On avoidance of V- and Λ-patterns in permutations

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Abstract

We study V- and Λ-patterns which generalize valleys and peaks, as well as increasing and decreasing runs, in permutations. A complete classification of permutations (multi)-avoiding V- and Λ-patterns of length 4 is given. We also establish a connection between restricted permutations and matchings in the coronas of complete graphs.

Keywords: valleys and peaks, permutations, partially ordered patterns, avoidance, matching, corona of a complete graph

1 Introduction

A *valley* (resp. *peak*) of a permutation $a_1 \cdots a_n$ is a value of i, $2 \leq i \leq n-1$, such that $a_{i-1} > a_i < a_{i+1}$ (resp. $a_{i-1} < a_i > a_{i+1}$). In the literature, valleys are sometimes called *minima* or *local minima*, and peaks are called *maxima* or *local maxima*. Further one says that a_i is a *modified maximum* if $a_{i-1} < a_i > a_{i+1}$ and a *modified minimum* if $a_{i-1} > a_i < a_{i+1}$, for $i = 1, \ldots, n$, where $a_0 = a_{n+1} = 0$.

An *increasing run* in a permutation $a = a_1 \cdots a_n$ is an increasing subsequence $a_i < a_{i+1} < \ldots < a_j$ that is not contained in a larger such subsequence. *Decreasing runs* are defined similarly. An n-permutation is *monotone* if it has increasing or decreasing run of length n. We say that i is an *accent* (resp. *descent*) in a permutation a if $a_i < a_{i+1}$ (resp. $a_i > a_{i+1}$).

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We call i a *double rise* (resp. *double fall*) if $i-1$ and i are both accents (resp. descents).

Before defining V- and Λ-patterns studied in this paper, we provide some known results related to the objects defined above. For some other results related to the topic of our discussion see, e. g., [7] and [9].

It is shown in [4] (problem 3.3.46(c) on page 195) that the number of permutations in $\mathcal{S}_n$ with i_1 modified minima, i_2 modified maxima, i_3 double rises, and i_4 double falls is

$$\left[u_1^{i_1}u_2^{i_2-1}u_3^{i_3}u_4^{i_4}\frac{x^n}{n!}\right]\frac{e^{\alpha_2 x}-e^{\alpha_1 x}}{\alpha_2 e^{\alpha_1 x}-\alpha_1 e^{\alpha_2 x}}$$

where $\alpha_1\alpha_2 = u_1u_2$, $\alpha_1+\alpha_2 = u_3+u_4$.

The bivariate generating function for the distribution of peaks (valleys) in permutations is obtained in [6, Cor 23]:

$$1-\frac{1}{y}+\frac{1}{y}\sqrt{y-1}\cdot\tan\left(x\sqrt{y-1}+\arctan\left(\frac{1}{\sqrt{y-1}}\right)\right)$$

where y is responsible for the number of peaks in a permutation and x for the length of the permutation. This result is an analogue to a result in [3] where the circular case of permutations is considered, that is, when the first letter of a permutation is thought to be to the right of the last letter in the permutation. It is shown in [3] that if $M(n,k)$ denotes the number of circular permutations in $\mathcal{S}_n$ having k maxima, then

$$\sum_{n\geq 1}\sum_{k\geq 0}M(n,k)y^k\frac{x^n}{n!}=\frac{zx(1-z\tan xz)}{z-\tan xz}$$

where $z=\sqrt{1-y}$.

