On competition polysemy and m-competition polysemy*

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Abstract

In 2004, Fischermann et al. [2] generalized bound polysemy to competition polysemy by using digraphs instead of posets. They provided a characterization of competition polysemic pairs and a characterization of the connected graphs G for which there exists a tree T such that (G, T) is competition polysemic. In this paper we continue to study the competition polysemy and characterize the connected graphs G for which there exists a triangle-free unicyclic graph G' such that (G, G') is competition polysemic. Furthermore,

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we generalize competition polysemy to m-competition polysemy and prove a characterization of m-competition polysemic pairs.

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1 Introduction

All graphs considered in this paper are finite and simple. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V : uv \in E\}$. A *clique* of G is the vertex set of a (not necessarily maximal) complete subgraph of G. An *edge clique cover* of G is a collection $\mathscr{C}$ of cliques such that for every edge $uv \in E$ some clique in $\mathscr{C}$ contains both vertices u and v. An *m-edge clique cover* of G is a collection of subsets $C_1, C_2, \ldots, C_k$ of V such that $xy \in E$ if and only if there exist m of the sets C_i that contain both x and y. A *block* of $G = (V, E)$ is a maximal 2-connected subgraph of G and a vertex $u \in V$ for which $G - u = G[V \setminus \{u\}]$ (the subgraph of G induced by $V \setminus \{u\}$) has more components than G is a *cutvertex*.

In 2000, Tanenbaum [6] introduced the notion of *bound polysemy*. He called a pair (G_1, G_2) of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on a common set of vertices V *bound polysemic*, if there exists a reflexive poset $P = (V, \leq)$ on the set V such that for all $u, v \in V$ with $u \neq v, uv \in E_1$ if and only if there is some $w \in V$ such that $u \leq w$ and $v \leq w$ and $uv \in E_2$ if and only if there is some $w \in V$ such that $w \leq u$ and $w \leq v$. In this situation the graphs G_1 and G_2 are called the *upper bound graph* and the *lower bound graph* of P, respectively. Upper bound graphs were introduced by McMorris and Zaslavsky in [5].

In 2004, Fischermann et al. [2] generalized bound polysemy to *competition polysemy* by using digraphs instead of posets. They called a pair (G_1, G_2) of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on a common set of vertices V competition polysemic, if there exists a digraph $D = (V, A)$ on the same set of vertices such that for all $u, v \in V$ with $u \neq v, uv \in E_1$ if and only if $N_D^+(u) \cap N_D^+(v) \neq \emptyset$ and $uv \in E_2$ if and only if $N_D^-(u) \cap N_D^-(v) \neq \emptyset$. In this situation D is called a *realization* of (G_1, G_2). Clearly, the graphs G_1 and G_2 are the *competition graph* (see [1]) and *common enemy graph* (see [4]) of D, respectively. Competition graphs were introduced by Cohen [1] and have been studied by various authors. For any digraph $D = (V, A)$, in fact, the common enemy graph of D is the competition graph of $\overleftarrow{D}$, where $\overleftarrow{D} = (V, \overleftarrow{A})$ and $\overleftarrow{A} = \{\overrightarrow{vu} : \overrightarrow{uv} \in A\}$.

Fischermann et al. [2] provided a characterization of competition polysemic pairs and a characterization of the connected graphs G for which there exists a tree T such that (G, T) is competition polysemic. In this paper we continue to study the competition polysemy and generalize competition polysemy to *m-competition polysemy*. For positive integer $m \geq 1$, we call a pair (G_1, G_2) of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on a common set of vertices V m-competition polysemic, if there exists a digraph $D = (V, A)$ on the same set of vertices such that for all $u, v \in V$ with $u \neq v$, $uv \in E_1$ if and only if $|N_D^+(u) \cap N_D^+(v)| \geq m$ and $uv \in E_2$ if and only if $|N_D^-(u) \cap N_D^-(v)| \geq m$. Clearly, the graph G_1 is the m-competition graph (see [3]) of D. Furthermore, we call the graph G_2 the *m-common competition graph* of D. Obviously, 1-competition polysemic is just the competition polysemic.

