On fractional (g, f, n)-critical graphs *

Hongxia Liu[a,b]

[a]School of Mathematics and Informational Science, Yantai University
Yantai, Shandong 264005, P. R. China

[b]School of Mathematics, Shandong University
Jinan, Shandong 250100, P. R. China

e-mail: mqy7174@sina.com

Guizhen Liu

School of Mathematics, Shandong University
Jinan, Shandong 250100, P. R. China

e-mail: gzliu@sdu.edu.cn

Abstract

Let G be a graph. Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \le f(x)$ for any $x \in V(G)$. A spanning subgraph F of G is called a fractional (g, f)-factor if $g(x) \le d_G^h(x) \le f(x)$ for all $x \in V(G)$, where $d_G^h(x) = \sum_{e \in E_x} h(e)$ is the fractional degree of $x \in V(F)$ with $E_x = \{e : e = xy \in E(G)\}$. A graph G is said to be fractional (g, f, n)-critical if $G - N$ has a fractional (g, f)-factor for each $N \subseteq V(G)$ with $|N| = n$. In this paper, several sufficient conditions in terms of stability number and degree for graphs to be fractional (g, f, n)-critical are given. Moreover, we show that the results in this paper are best possible in some sense.

Keywords: graph, fractional (g, f)-factor, fractional (g, f, n)-critical graph, stability number, degree

Mathematics Subject Classification 2000: 05C70

*This work is supported by NSFC(10871119), SRFDP(200804220001) of China and the Project of Shandong Province Higher Educational Science and Technology Program.

1. Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d_G(x)$ the degree of a vertex x in G and by $N_G(x)$ the set of vertices adjacent to x in G. We use $N_G[x]$ to denote $N_G(x) \cup \{x\}$ and $\delta(G)$ to denote the minimum degree of G. For a subset $S \subseteq V(G)$, we denote by $N_G(S)$ the union of $N_G(x)$ for every $x \in S$, by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting the vertices in S together with the edges incident to the vertices in S.

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for every $x \in V(G)$. Let h be a function defined on $E(G)$ such that $h(e) \in [0,1]$ for every edge $e \in E(G)$. We call h a fractional (g, f)-indicator function if for each vertex x we have $g(x) \leq d_G^h(x) \leq f(x)$, where $d_G^h(x) = \sum_{e \in E_x} h(e)$ is the fractional degree of $x \in G$ with $E_x = \{e : e = xy \in E(G)\}$. Set $E_h = \{e : e \in E(G)$ and $h(e) \neq 0\}$. If G_h is a spanning subgraph of G such that $E(G_h) = E_h$, then G_h is called a fractional (g, f)-factor of G with indicator function h. In particular, if $g(x) = f(x)$ for every $x \in V(G)$, then a fractional (g, f)-factor is also referred as a fractional f-factor. If $f(x) = k$ for all $x \in V(G)$, then a fractional f-factor is a fractional k-factor. A graph G is said to be fractional (g, f, n)-critical if $G - N$ has a fractional (g, f)-factor for each $N \subseteq V(G)$ with $|N| = n$. If $g(x) = f(x)$ for every $x \in V(G)$, then a fractional (g, f, n)-critical graph is a fractional (f, n)-critical graph. If $f(x) = k$ for all $x \in V(G)$, then a fractional (f, n)-critical graph is fractional (k, n)-critical. If $k = 1$, then a fractional (k, n)-critical graph is simply called a fractional n-critical graph. The other terminologies and notations can be found in [1].

Many authors have investigated graph factors [3, 10, 11], fractional (g, f)-factors [6, 7, 9, 13], and critical graphs[5, 12, 14]. In [15], S. Zhou and H. Liu gave a neighborhood condition for a graph to have a fractional k-factor.

Theorem A. [15] *Let k be an integer such that $k \geq 1$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, and the minimum degree $\delta(G) \geq k$. If*

$$|N_G(x) \cup N_G(y)| \geq \max\left\{\frac{n}{2}, \frac{1}{2}(n + k - 2)\right\}$$

for each pair of non-adjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.

In [4], J. Cai and G. Liu gave a stability number and minimum degree condition for the existence of fractional f-factor in a graph.