In this paper we generalize the concepts of valleys, peaks, and increasing and decreasing runs of given length by introducing V- and Λ-patterns in permutations. We say that a factor $a_{i-k}\cdots a_i\cdots a_{i+\ell}$ of a permutation $a_1\cdots a_n$ is an occurrence of the pattern $V(k,\ell)$ (resp. $\Lambda(k,\ell)$) if $a_{i-k} > a_{i-k+1} > \cdots > a_i < a_{i+1} < \cdots < a_{i+\ell}$ (resp. $a_{i-k} < a_{i-k+1} < \cdots < a_i > a_{i+1} > \cdots > a_{i+\ell}$). As the matter of fact, $V(k,\ell)$ and $\Lambda(k,\ell)$ are instances of so called *partially ordered patterns* (*POPs*) (see [6]). Moreover, $V(k,\ell)$ is nothing else but a *co-unimodal* pattern introduced in [1]. A specific result from [1] on a joint distribution that involves co-unimodal patterns can be found in [6, Section 2.1]. So, our work is a contribution into the POPs theory. However, we do not define POPs in the paper.

If $\pi = a_1a_2\cdots a_n$ is an n-permutation, then the *reverse* of π is $\pi^r := a_n\cdots a_2a_1$, and the *complement* of π is the permutation π^c such that $\pi^c_i = n+1-a_i$ for all $i\in\{1,\ldots,n\}$. We call π^r, π^c, and $(\pi^r)^c = (\pi^c)^r$ *trivial* *bijections.* Note that $V(k,\ell)$ and $\Lambda(k,\ell)$ have the same distribution (simply use the complement operation). In particular, these patterns are equivalent in sense of avoidance, study of which is the goal of this paper.

Our main concern is (multi-)avoidance of $V(k,\ell)$ and $\Lambda(k,\ell)$ patterns of length 4, and we give a complete classification in this case. Many patterns or combinations of patterns to avoid are equivalent because of the trivial bijections. This reduces significantly the number of cases to consider in our classification. We provide in all but one cases exponential generating functions (see Sections 2 and 3); in the remaining case we give a recurrence relation to calculate the number of permutations in question (see Theorem 6). However, in Subsection 2.2 we discuss a general approach to study avoidance of a single pattern $V(k,\ell)$ or $\Lambda(k,\ell)$. Independent interest in study V- and Λ-patterns arises in their connection to matchings in certain graphs discussed in Section 4.

Throughout the paper, we call a permutation *good* if it avoids all the patterns under consideration in a given subsection. Also, given any sequence $b = b_1 \cdots b_n$ of distinct integers, we let the *reduced form* of b be the permutation that results by replacing the i-th largest integer that appears in b by i.

2 Avoidance of $V(k,\ell)$

The case $k = \ell = 1$ was studied in [5] where it was shown that there are 2^{n-1} good n-permutations. Thus, the exponential generating function (EGF) for the number of permutations avoiding $V(1,1)$ is $(e^{2x}+1)/2$.

In Subsection 2.1 we consider the case $k = 2$ and $\ell = 1$. The pattern $V(2,1)$ is a representative for the only equivalence class when avoiding a single pattern of length 4. In Subsection 2.2 we discuss a general approach to study avoidance of $V(k,\ell)$.

2.1 Permutations avoiding $V(2,1)$

The following theorem gives the EGF for the number of permutations avoiding $V(2,1)$ thus settling one of the problems in [6, Table 1] (see the pattern $ijkm$ there).

Theorem 1. *The EGF $A(x)$ for the number of permutations avoiding $V(2,1)$ is given by*

$$1+\exp\left(\frac{3x}{2}\right)\sec\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)\int_0^x \exp\left(-\frac{3u}{2}\right)\cos\left(\frac{\sqrt{3}u}{2}+\frac{\pi}{6}\right)du.$$

Expanding $A(x)$ one gets the initial values for the number of good permutations: 1, 2, 6, 21, 90, 450, 2619, 17334, ...

Proof. Denote by $\mathcal{A}_n$ the set of good permutations in this case, and by $\mathcal{B}_n$ — the set of good permutations ending with an accent. Let $A_n = |\mathcal{A}_n|$ and $B_n = |\mathcal{B}_n|$. We define $A_0 = A_1 = B_0 = B_1 = 1$.