Section 2 provides a characterization of the connected graphs G for which there exists a triangle-free unicyclic graph G' such that (G, G') is competition polysemic. Section 3 generalizes competition polysemy to m-competition polysemy and proves a characterization of m-competition polysemic pairs.

2 Competition polysemy

For given graph $G = (V, E)$, if $\mathscr{C} = \{C_1, C_2, \ldots, C_p\}$ is an edge clique cover of G with $p \leq |V|$, then we can choose a set of p different vertices $R = \{v_1, v_2, \ldots, v_p\} \subseteq V$. We call R a set of distinct representatives of the cliques in $\mathscr{C}$ (Note that we do not require $v_i \in C_i$ for $1 \leq i \leq p$). Fischermann et al. [2] provided a characterization of competition polysemic pairs of graphs as follows.

Theorem 1 (Fischermann et al. [2]) *A pair (G_1, G_2) of graphs with $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is competition polysemic if and only if there exist edge clique covers $\mathscr{C}_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,p}\}$ of G_1 and $\mathscr{C}_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,q}\}$ of G_2 for which there exist sets of distinct representatives $R_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,p}\}$ and $R_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,q}\}$, $p, q \leq |V|$, such that*

(i) $v_{2,i} \in C_{1,j}$ if and only if $v_{1,j} \in C_{2,i}$,

(ii) if $C_{1,i} \cap C_{1,j} \neq \emptyset$, then there is some $1 \leq l \leq q$ such that $v_{1,i}, v_{1,j} \in C_{2,l}$ and

(iii) if $C_{2,i} \cap C_{2,j} \neq \emptyset$, then there is some $1 \leq l \leq p$ such that $v_{2,i}, v_{2,j} \in C_{1,l}$.

Recall that a *unicyclic graph* is a graph which is connected and has exactly one cycle, and a *spiked cycle* is a graph that is a cycle upon removal of all pendant edges. In the following we characterized the connected graphs

G for which there exists a triangle-free unicyclic graph G' such that (G, G') is competition polysemic.

Theorem 2 *Let $G = (V, E_G)$ be a connected graph. There is a triangle-free unicyclic graph $G' = (V, E_{G'})$ such that (G, G') is competition polysemic if and only if*

(i) exactly one block of G is not complete,

(ii) every cutvertex of G lies in exactly two blocks of G and

(iii) if B_0 is the block of G that is not complete, then the vertex set of B_0 is the union of some cliques of G, these cliques can be labeling as $C_1, C_2, \ldots, C_t$ such that $|C_i \cap C_{i+1}| = 1$ for $1 \le i \le t$, and the vertex in $C_i \cap C_{i+1}$ for each $1 \le i \le t$ lies in no other block of G, where $t \ge 4$ and $C_{t+1} = C_1$.

Proof. Suppose that (G, G') is competition polysemic with realization D, where $G = (V, E_G)$ is a connected graph and $G' = (V, E_{G'})$ is a triangle-free unicyclic graph. Let $V = \{v_1, v_2, \ldots, v_n\}$ and, for $1 \le i \le n$ let $v_{1,i} = v_{2,i} = v_i$, $C_{1,i} = N_D^-(v_{1,i})$ and $C_{2,i} = N_D^+(v_{2,i})$. Let $\mathscr{C}_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,n}\}$ and $\mathscr{C}_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,n}\}$. As in the proof of Theorem 5 is follows that $\mathscr{C}_1, \mathscr{C}_2, R_1$ and R_2 are as in the statement of Theorem 5 for the case $m = 1$.

