) = mn + 1$.

Let $P$ be the Hamiltonian path of $K_m \times K_n$ starting from $x = (1,1)$

such that if $m$ is odd (resp. even) then the terminating vertex $y$ of $P$ is equal to $(m, n)$ (resp. $(m, 2)$). Let $D$ be a strong orientation of $K_m \times K_n$ such that the path $P$ is a shortest directed $(x, y)$-path. Let $D_{xy} = P' \cup Q'$ be a $(x, y)$-geodesic, where $P'$ is a directed $(x, y)$-path and $Q'$ is a directed $(y, x)$-path. Since $xy \notin E(K_m \times K_n)$, $sd_D(x, y) = |A(P')| + |A(Q') \setminus A(P')| \geq |A(P)|+2 = mn+1$. On the other hand, by Lemma 13, $sd_D(x, y) \leq mn+1$. So $sd_D(x, y) = mn + 1$, consequently, $sdiam(D) = mn + 1$, which implies that $SDIAM(K_m \times K_n) = mn + 1$ for $n \geq m \geq 2$ and $n > 2$. □

Finally, by Theorem 1, Theorem 4 and Theorem 14, we can give the bounds on the upper orientable strong radius of $K_m \times K_n$.

**Theorem 15.** Let $n \geq m \geq 2$ and $n > 2$. Then
$\lceil \frac{mn+1}{2} \rceil \leq SRAD(K_m \times K_n) \leq mn.$

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