Consider an arbitrary permutation $a \in \mathcal{A}_{n+1}$ where $n \geq 1$ and suppose that $a_{i+1} = 1$ for some i. Let a_L and a_R be the reduced forms of $a_1 \cdots a_i$ and $a_{i+2} \cdots a_{n+1}$ respectively. If $i = n$ then $a_R = \emptyset$ and $a_L \in \mathcal{A}_n$. Otherwise, we have a valley at $i+1$ and in order to avoid $V(2,1)$, a_L is either of length less than 2, or a_L must end with an accent (i. e., $a_L \in \mathcal{B}_i$) while $a_R \in \mathcal{A}_{n-i}$. There are $\binom{n}{i}$ ways to choose elements of a_L. Thus,

$$A_{n+1} = \sum_{i=0}^{n-1} \binom{n}{i} B_i A_{n-i} + A_n. \tag{1}$$

Let us now find the EGF $B(x) = \sum_{n\geq 0} B_n x^n/n!$. Let $b \in \mathcal{B}_{n+1}$ and $b_{i+1} = 1$ where, clearly, $i \leq n-1$. Define b_L and b_R similarly to a_L and a_R above. Then $b_L \in \mathcal{B}_i$ by the same reason as above and $b_R \in \mathcal{B}_{n-i}$ since b must end with an accent. Therefore,

$$B_{n+1} = \sum_{i=0}^{n-1} \binom{n}{i} B_i B_{n-i} = \sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} - B_n.$$

The formula above works for all $n \geq 1$. We multiply both parts of the equation above by $x^n/n!$ and sum over all $n \geq 1$ to get

$$B'(x) = B^2(x) - B(x) + 1 \tag{2}$$

subject to the initial condition $B(0) = 1$. Solution to (2) is given by

$$B(x) = \frac{1}{2} + \frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}x}{2} + \frac{\pi}{6}\right). \tag{3}$$

The initial terms of B_n for $n \geq 1$ are 1, 1, 3, 9, 39, 189, 1107, ... and this sequence is [8, A080635]. We can now solve (1):

$$A_{n+1} = \sum_{i=0}^{n-1} \binom{n}{i} B_i A_{n-i} + A_n = \sum_{i=0}^{n} \binom{n}{i} B_i A_{n-i} + A_n - B_n,$$

$$\sum_{n\geq 0} A_{n+1} \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{i=0}^{n} \frac{B_i x^i}{i!} \frac{A_{n-i} x^{n-i}}{(n-i)!} + \sum_{n\geq 0} (A_n - B_n) \frac{x^n}{n!},$$

$$A'(x) = A(x)B(x) + A(x) - B(x). \tag{4}$$

The initial condition for differential equation (4) is $A(0) = 1$ and $B(x)$ is given by (3). We solve this equation to get the desired formula. □

2.2 General approach to study avoidance of $V(k,\ell)$

Note that the EGFs for the number of permutations avoiding $V(k,\ell)$ for arbitrary k and ℓ can be found by a similar way to that we proceed with $V(2,1)$. However, the larger k and ℓ become, more and more functions get involved in the recurrences, and therefore the systems of differential equations in question are harder and harder to solve. So, when illustrating the general approach, we will only construct the recurrences for the number of good permutations. These recurrences can be easily turned to differential equations involving EGFs.

Example 1. In order to increase k, we have to consider not only the set of good permutations ending with an accent, but also the set of good permutations having the rightmost accent in position $n-i$ for all $i = 1,\dots,k-1$. Let us construct the recurrences for the case $k=3$. Let $\mathcal{A}_n$ be the set of good permutations (i. e., permutations avoiding $V(3,1)$), $\mathcal{B}_n$ be the set of good permutations ending with an accent, and $\mathcal{C}_n$ be the set of good permutations with the last accent at position $n-2$ (i. e., $c_{n-2} < c_{n-1} > c_n$). Let $A_n = |\mathcal{A}_n|, B_n = |\mathcal{B}_n|$, and $C_n = |\mathcal{C}_n|$. Clearly $A_0 = A_1 = 1$, $A_2 = 2$, and $B_2 = 1$. Also, we define $B_0 = B_1 = C_2 = 1$ and $C_0 = C_1 = 0$. We choose the initial values for B_n and C_n in such a way that $\mathcal{C}_n \cap \mathcal{B}_n = \emptyset$ for all n and $\mathcal{C}_n \cup \mathcal{B}_n = \mathcal{A}_n$ for $n \leq 2$. In particular, $\mathcal{C}_2 = \{21\}$. Now consider $n \geq 2$.