Since G' is a triangle-free unicyclic graph, $\mathscr{C}_2$ contains exactly n different cliques of cardinality 2. Therefore,

$$|C_{2,i} \cap C_{2,j}| \le 1, \text{ where } i, j \in \{1, 2, \ldots, n\} \text{ and } i \ne j. \tag{1}$$

If $v_{2,i} \in C_{1,j} \cap C_{1,k} \cap C_{1,l}$ for some $1 \le i \le n$ and $1 \le j < k < l \le n$, then $v_{1,j}, v_{1,k}, v_{1,l} \in C_{2,i}$, which implies a contradiction to $|C_{2,i}| = 2$. Hence,

$$\text{every vertex of } G \text{ lies in at most two cliques of } \mathscr{C}_1. \tag{2}$$

If $v_{2,s}, v_{2,t} \in C_{1,i} \cap C_{1,j}$ for some $1 \le i < j \le n$ and $1 \le s < t \le n$, then $v_{1,i}, v_{1,j} \in C_{2,s} \cap C_{2,t}$, a contradiction to (1). Thus,

$$|C_{1,i} \cap C_{1,j}| \le 1, \text{ where } 1 \le i < j \le n. \tag{3}$$

Suppose $v_{j_1} v_{j_2} \ldots v_{j_t} v_{j_1}$ is the unique cycle of G'. Then $v_{j_1} v_{j_t}, v_{j_i} v_{j_{i+1}} \in E(G')$ for $1 \le i \le t-1$, and $t \ge 4$ since G' is triangle-free. Let $C_{2,f_i} = \{v_{j_i}, v_{j_{i+1}}\}$ for $1 \le i \le t-1$ and $C_{2,f_t} = \{v_{j_1}, v_{j_t}\}$. By Theorem 1 and (3), $\{v_{2,f_i}\} = C_{1,j_i} \cap C_{1,j_{i+1}}$ for $1 \le i \le t-1$ and $\{v_{2,f_t}\} = C_{1,j_1} \cap C_{1,j_t}$. So G contains a cycle $v_{2,f_1} v_{2,f_2} \cdots v_{2,f_t} v_{2,f_1}$ which is covered by t cliques $C_{1,j_1}, C_{1,j_2}, \ldots, C_{1,j_t}$.

For any cycle of G that is not covered by a single clique in $\mathscr{C}_1$, there are $s \geq 2$ cliques $C_{1,k_1}, C_{1,k_2}, \ldots, C_{1,k_s} \in \mathscr{C}_1$ such that $C_{1,k_i} \neq C_{1,k_{i+1}}$ for every $1 \leq i \leq s-1$ and $C_{1,k_s} \neq C_{1,k_1}$ and s vertices $v_{g_1}, v_{g_2}, \ldots, v_{g_s}$ such that $\{v_{g_i}\} = C_{1,k_i} \cap C_{1,k_{i+1}}$ for every $1 \leq i \leq s-1$ and $\{v_{g_s}\} = C_{1,k_s} \cap C_{1,k_1}$ with $g_i \neq g_j$ for $i \neq j$.

We obtain $\{v_{1,k_i}, v_{1,k_{i+1}}\} = C_{2,g_i}$ and $\{v_{1,k_s}, v_{1,k_1}\} = C_{2,g_s}$, where $1 \leq i \leq s-1$. Therefore $v_{1,k_i}v_{1,k_{i+1}} \in E(G')$ for every $1 \leq i \leq s-1$ and $v_{1,k_s}v_{1,k_1} \in E(G')$. Since G' has exactly one cycle, we have that $s = t$ and $v_{1,k_1}v_{1,k_2} \cdots v_{1,k_s}v_{1,k_1}$ is the unique cycle $v_{j_1}v_{j_2} \ldots v_{j_t}v_{j_1}$ of G'.

Hence, every cycle in G that is not covered by a single clique in $\mathscr{C}_1$ is covered by the t unique cliques $C_{1,j_1}, C_{1,j_2}, \ldots, C_{1,j_t}$ for which $v_{1,j_1}v_{1,j_2} \cdots v_{1,j_t}v_{1,j_1}$ is the unique cycle of G'.

This implies that every clique $C_{1,i}$ with $v_{1,i} \notin \{v_{1,j_1}, v_{1,j_2}, \ldots, v_{1,j_t}\}$ is the vertex set of a complete block in G, and if some block B of G is not complete, then $V(B) \subseteq C_{1,j_1} \cup C_{1,j_2} \cup \cdots \cup C_{1,j_t}$. Since every block of G which contains two vertices of a clique contains the whole clique, we obtain that $V(B) = C_{1,j_1} \cup C_{1,j_2} \cup \cdots \cup C_{1,j_t}$. Thus, exactly one block of G is not complete and Condition (i) holds.