Theorem B. [4] *Let G be a graph, a, b be nonnegative integers such that $a \leq b$, and f be a nonnegative integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq b$, and stability number $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then G has a fractional f-factor.*

In [6], G. Liu and L. Zhang gave a degree condition for a graph to have a fractional (g, f)-factor.

Theorem C. [6] *Let G be a graph. If*

$$\frac{g(w)}{d_G(w)} \leq \frac{f(v)}{d_G(v)}$$

for every pair of vertices $v, w \in V(G)$, then G has a fractional (g, f)-factor.

In [16], S. Zhou and Q. Shen gave a sufficient condition in terms of binding number for a graph to be fractional (f, n)-critical.

Theorem D. [16] *Let G be a graph of order p, and let a, b and n be nonnegative integers such that $2 \leq a \leq b$, and let f be an integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for each $x \in V(G)$. If $bind(G) > \frac{(a+b-1)(p-1)}{(ap-(a+b)-bn+2)}$ and $p \geq \frac{(a+b)(a+b-3)}{a} + \frac{bn}{a-1}$, then G is fractional (f, n)-critical.*

We extend Theorem B and Theorem C to fractional (g, f, n)-critical graphs and obtain the following results.

Theorem 1. *Let G be a graph, a, b and n be nonnegative integers such that $a \leq b$, and $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq b$, and stability number $\alpha(G) \leq \frac{4(a(\delta-b+1)-bn)}{(b+1)^2}$, then G is fractional (g, f, n)-critical.*

Theorem 2. *Let G be a graph, a, b and n be nonnegative integers such that $a \leq b$, and $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$. If*

$$\frac{g(w) + bn}{d_G(w)} \leq \frac{f(v)}{d_G(v)} \tag{1}$$

for every pair of vertices $v, w \in V(G)$, then G is fractional (g, f, n)-critical.

In Theorem 1, if $g(x) = f(x)$ for every $x \in V(G)$, then we get the following corollary.

Corollary 1. *Let G be a graph, a, b and n be nonnegative integers such that $a \leq b$, and $f(x)$ be a nonnegative integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq b$, and stability number $\alpha(G) \leq \frac{4(a(\delta-b+1)-bn)}{(b+1)^2}$, then G is fractional (f, n)-critical.*

In Corollary 1, if $n = 0$, then we get the following corollary.

Corollary 2. *Let G be a graph, a and b be nonnegative integers such that $a \leq b$, and $f(x)$ be a nonnegative integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq b$, and stability number $\alpha(G) \leq \frac{4a(\delta-b+1)}{(b+1)^2}$, then G has a fractional f-factor.*

Since $\frac{4a(\delta-b)}{(b+1)^2} \leq \frac{4a(\delta-b+1)}{(b+1)^2}$, it is easy to see Theorem B is a special case of Corollary 2.

It is easy to see that Theorem C is a special case of Theorem 2 for $n = 0$ when $a \leq g(x) \leq f(x) \leq b$.

2. Preliminary lemmas

Let Z be the set of integers, and let S and T be disjoint subsets of $V(G)$. For $g, f : V(G) \to Z$, we denote $f(S) = \sum_{x \in S} f(x)$, $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$, and $g(T) = \sum_{x \in T} g(x)$. Let

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T),$$

and

$$f_n(S) = \max\{f(N) : N \subseteq S \text{ and } |N| = n\}$$

for $|S| \geq n$.

The following lemma is due to Anstee [2], G. Liu and L. Zhang[7].

Lemma 1.[2, 7] *Let G be a graph, and $g, f : V(G) \to Z$ be two functions such that $g(x) \leq f(x)$ for each $x \in V(G)$. A graph G has an fractional (g, f)-factor if and only if for any subset $S \subseteq V(G)$, we have*

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq 0, \tag{2}$$

where $T = \{x : x \in V(G) - S$, *and* $d_{G-S}(x) < g(x)\}$.

Lemma 2. *Let* G *be a graph, and let* $n \geq 0$ *be an integer. Let* $g, f : V(G) \to Z$ *be two functions such that* $g(x) \leq f(x)$ *for each* $x \in V(G)$. *A graph* G *is fractional* (g, f, n)*-critical if and only if*

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq f_n(S) \tag{3}$$

for any subset $S \subseteq V(G)$ *with* $|S| \geq n$, *where* $T = \{x : x \in V(G) - S$, *and* $d_{G-S}(x) < g(x)\}$.