Let $a \in \mathcal{A}_{n+1}$ and $a_{i+1} = 1$. If $i = n$ then $a_L \in \mathcal{A}_n$. Otherwise, in order to avoid $V(3,1)$ we must require that a_{i-1} is not a double fall. This means that $a_L \in \mathcal{B}_i$ or $a_L \in \mathcal{C}_i$ while $a_R \in \mathcal{A}_{n-i}$. We have

$$A_{n+1} = \sum_{i=0}^{n-1} \binom{n}{i} (B_i + C_i) A_{n-i} + A_n. \tag{5}$$

If $b \in \mathcal{B}_{n+1}$ and $b_{i+1} = 1$ then $i \leq n-1$. Analogously, we get a recurrence for B_{n+1}:

$$B_{n+1} = \sum_{i=0}^{n-1} \binom{n}{i} (B_i + C_i) B_{n-i}. \tag{6}$$

If $c \in \mathcal{C}_{n+1}$ and $c_{i+1} = 1$ then $i \neq n-1$. If $i = n$ then c_L must end with an accent, so $c_L \in \mathcal{B}_n$. The case $i < n-1$ is dealt similarly as above. We have

$$C_{n+1} = \sum_{i=0}^{n-2} \binom{n}{i} (B_i + C_i) C_{n-i} + B_n. \tag{7}$$

Example 2. In order to increase ℓ we have to consider not only the set of good permutations having the rightmost accent in position $n-i$ for all $i = 1, \ldots, k-1$, but also the set of good permutations having the leftmost descent in position j for all $j = 1, \ldots, \ell - 1$, and all their mutual combinations. Thus, in case $k = \ell = 2$ we need four sets:

$\mathcal{A}_n$ — the set of good (avoiding $V(2,2)$) permutations;

$\mathcal{B}_n$ — the set of good permutations starting with a descent;

$\mathcal{C}_n$ — the set of good permutations ending with an accent;

$\mathcal{D}_n$ — the set of good permutations starting with a descent and ending with an accent.

Again, we define $A_n = |\mathcal{A}_n|, B_n = |\mathcal{B}_n|, C_n = |\mathcal{C}_n|$, and $D_n = |\mathcal{D}_n|$. Let also $A_i = B_i = C_i = D_i = 1$ for $i = 0, 1$.

Consider $a \in \mathcal{A}_{n+1}$ with $a_{i+1} = 1$ for some $n \geq 1$. If a_L ends with an accent (i. e., $a_L \in \mathcal{C}_i$) then a_R may be an arbitrary good permutation of length $n-i$. Otherwise, a_R must start with a descent in order to avoid $V(2,2)$. So,

$$A_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (C_i A_{n-i} + (A_i - C_i) B_{n-i}). \tag{8}$$

Let $b \in \mathcal{B}_{n+1}$ and $b_{i+1} = 1$ where clearly $i \neq 0$. Then either b_L ends with an accent (this is equivalent to $b_L \in \mathcal{D}_i$) and $b_R \in \mathcal{A}_{n-i}$ or $b_L \in \mathcal{B}_i \setminus \mathcal{D}_i$ and b_R starts with a descent (i. e., $b_R \in \mathcal{B}_{n-i}$). We have

$$B_{n+1} = \sum_{i=1}^{n} \binom{n}{i} (D_i A_{n-i} + (B_i - D_i) B_{n-i}). \tag{9}$$