Since every cutvertex of G lies in at least two blocks of G, we get, by (2), that every cutvertex of G lies in exactly two blocks of G and Condition (ii) holds.

Now, Suppose B_0 is the block of G that is not complete. Then, $V(B_0) = C_{1,j_1} \cup C_{1,j_2} \cup \cdots \cup C_{1,j_t}$ and $C_{2,f_i} = \{v_{1,j_i}, v_{1,j_{i+1}}\}$ for every $1 \leq i \leq t-1$ and $C_{2,f_t} = \{v_{1,j_t}, v_{1,j_1}\}$. Therefore, $\{v_{f_i}\} = C_{1,j_i} \cap C_{1,j_{i+1}}$ for every $1 \leq i \leq t-1$ and $\{v_{f_t}\} = C_{1,j_t} \cap C_{1,j_1}$. By (2), for $i \in \{1, 2, \ldots, t\}$, v_{f_i} lie in no clique $C_{1,k}$ with $k \neq j_i, j_{i+1}$ and in no block of G besides B_0, where $j_{t+1} = j_1$. Hence Condition (iii) holds. This completes the first part of the proof.

Now, let $G = (V, E_G)$ be a connected graph such that the Conditions (i)–(iii) hold. Let S be the set of cutvertices of G.

Let B_0 be the block of G that is not complete, let $C_1, C_2, \ldots, C_t$ be the cliques of G such that $V(B_0) = C_1 \cup C_2 \cup \cdots \cup C_t$ and $|C_i \cap C_{i+1}| = 1$ for every $1 \leq i \leq t-1$ and $|C_t \cap C_1| = 1$. Let $\{x_i\} = C_i \cap C_{i+1}$ and $\{x_t\} = C_t \cap C_1$, where $1 \leq i \leq t-1$. Define $N_i = C_i$ for $1 \leq i \leq t$.

It is easy to see that for $1 \leq i \leq |S|$ we can (recursively) choose vertices $x_{t+i} \in S \setminus \{x_j : t+1 \leq j \leq t+i-1\}$ and define sets

$$N_{t+i} = \{x_{t+i}\} \cup \left(\{u \in V : ux_{t+i} \in E_G\} \setminus \bigcup_{j=1}^{t+i-1} N_j \right)$$

such that every set N_i for $0 \leq i \leq t + |S|$ is a clique of G and if $i \geq t+1$, then N_i is the vertex set of a block in G. Furthermore, for $i \geq t+1$ every cutvertex x_i of G lies in N_i and N_j for some unique $j < i$.

Now, we define the digraph $D = (V, A)$ with vertex set V and arc set

$$A = \{\overrightarrow{ux_i} : u \in N_i, 1 \le i \le t + |S|\} \cup \{\overrightarrow{uu} : u \in V\}$$

Let E_1 and E_2 be the edge sets of the competition graph and the common enemy graph of D, respectively. Note, that for every $x \in V$ we have $x \in N_i \setminus \{x_i\}$ and $N_D^+(x) = \{x, x_i\}$ for some $1 \le i \le t + |S|$. Especially $N_D^+(x_i) = \{x_i, x_{i+1}\}$ for every $1 \le i \le t-1$ and $N^+(x_t) = \{x_t, x_1\}$. Thus, for $u, v \in V$ with $u \neq v$ we obtain that $uv \in E_2$ if and only if $\{u, v\} = N_D^+(x)$ for some $x \in V$ if and only if $\{u, v\} = \{x, x_i\}$ and $x \in N_i \setminus \{x_i\}$ for some $1 \le i \le t + |S|$. Hence, we obtain that $G_2 = (V, E_2)$ is a triangle-free unicyclic graph, since $t \ge 4$ and for every block B of G the subgraph $G_2[V(B)]$ induced by $V(B)$ in G_2 is a star, if B is complete and a spiked cycle, if $B = B_0$ the block not complete.