Proof. Suppose that G is fractional (g, f, n)-critical. Let $S \subseteq V(G)$ be any subset such that $G[S]$ has a subgraph N such that S has a subset N with $f(N) = f_n(S)$. Then $G - N$ has a fractional (g, f)-factor. Let $G' = G - N$, $S' = S - N$, and $T' = \{x : x \in V(G') - S'$, and $d_{G'-S'}(x) < g(x)\}$. Then $G' - S' = G - S$. By Lemma 1, G' has a fractional (g, f)-factor if and only if

$$\delta_{G'}(S',T') = f(S') + d_{G'-S'}(T') - g(T') \geq 0$$

for all disjoint subsets S' and T' of $V(G')$, where $T' = \{x : x \in V(G') - S'$, and $d_{G'-S'}(x) < g(x)\}$. Since $T' = T$ from the definition of T, and $d_{G'-S'}(T') = d_{G-S}(T)$, we have

$$\begin{aligned} 0 \leq \delta_{G'}(S',T') &= f(S') + d_{G'-S'}(T') - g(T') \\ &= f(S) - f(N) + d_{G-S}(T) - g(T) \\ &= \delta_G(S,T) - f(N) = \delta_G(S,T) - f_n(S). \end{aligned}$$

Therefore $\delta_G(S,T) \geq f_n(S)$ holds.

Conversely, we suppose that $\delta_G(S,T) \geq f_n(S)$ holds for any subset $S \subseteq V(G)$ with $|S| \geq n$. Let $N \subseteq V(G)$ with $|N| = n$, and $G' = G - N$. For every subset $S' \subseteq G'$, let $S = S' \bigcup N$. Define $T' = \{x : x \in V(G') - S'$, and $d_{G'-S'}(x) < g(x)\}$. Then $G' - S' = G - S$, $T' = T$ and $d_{G'-S'}(T') = d_{G-S}(T)$. From the definition of $f_n(S)$, we have

$$\begin{aligned} \delta_{G'}(S',T') &= f(S') + d_{G'-S'}(T') - g(T') \\ &= f(S) - f(N) + d_{G-S}(T) - g(T) \\ &= \delta_G(S,T) - f(N) \geq \delta_G(S,T) - f_n(S) \geq 0. \end{aligned}$$

Thus G' has a fractional (g, f)-factor by Lemma 1. Consequently G is fractional (g, f, n)-critical. □

3. The proof of main theorems

Proof of Theorem 1. We prove the theorem by contradiction. Suppose that G is not fractional (g, f, n)-critical. Then, by Lemma 2, there exists some subset $S \subseteq V(G)$ with $|S| \geq n$ such that

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \leq f_n(S) - 1, \quad (4)$$

where $T = \{x : x \in V(G) - S, \text{ and } d_{G-S}(x) < g(x)\}$.

Claim 1. $T \neq \emptyset$.

Proof. If $T = \emptyset$, then by (4) and the definition of $f_n(S)$, $f(S) = \delta_G(S,T) \leq f_n(S) - 1 \leq f(S) - 1$. This is a contradiction. □

Since $T \neq \emptyset$, define $h = \min\{d_{G-S}(x) \mid x \in T\}$. Then

$$| S | \geq \delta(G) - h. \quad (5)$$

Claim 2. $0 \leq h \leq b - 1$.
Proof. According to the definition of T, we know that any $x \in T$ satisfies $d_{G-S}(x) < g(x) \leq b$. Hence $0 \leq h \leq g(x) - 1 \leq b - 1$. □

Since $T \neq \phi$, in the following we shall construct a sequence $x_1, x_2, ... x_k$ of vertices of T. We take $x_1 \in T$ such that x_1 is the vertex with the least degree in $G[T]$. Let $N_1 = N_G[x_1] \bigcap T$ and $T_1 = T$. For $i \geq 2$, if $T - \bigcup_{j<i} N_j \neq \emptyset$, let $T_i = T - \bigcup_{j<i} N_j$. Then take $x_i \in T_i$ such that x_i is the vertex with the least degree in $G[T_i]$, and set $N_i = N_G[x_i] \bigcap T_i$. We continue this procedures until we reach the situation in which $T_i = \emptyset$ for some i, say for $i = k + 1$. Then from the above definition we know that $x_1, x_2, \cdots, x_k$ is an independent set of G. Since $T \neq \emptyset$, we have $k \geq 1$.