The recurrences for C_{n+1} and D_{n+1} are obtained in a similar way. Omitting the details, we provide the following formulae:

$$C_{n+1} = \sum_{i=0}^{n-1} \binom{n}{i} (C_i C_{n-i} + (A_i - C_i) D_{n-i}). \tag{10}$$

$$D_{n+1} = \sum_{i=1}^{n-1} \binom{n}{i} (D_i C_{n-i} + (B_i - D_i) D_{n-i}). \tag{11}$$

3 Avoidance of V- and Λ-patterns of length 4

A permutation $\pi_1 \pi_2 \ldots \pi_n$ is *alternating* (resp., *reverse alternating*) if $\pi_1 > \pi_2 < \pi_3 > \cdots$ (resp., $\pi_1 < \pi_2 > \pi_3 < \cdots$). It is well known that the EGF for the number of (reverse) alternating permutations is $E(x) = \tan x + \sec x$.

Let E_n denote the number of (reverse) alternating n-permutations; E_n are known as *Euler (zigzag) numbers.*

If we prohibit all four V- and Λ-patterns of length 4 ($V(1,2)$, $V(2,1)$, $\Lambda(1,2)$, and $\Lambda(2,1)$), then a good permutation is nothing else but either monotone or alternating or reverse alternating permutation. Thus the EGF for the number of good permutations in this case is given by

$$2(\tan x + \sec x) + 2e^x - x^2 - 3x - 3.$$

In Subsections 3.1 and 3.2 we consider avoidance of three or two patterns of length 4 respectively.

3.1 Avoidance of three patterns

In this subsection we study simultaneous avoidance of $V(1,2)$, $V(2,1)$, and $\Lambda(1,2)$. All other combinations of three patterns are equivalent to this one due to the trivial bijections.

Theorem 2. *The EGF $A(x)$ for the number of permutations avoiding simultaneously the patterns $V(1,2)$, $V(2,1)$, and $\Lambda(1,2)$ is given by*

$$\frac{1}{2}(e^x + (\tan x + \sec x)(e^x + 1) - (1 + 2x + x^2)).$$

The initial values in question are $1, 2, 6, 15, 47, 178, 791, 4025, 23057, \ldots$.

Proof. One can easily see that any good permutation $\pi = a_1 \cdots a_n$ is either the monotone decreasing permutation $n(n-1)\cdots 1$ or it has the following structure: π starts with an increasing run of length i, $1 \leq i \leq n$ (followed by a descent $a_i > a_{i+1}$) and $a_{i+1} \cdots a_n$ forms a reverse alternating permutation.

Suppose A_n^i denotes the number of good n-permutations starting with the increasing run of length i. Let A_n denote the number of (all) good permutations. Then $A_0 = A_1 = 1$, $A_2 = 2$, $A_3 = 6$, and for $n \geq 3$, $A_n = 1 + \sum_{i=1}^{n} A_n^i$, where 1 is responsible for the decreasing permutation.

Clearly, $A_n^1 = A_n^2 = E_n$, $A_n^{n-1} = n - 1$ and $A_n^n = 1$. Moreover, for $1 \leq i \leq n-2$ and $n \geq 3$, we have

$$A_n^i = \binom{n}{i} E_{n-i} - A_n^{i+2}.$$

Indeed, we can choose the first i letters for the increasing run in $\binom{n}{i}$ ways and there are E_{n-i} ways to choose a reverse alternating permutation on the remaining $n - i$ letters. However, we do not necessarily have a descent in position i after that. If not, then we get a permutation counted by A_n^{i+2}.

So, we subtract A_n^{i+2} since all the good n-permutations starting with the increasing run of length $i+2$ can be obtained in this way.