Now, it remains to prove that $G_1 = (V, E_1) = (V, E_G) = G$. Note that $N_D^-(x) = N_i$ if $x = x_i$ for $1 \le i \le t + |S|$ and $N_D^-(x) = \{x\}$ if $x \in V \setminus \{x_1, x_2, \dots, x_{t+|S|}\}$. Let uv be an edge of G. If $uv \in E(B_0)$, then $u, v \in N_i$ for some $i \in \{1, 2, \dots, t\}$ which implies that $u, v \in N_D^-(x_i)$ for some $i \in \{1, 2, \dots, t\}$ and thus $uv \in E_1$. If $uv \in E(B)$ for some block $B \neq B_0$, then B is complete and contains at least one cutvertex. If $i = \min\{t+1 \le j \le t + |S| : x_j \in V(B)\}$, then $u, v \in N_i = V(B)$ and $u, v \in N_D^-(x_i)$ which implies that $uv \in E_1$. This yields that $E_G \in E_1$.

Conversely, let $uv \in E_1$. We have $u, v \in N_D^-(x)$ for some vertex $x \in V$ with $|N_D^-(x)| \ge 2$. This implies that $x = x_j$ and $u, v \in N_j$ for some $1 \le j \le t + |S|$. Since N_j is a clique in G, we obtain that $uv \in E_G$. Hence $E_G = E_1$ and the proof is complete. ∎

Let $V = \{v_1, v_2, \dots, v_n\}$, $E_S = \{v_1v_i | 2 \le i \le n\}$, $E_F = E_S \cup \{v_iv_{i+1} | 2 \le i \le n-1\}$ and $E_W = E_F \cup \{v_2v_n\}$. We call graphs $S = (V, E_S)$, $F = (V, E_F)$ and $W = (V, E_W)$ the *star graph*, the *fan graph* and the *wheel graph* of order n, respectively. Fischermann et al. [2] proved that $(\overline{K_n}, K_n)$ is competition polysemic. Now we generalize it as follows.

Theorem 3 *For $n \ge 2$, the pair (G, K_n) is competition polysemic if G is the empty graph, the star graph, the fan graph or the wheel graph of order n, respectively.*

Proof. Let $V = \{v_1, v_2, \dots, v_n\}$, $A_S = \{\overrightarrow{v_1v_i}, \overrightarrow{v_iv_i} | 1 \le i \le n\}$, $A_F = A_S \cup \{\overrightarrow{v_iv_{i+1}} | 2 \le i \le n-1\}$ and $A_W = A_F \cup \{\overrightarrow{v_nv_2}\}$. It is straightforward to verify that the pair (S, K_n) is competition polysemic with realization $D_S = (V, A_S)$, the pair (F, K_n) is competition polysemic with realization $D_F = (V, A_F)$, and the pair (W, K_n) is competition polysemic with realization $D_W = (V, A_W)$, respectively. ∎

3 m-Competition polysemy

We start this section with the following characterization of m-competition graphs due to Kim et al. [3].

Theorem 4 (Kim et al. [3]) *A graph $G = (V, E)$ is the m-competition graph of some digraph if and only if there exists an m-edge clique cover $\mathscr{C} = \{C_1, C_2, \ldots, C_p\}$ of G with $p \leq |V|$.*

The following theorem is the main result in this section.

Theorem 5 *A pair (G_1, G_2) of graphs with $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is m-competition polysemic if and only if there exist m-edge clique covers $\mathscr{C}_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,p}\}$ of G_1 and $\mathscr{C}_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,q}\}$ of G_2 for which there exist sets of distinct representatives $R_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,p}\}$ and $R_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,q}\}$, $p, q \leq |V|$, such that*

(*i*) *$v_{2,i} \in C_{1,j}$ if and only if $v_{1,j} \in C_{2,i}$,*

(*ii*) *if $v_{2,s}, v_{2,t} \in \cap_{k=1}^{l} C_{1,i_k}$ and $|C_{2,s} \cap C_{2,t} \setminus \{v_{1,i_1}, v_{1,i_2}, \ldots, v_{1,i_l}\}| \geq m-l$, then there exist $C_{1,i_1}, C_{1,i_2}, \ldots, C_{1,i_m}$ such that $i_1, i_2, \ldots, i_m$ are distinct and $v_{2,s}, v_{2,t} \in C_{1,i_j}$, where $0 \leq l \leq m$ and $j = 1, 2, \ldots, m$, and*