Let $| N_i | = n_i$. We can easily get the following properties by the definition of N_i.

$$\alpha(G[T]) \geq k, \quad (6)$$

$$| T | = \sum_{1 \leq i \leq k} n_i, \quad (7)$$

$$\sum_{1 \leq i \leq k} (\sum_{x \in N_i} d_{T_i}(x)) \geq \sum_{1 \leq i \leq k} (n_i^2 - n_i). \quad (8)$$

It follows that

$$d_{G-S}(T) \geq \sum_{1 \leq i \leq k} (n_i^2 - n_i) + \sum_{1 \leq i < j \leq k} e_G(N_i, N_j) \geq \sum_{1 \leq i \leq k} (n_i^2 - n_i). \quad (9)$$

Define $f(n_i) = n_i(n_i - b - 1)$. Then $n_i(n_i - b - 1)$ attains its minimum value $-(b+1)^2/4$ at $\frac{b+1}{2}$ by derivation. Thus $n_i(n_i - b - 1) \geq -(b+1)^2/4$. Combining this inequality with (4), (7), (9), we get

$$\begin{aligned} f_n(S) - 1 \geq \delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\ &\geq a|S| + d_{G-S}(T) - b|T| \\ &\geq a|S| + \sum_{1\leq i\leq k} (n_i(n_i - b - 1)) \\ &\geq a|S| - (b+1)^2 k/4. \end{aligned}$$

Since $\alpha(G) \geq \alpha(G[T]) \geq k$, by (5), (6), and the conditon $\alpha(G) \leq \frac{4(a(\delta-b+1)-bn)}{(b+1)^2}$, we obtain

$$\begin{aligned} f_n(S) - 1 \geq \delta_G(S,T) &\geq a|S| - (b+1)^2 k/4 \\ &\geq a(\delta(G) - h) - \frac{(b+1)^2}{4}\frac{4(a(\delta - b + 1) - bn)}{(b+1)^2} \\ &= a(\delta(G) - h) - (a(\delta(G) - b + 1) - bn) \\ &= a(b - 1 - h) + bn \end{aligned}$$

Since $h \leq b - 1$ by Claim 2, we have

$$bn - 1 \geq f_n(S) - 1 \geq \delta_G(S,T) \geq bn.$$

That is a contradiction. This completes the proof of Theorem 1. □

Proof of Theorem 2. To prove the theorem , we need only to verify condition (3) of Lemma 2. For any subset $S \subseteq V(G)$ with $|S| \geq n$, let $T = \{x : x \in V(G) - S, \text{ and } d_{G-S}(x) < g(x)\}$, $\bar{S} = V(G) - S$. Note that $0 \leq g(x) \leq f(x) \leq d_G(x)$. If $S = \emptyset$, then (3) holds trivially. So we assume $S \neq \emptyset$. By (1) we get $f(S)d_G(T) - d_G(S)g(T) \geq bn|T|d_G(S)$. Note that $d_G(\bar{S}) - d_{G-S}(\bar{S}) \leq d_G(S)$ and $d_G(S) \geq f(S)$. Thus

$$\begin{aligned} f(S) + d_{G-S}(T) - g(T) &\geq \frac{f(S)}{d_G(S)}(d_G(\bar{S}) - d_{G-S}(\bar{S})) - (g(T) - d_{G-S}(T)) \\ &= \frac{1}{d_G(S)}(f(S)d_G(T) - d_G(S)g(T) \\ &\quad + \frac{1}{d_G(S)}(d_G(S) - f(S))d_{G-S}(T) \\ &\quad + \frac{f(S)}{d_G(S)}(d_G(V(G) - (S \cup T)) \\ &\quad - d_{G-S}(V(G) - (S \cup T))) \\ &\geq \frac{1}{d_G(S)}(bn|T|d_G(S)). \end{aligned}$$

If $T = \emptyset$, then (3) holds trivially. If $T \neq \emptyset$, then by the above inequality, we have $f(S) + d_{G-S}(T) - g(T) \geq \frac{1}{d_G(S)}(bn|T|d_G(S)) \geq \frac{1}{d_G(S)}(bnd_G(S)) = bn \geq f_n(S)$. Thus (3) holds. Therefore G is fractional (g, f, n)-critical. This completes the proof of Theorem 2. □

4. Remarks

Remark 1. The condition of Theorem 1 is sharp. The upper bound on the stability number condition $\alpha(G) \leq \frac{4(a(\delta-b+1)-bn)}{(b+1)^2}$ is best possible in the following sense. We cannot replace $\alpha(G) \leq \frac{4(a(\delta-b+1)-bn)}{(b+1)^2}$ by $\alpha(G) \leq \frac{4(a(\delta-b+1)-bn)}{(b+1)^2} + 1$ in Theorem 1, which is shown by the following example.