Summing over i from 1 to $n-2$ both parts of the formula above, one gets

$$\sum_{i=1}^{n-2}\binom{n}{i}E_{n-i}=\sum_{i=1}^{n-2}A_n^i+\sum_{i=3}^{n}A_n^i.$$

Thus

$$2A_n=\sum_{i=1}^{n-2}\binom{n}{i}E_{n-i}+2+A_n^{n-1}+A_n^n+A_n^1+A_n^2=\sum_{i=0}^{n}\binom{n}{i}E_{n-i}+E_n+1.$$

We used $E_0=E_1=1$ above. We now multiply both parts of the equation above by $x^n/n!$ and sum over all $n\geq 3$ to get

$$2(A(x)-1-x-x^2)=(E(x)e^x-1-2x-2x^2)+(E(x)-1-x-\frac{x^2}{2})+(e^x-1-x-\frac{x^2}{2}),$$

which leads to the desired formula. □

3.2 Avoidance of two patterns

There are three classes of equivalence in case of avoidance of two patterns. We choose the following representatives for the classes: $V(1,2)$ and $\Lambda(1,2)$, $V(1,2)$ and $\Lambda(2,1)$, and $V(1,2)$ and $V(2,1)$.

3.2.1 Avoidance of $V(1,2)$ and $\Lambda(1,2)$ simultaneously.

One can see that a good permutation in this case belongs to one of the following classes. We say that a permutation belongs to the first class if it starts with an increasing run followed by a reverse alternating permutation (possibly of the empty length). If a permutation starts with a decreasing run followed by an alternating permutation (possibly of the empty length) then it belongs to the second class. Note that reverse alternating and alternating permutations belong to both of these classes at the same time.

If we remove alternating and reverse alternating permutations from the classes then the complement operation sets a natural bijection between the sets of remaining good permutations in these classes. Moreover, we have already counted the number of permutations of the first class in Theorem 2. Thus, the EGF in our case is $2(A(x)-e^x-E(x))+3+3x+x^2$. Here, $A(x)$ is given by Theorem 2; subtracting e^x corresponds to not counting the decreasing permutation; subtracting $E(x)$ corresponds to not counting the alternating and the reverse alternating permutations twice; adding $3+3x+x^2$ corresponds to adjusting the initial values for $n\leq 2$ since the fact that the number of good n-permutations is given by $2(A_n - 1 - E_n)$, where A_n is as in Theorem 2, works only for $n \geq 3$.

We have just proved the following

Theorem 3. *The EGF $A(x)$ for the number of permutations avoiding simultaneously the patterns $V(1,2)$ and $\Lambda(1,2)$ is given by*

$$1 + x + (\tan x + \sec x - 1)(e^x - 1).$$

The initial values in question are $1, 2, 6, 18, 60, 232, 1036, 5278, 30240, \ldots$.

3.2.2 Avoidance of $V(1,2)$ and $\Lambda(2,1)$ simultaneously.

If a good permutation has a double rise then it cannot have any peak or valley. So, it must be a monotone increasing permutation. On the other hand, any permutation without double rises clearly avoids $V(1,2)$ and $\Lambda(2,1)$. The number of permutations without double rises is given by the following theorem, proved in [2].

Proposition 4. *The EGF for the number of permutations with no double rises is given by*

$$\frac{\sqrt{3}}{2} \exp\left(\frac{x}{2}\right) \sec\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}\right).$$

Using this result and adjusting the values for $n \leq 2$, we get the following corollary.

Corollary 5. *The EGF $A(x)$ for the number of permutations avoiding simultaneously the patterns $V(1,2)$ and $\Lambda(2,1)$ is given by*

$$\frac{\sqrt{3}}{2} \exp\left(\frac{x}{2}\right) \sec\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}\right) + e^x - \left(1 + x + \frac{x^2}{2}\right).$$

The initial values in question are $1, 2, 6, 18, 71, 350, 2018, 13359, 99378, \ldots$.

3.2.3 Avoidance of $V(1,2)$ and $V(2,1)$ simultaneously.

It is easy to see that any good permutation has the shape similar to that in Figure 1. That is, a good permutation consists of three blocks (the second of them may be empty): it begins with an increasing run followed by a reverse alternating factor of odd length, and it ends with a decreasing run.