(*iii*) *if $v_{1,s}, v_{1,t} \in \cap_{k=1}^{l} C_{2,i_k}$ and $|C_{1,s} \cap C_{1,t} \setminus \{v_{2,i_1}, v_{2,i_2}, \ldots, v_{2,i_l}\}| \geq m-l$, then there exist $C_{2,i_1}, C_{2,i_2}, \ldots, C_{2,i_m}$ such that $i_1, i_2, \ldots, i_m$ are distinct and $v_{1,s}, v_{1,t} \in C_{2,i_j}$, where $0 \leq l \leq m$ and $j = 1, 2, \ldots, m$.*

Proof. First, we assume that (G_1, G_2) with $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is m-competition polysemic with realization $D = (V, A)$ and prove the existence of $\mathscr{C}_1, \mathscr{C}_2, R_1$ and R_2 as in the statement of the theorem.

Let $V = \{v_1, v_2, \ldots, v_n\}$ and for $1 \leq i \leq n$ let $v_{1,i} = v_{2,i} = v_i, C_{1,i} = N_D^-(v_{1,i})$ and $C_{2,i} = N_D^+(v_{2,i})$. Clearly, $u, v \in C_{1,i} = N_D^-(v_{1,i})$ holds for $u, v \in V$ with $u \neq v$ and $1 \leq i \leq n$ if and only if $v_{1,i} \in N_D^+(u) \cap N_D^+(v)$. Furthermore, there exist m sets $C_{1,i_1}, C_{1,i_2}, \ldots, C_{1,i_m}$ in $\mathscr{C}_1$ such that $u, v \in \cap_{k=1}^{m} C_{1,i_k}$ if and only if $v_{1,i_k} \in N_D^+(u) \cap N_D^+(v)$ for $k = 1, 2, \ldots, m$, or equivalently $uv \in E_1$. This implies that $\mathscr{C}_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,n}\}$ is an m-edge clique cover of G_1. By symmetry, $\mathscr{C}_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,n}\}$ is an m-edge clique cover of G_2.

By the definitions of $C_{1,i}$ and $C_{2,j}$, it is easy to see that $v_{2,j} \in C_{1,i} = N_D^-(v_{1,i})$ holds if and only if $v_{1,i} \in N_D^+(v_{2,j}) = C_{2,j}$, which implies (i).

Suppose $v_{2,s}, v_{2,t} \in \cap_{k=1}^{l} C_{1,i_k}$ and $|C_{2,s} \cap C_{2,t} \setminus \{v_{1,i_1}, v_{1,i_2}, \ldots, v_{1,i_l}\}| \geq m - l$ for some $0 \leq l \leq m$. Then there are $x_1, x_2, \ldots, x_{m-l} \in C_{2,s} \cap C_{2,t} \setminus \{v_{1,i_1}, v_{1,i_2}, \ldots, v_{1,i_l}\}$. Let $v_{1,i_{l+j}} = x_j$ for $j = 1, 2, \ldots, m - l$ and by (i) we have $v_{2,s}, v_{2,t} \in C_{1,i_{l+j}}$, which implies (ii) and, by symmetry, also (iii). This completes the first part of the proof.

Now, let (G_1, G_2) be a pair of graphs with $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ and let $\mathscr{C}_1, \mathscr{C}_2, R_1$ and R_2 be as in the statement of the theorem. Let the digraph D have vertex set V and arc set $A = A_1 \cup A_2$ where $A_1 = \{\overrightarrow{uv_{1,i}} | u \in C_{1,i}, 1 \leq i \leq p\}$ and $A_2 = \{\overrightarrow{v_{2,j}u} | u \in C_{2,j}, 1 \leq i \leq q\}$. We prove that (G_1, G_2) is m-competition polysemic with realization D.