Let $k = \lfloor \frac{4(a(\delta-b+1)-bn)}{(b+1)^2} \rfloor \geq 2$. We let $G_1 = K_2$ and $G_2 = \bigcup_{i=1}^{k+1} K_b^i$, where K_b^i is a complete graph with b vertices$(1 \leq i \leq k+1)$. Then let $G = G_1 + G_2$ be the join of G_1 and G_2(that is, $V(G) = V(G_1) \bigcup V(G_2)$ and $E(G) = E(G_1) \bigcup E(G_2) \bigcup \{uv : u \in V(G_1), v \in V(G_2)\}$). Then we have $\alpha(G) = \lfloor \frac{4(a(\delta-b+1)-bn)}{(b+1)^2} \rfloor + 1$ and $\delta(G) = b + 1 \geq b$. We take $S = V(G_1)$, $g(x) = a$ and $f(x) = a$ for $x \in V(G_1)$; $T = V(G_2)$, $g(x) = b$ and $f(x) = b$ for $x \in V(G_2)$. It follows that

$$\begin{aligned} \delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\ &= 2a + b \cdot (b-1)(k+1) - bb(k+1) \\ &= 2a - b(k+1) \\ &< an = f_n(S). \end{aligned}$$

Therefore, according to Lemma 2, G is not fractional (g, f, n)-critical.

References

[1] B. Alspach, K. Heinrich, and G. Liu, Contemporary design theory-A Collection of Surveys, Wiley, New York, pp. (1992), 13-37.

[2] R. P. Anstee, An algorithmic proof of Tuttes f-factor theorem, J. Algorithms 6 (1985) (1), 112-131.

[3] Q. Bian, On toughness and (g, f)-factors in bipartite graphs, J. Appl. Math. and Computing, 22 (2006), 299-304.

[4] J. Cai and G. Liu, Stability number and fractional f-factors in graphs, Ars Comb., 80(2006), 141-146.

[5] G. Liu and J. Wang, (a, b, k)-critical graphs, Advances in Mathematics(in China), 27(6)(1998), 536-540.

[6] G. Liu and L. Zhang, Maximum fractional $(0, f)$-Factors of Graphs, Mathematica Applicata, 13(2000), 31-35.

[7] G. Liu and L. Zhang, Fractional $(g.f)$-factors of graphs, Acta Math. Scientia Ser B, 21(2001), 541-545.

[8] H. Liu and G. Liu, Binding number and minimum degree for the existence of (g, f, n)-critical graphs, J. Appl. Math. Comput., 29(2009), 207-216.

[9] H. Liu and G. Liu, Neighborhood union and fractional f-factors in graphs, J. Appl. Math. and Informatics, 27 (2009), 1157-1163.

[10] L. Lovász, On the structure of factorizable graph, Acta Math. Acad. Sci. Hungar 23(1972), 179-195.

[11] W. T. Tutte, The factors of graphs, Can. J. Math., 4(1952), 314-328.

[12] Q. Yu, Characterizations of various matching extension graphs, Australasian J. Combin. 7 (1993), 55-64.

[13] L. Zhang and G. Liu, Fractional k-factors of graphs , J. Sys. Sci. Math. Scis. 21(2001), 88-92.

[14] S. Zhou, Independence number, connectivity and (a, b, k)-critical graphs, Discrete Math., 309(2009), 4144-4148.

[15] S. Zhou and H. Liu, Neighborhood conditions and fractional k-factors, BULLETIN of the Malaysian Mathematical Sciences Society, 1(2009), 37-45.

[16] S. Zhou and Q. Shen, On fractional (f, n)-critical graphs, Information Processing Letters, 109(2009), 811-815.