We let $A^n_{i,j}$ denote the number of good permutations having the initial increasing run of length i and the final decreasing run of length j. For example, the permutation corresponding to the shape in Figure 1 would be counted by $A^{10}_{3,2}$. By definition, i and j are non-zero. Also, if the reverse-alternating block is empty, then $i + j = n + 1$ since the peak element is counted twice in this case.

Figure 1: The shape of a permutation avoiding $V(1,2)$ and $V(2,1)$.

Theorem 6. *The number of n-permutations avoiding $V(1,2)$ and $V(2,1)$ simultaneously is given by*

$$A_n = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ n-i-j \text{ is odd}}}^{n-i+1} A^n_{i,j}$$

with

$$A^n_{i,j} = \begin{cases} \binom{n-1}{i-1} & \text{if } n \geq i \geq 1 \text{ and } n-i-j=-1, \\ \binom{n}{i}\binom{n-i}{j} - \binom{n-1}{i-1} - \binom{n-1}{i} - \binom{n-1}{i+1} & \text{if } n-i-j=1, \\ \binom{n}{i}\binom{n-i}{j}E_{n-i-j} - A^n_{i+2,j} - A^n_{i,j+2} - A^n_{i+2,j+2} & \text{if } n-i-j \geq 3 \text{ is odd} \end{cases}$$

where recall that E_n is the number of alternating permutations. The initial values for A_n are given by $1, 2, 6, 18, 66, 252, 1176, 5768, 34216, \ldots$.

Proof. Suppose a good permutation $a_1 \cdots a_n$ has the initial increasing run of length i and the final decreasing run of length j. In particular, we have $a_i > a_{i+1}$ and $a_{j-1} < a_j$. Since the reverse-alternating block must end with a descent, it is of odd length. Hence, $n-i-j$ must be odd. Using the structure of good permutations, we distinguish three cases:

1. The reverse-alternating block is empty, that is, $i+j=n+1$. The element n is both in the increasing and decreasing runs, so we choose the remaining elements of the increasing run in $\binom{n-1}{i-1}$ ways and (uniquely) order all the elements in the permutation.

2. The reverse-alternating block contains one element, that is, $n-i-j=1$. We use an inclusion-exclusion argument to count $A^n_{i,j}$ in this case. We claim that

$$A^n_{i,j} = \binom{n}{i}\binom{n-i}{j} - A^n_{i+1,j+1} - A^n_{i+2,j} - A^n_{i,j+2}$$

where $A^n_{i+1,j+1}$, $A^n_{i+2,j}$ and $A^n_{i,j+2}$ can be found using Case 1. Indeed, $\binom{n}{i}\binom{n-i}{j}$ is the number of ways to choose elements in the increasing run, and then in the decreasing run. The remaining element then goes into the reverse-alternating block. However, some of the permutations we count are "bad." They are those where either $a_i < a_{i+1}$, or $a_{j-1} > a_j$, or both. The number of bad permutations corresponding to these cases is given by $A^n_{i+2,j}$, $A^n_{i,j+2}$ and $A^n_{i+1,j+1}$ respectively.

3. The reverse-alternating block contains more than 2 elements, that is, $n-i-j \geq 3$ and $n-i-j$ is odd. We use the same considerations as in Case 2. Namely, we choose elements in the increasing and decreasing blocks to get $\binom{n}{i}\binom{n-i}{j}E_{n-i-j}$ permutations, and then we subtract $A^n_{i+2,j}$, $A^n_{i,j+2}$ and $A^n_{i+2,j+2}$ corresponding to the bad overcounted permutations where either $a_i < a_{i+1}$, or $a_{j-1} > a_j$, or both.

□

4 Restricted permutations and matchings in the coronas of complete graphs

It is always interesting to establish a connection between restricted permutations and other combinatorial objects. Such connections are not only interesting from finding a bijection point of view, but also, they sometimes allow enumerating certain statistics.