Let $uv \in E_1$ for $u, v \in V$ with $u \neq v$. Since $\mathscr{C}_1$ is an m-edge clique cover of G_1, there are some $1 \leq i_1 < \cdots < i_m \leq p$ such that $u, v \in \cap_{k=1}^{m} C_{1,i_k}$. This implies that $\overrightarrow{uv_{1,i_k}}, \overrightarrow{vv_{1,i_k}} \in A_1$ and $v_{1,i_k} \in N_D^+(u) \cap N_D^+(v)$ for $k = 1, 2, \ldots, m$, i.e., $|N_D^+(u) \cap N_D^+(v)| \geq m$.

Now, let $x_1, x_2, \ldots, x_m \in N_D^+(u) \cap N_D^+(v)$ for $u, v \in V$ with $u \neq v$. We have that $\overrightarrow{ux_i}, \overrightarrow{vx_i} \in A_1 \cup A_2$ for $i = 1, 2, \ldots, m$. For $i \in \{1, 2, \ldots, m\}$, if $\overrightarrow{ux_i}, \overrightarrow{vx_i} \in A_1$, then $x_i = v_{1,j}$ and $u, v \in C_{1,j}$ for some $1 \leq j \leq p$. If $\overrightarrow{ux_i} \in A_1$ and $\overrightarrow{vx_i} \in A_2$, then $x_i = v_{1,j}$ and $u \in C_{1,j}$ for some $1 \leq j \leq p$; $v = v_{2,k}$ and $x_i = v_{1,j} \in C_{2,k}$ for some $1 \leq k \leq q$. Condition (i) implies that $v = v_{2,k} \in C_{1,j}$. Thus $u, v \in C_{1,j}$. By symmetry, if $\overrightarrow{ux_i} \in A_2$ and $\overrightarrow{vx_i} \in A_1$, then $x_i = v_{1,j}$ and $u, v \in C_{1,j}$ for some $1 \leq j \leq p$. Without loss of generality, suppose that for each $i \in \{l+1, l+2, \ldots, m\}$ we have $\overrightarrow{ux_i}, \overrightarrow{vx_i} \in A_2$, and for each $i \in \{1, 2, \ldots, l\}$ at most one of $\overrightarrow{ux_i}$ and $\overrightarrow{vx_i}$ is in A_2, where $0 \leq l \leq m$. Write $x_i = v_{1,j_i}$ for $i \in \{1, 2, \ldots, l\}$. If $l = m$, then by the discusses above, $u, v \in C_{1,j_i}$ for each $i \in \{1, 2, \ldots, m\}$, which implies $uv \in E_1$. If $l < m$, then there exist $1 \leq s < t \leq q$ such that $u = v_{2,s}, v = v_{2,t}$ and $v_{2,s}, v_{2,t} \in \cap_{k=1}^{l} C_{1,j_k}$ and $x_i \in C_{2,s} \cap C_{2,t}$ for each $i \in \{l+1, l+2, \ldots, m\}$. Condition (ii) implies that there exist $C_{1,i_1}, C_{1,i_2}, \ldots, C_{1,i_m}$ such that $i_1, i_2, \ldots, i_m$ are distinct and $v_{2,s}, v_{2,t} \in C_{1,i_j}$, where $j = 1, 2, \ldots, m$, which implies $uv \in E_1$.

Till now we obtain that $uv \in E_1$ for $u, v \in V$ with $u \neq v$ if and only if $|N_D^+(u) \cap N_D^+(v)| \geq m$. Which means that G_1 is the m-competition graph of D. By symmetry, G_2 is the m-common enemy graph of D and hence (G_1, G_2) is m-competition polysemic with realization D. The proof is complete. ∎

For competition polysemy, M. Fischermann et al. [2] proved the following result.

Theorem 6 (M. Fischermann et al. [2]) *Let $G = (V_G, E_G)$ be a graph. There exists a graph $H = (V_H, E_H)$ of order at most $|E_G|$ such that $(G \cup H, G \cup H)$ is competition polysemic where $G \cup H = (V_G \cup V_H, E_G \cup E_H)$ and $V_G \cap V_H = \emptyset$.*

For m-competition polysemy, we have a similar result as follows.