There are many combinatorial objects that appear in the literature in connection with restricted permutation (Catalan numbers, Dyck paths, Motzkin paths, involutions, two-dimensional faces in a unit cube, etc) and our example below contributes yet another connection. However, this connection beyond involving a V-pattern involves the *consecutive* pattern 213. One says that $a_ia_{i+1}a_{i+2}$ is an occurrence of the consecutive pattern 213 in a permutation $a_1 \cdots a_n$ if $a_{i+1} < a_i < a_{i+2}$.

Let K'_n denote the *corona* of the complete graph K_n (it is the graph constructed from K_n by adding for each vertex v a new vertex v' and the edge vv'). Let M_n denote the number of matchings in K'_n including the empty one. For example, $M_3 = 14$ because K'_3 has one empty matching, six matchings with one edge, six matchings of cardinality two and one matching containing three edges. Sequence M_n appears as [8, A005425].

Theorem 7. *The set of $(n+1)$-permutations avoiding simultaneously the patterns 213 and $V(1,2)$ is in one-to-one correspondence with the set of all matchings of K'_n.*

Proof. To describe a recursive bijection between the objects under consideration, we note that the good permutations involved can be calculated using the recurrence $A_{n+1} = 2A_n + (n-1)A_{n-1}$, and the matchings can be calculated using $M_n = 2M_{n-1} + (n-1)M_{n-2}$. We now explain where the recurrences come from and then provide a bijective map.

One can see that, in a good $(n+1)$-permutation, the elements of the permutation to the left of $(n+1)$ must be in increasing order. Moreover, if $n+1$ occupies position i, then n is either in the position $i-1$, or $i+1$, or $i+2$. In the first two cases we can safely remove $n+1$ to get an n-permutation which is good (basically we can identify n and $n+1$ together and denote this new letter by n). In the third case above, we choose a letter between $n+1$ and n in $n-1$ ways, and then identify these three letters denoting the resulting letter $n-1$ to get a good $(n-1)$-permutation. We may do this since the letters $n+1$ and n are the largest letters in the permutation. So, $A_{n+1} = 2A_n + (n-1)A_{n-1}$.

To see that the same recurrence works for K'_n, we mark the vertices of K_n by 1,2,...,n and the other vertices adjacent to them by $1'$, $2'$,...,n' respectively. Let us count the number of matchings in K'_n. Clearly, there are M_{n-1} matchings that do not involve n, M_{n-1} matchings involving the edge nn' and M_{n-2} matchings involving the edge ni for every $i \in \{1, 2, \ldots, n-1\}$. Hence, $M_n = 2M_{n-1} + (n-1)M_{n-2}$.

It is easy now to describe a recursive bijective map. If the node n is not involved in a matching of K'_n then the good permutation corresponding to this matching contains $(n+1)n$ (n stays next to the right of $(n+1)$). If the edge nn' is involved in a matching then the permutation contains $n(n+1)$ (n stays next to the left of $(n+1)$). Finally, if the edge ni is involved in a matching then the good permutation contains the segment $(n+1)in$. Then we proceed by induction. The reverse to this map is easily seeing.

For example, the matching $\{11', 22', 33', 44'\}$ corresponds to the permutation 12345, whereas the matching $\{13, 22'\}$ corresponds to the permutation 25413. □

We now get the following corollary to Theorem 7.

Corollary 8. *The EGF for the number of permutations avoiding the patterns* 213 *and* $V(1,2)$ *is given by*

$$A(x) = 1 + \int_0^x e^{2t+\frac{t^2}{2}}\, dt.$$

The initial values in question are $1, 2, 5, 14, 43, 142, 499, 1850, 7193, \ldots$

Proof. One can either use the EGF for the matchings which is known ([8, A005425], observe the shift coursing appearance of the integral sign), or to derive the EGF from the recurrence for the number of good permutations given in the proof of Theorem 7. □

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