Theorem 7 *Let $G = (V_G, E_G)$ be a graph. There exists a graph $H = (V_H, E_H)$ of order at most $m|E_G|$ such that $(G \cup H, G \cup H)$ is m-competition* *polysemic where* $G \cup H = (V_G \cup V_H, E_G \cup E_H)$ *and* $V_G \cap V_H = \emptyset$.

Proof. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_p\}$ be an m-edge clique cover of $G = (V_G, E_G)$ such p is the minimum. Since collection $\{m \cdot \{u,v\} : uv \in E_G\}$ ($m \cdot \{u,v\}$ denotes that $\{u,v\}$ is repeated m times in the collection) is also an m-edge clique cover of $G = (V_G, E_G)$, then $p \leq m|E_G|$. Let $D = (V_D, A_D)$ be the digraph with vertex set $V_D = V_G \cup \{v_1, v_2, \ldots, v_p\}$, where $V_G \cap \{v_1, v_2, \ldots, v_p\} = \emptyset$, and arc set $A_D = \cup_{i=1}^{p} \{\overrightarrow{wv_i}, \overrightarrow{v_iw} : w \in C_i\}$. Let $G_1 = (V_D, E_1)$ and $G_2 = (V_D, E_2)$ be the m-competition graph and m-common enemy competition graph of D, respectively. Since $N_D^+(v) = N_D^-(v)$ for every vertex $v \in V_D$, we have $G_1 = G_2$. For $u, v \in V_G$ with $u \neq v$ we have $uv \in E_G$ if and only if $u, v \in \cap_{j=1}^{m} C_{i_j}$ for some $1 \leq i_1 < \cdots < i_m \leq p$ if and only if $|N_D^+(u) \cap N_D^+(v)| \geq m$ if and only if $uv \in E_1$. For $u \in V_G$ and $v \in \{v_1, v_2, \ldots, v_p\}$ we have $|N_D^+(u) \cap N_D^+(v)| = \phi$ and hence $uv \notin E_1$. Let $H = (V_D \setminus V_G, E_1 \setminus E_G)$. Then H has $p \leq m|E_G|$ vertices and $G_1 = G_2 = G \cup H$. This completes the proof. ∎

Suppose (G_1, G_2) with $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is m-competition polysemic with realization $D = (V, A)$. Let $D' = (V, A')$, where $A' = A \cup \{\overrightarrow{vu} : \overrightarrow{uv} \in A, u \neq v\}$. Then $\overleftarrow{A'} = A'$ and the following theorem follows.

Theorem 8 *If a pair* (G_1, G_2) *of graphs with* $G_1 = (V, E_1)$ *and* $G_2 = (V, E_2)$ *is* m*-competition polysemic, then* $G = (V, E_1 \cup E_2)$ *is a graph such that* (G, G) *is* m*-competition polysemic.*

The following result is also easy to be proved.

Theorem 9 *For* $n \geq 3$ *the pairs* $(K_{1,n}, C_n \cup I_1)$ *is* m*-competition polysemic for* $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$, *where* $K_{1,n}, C_n$ *and* I_1 *denote the star of order* $n+1$, *the cycle of order* n *and one isolated vertex, respectively.*

Proof. Let $V = \{x, v_0, v_1, \ldots, v_{n-1}\}$, denote $K_{1,n} = (V, E_1)$ and $C_n \cup I_1 = (V, E_2)$ in which $E_1 = \cup_{i=0}^{n-1} \{xv_i\}$ and $E_2 = \cup_{i=0}^{n-2} \{v_iv_{i+1}\} \cup \{v_0v_{n-1}\}$. Then it is easy to check that $(K_{1,n}, C_n \cup I_1)$ is m-competition polysemic with realization $D = (V, A_m)$ for $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$, where $A_m = \cup_{i=0}^{n-1} (\cup_{j=1}^{m} \{\overrightarrow{xv_i}, \overrightarrow{v_iv_{i+j-1}}\})$ and $i + j - 1$ is taken modulo n. ∎

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