Uniformly Least Reliable Graphs in Class $\Omega(n, e)$ as $e \leq n + 1$

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Abstract

We consider the undirected simple connected graph for which edges fail independently of each other with equal probability $1 - p$ and nodes are perfect. The all-terminal reliability of a graph G is the probability that the spanning subgraph of surviving edges is connected, denoted as $R(G, p)$. Graph $G \in \Omega(n, e)$ is said to be uniformly least reliable if $R(G, p) \leq R(G', p)$ for all $G' \in \Omega(n, e)$, and for all edge failure probabilities $0 < 1 - p < 1$. In this paper, we prove the existence of uniformly least reliable graphs in the class $\Omega(n, e)$ for $e \leq n + 1$ and give their topologies.

Keywords : all-terminal reliability; Boesch's conjecture; uniformly least reliable graphs

1 Introduction

Let $G(V, E)$ be an undirected simple connected graph with a set V of nodes and a set E of edges. Given a graph whose nodes are perfect but whose edges fail independently of each other with a constant probability $1 - p$, the *All-terminal reliability* of the graph $G(V, E)$, denoted as $R(G, p)$, is defined as the probability that the spanning subgraph of G on the surviving edges is connected[1, 2]. The uniformly least reliable graphs problem is to find the connected graph in $\Omega(n, e)$ having the lowest ATR for all p, $0 < p < 1$.

Let $\Omega(n, e)$ denote the class of undirected simple graphs with n nodes and e edges. A graph $G \in \Omega(n, e)$ is said to be uniformly least reliable if $R(G, p) \leq R(G', p)$ among all $G' \in \Omega(n, e)$, and for all edge failure probabilities $0 < 1 - p < 1$. Boesch, Satyanarayana and Suffel [3] presented a conjecture that any graph $L \in \Omega(n, e)$ is uniformly least reliable if and only if one block of L, say H, has a clique of size $|V(H)| - 1$ while the other blocks are single edges, where

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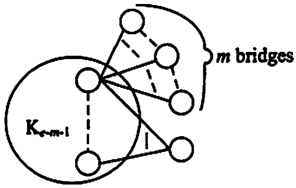
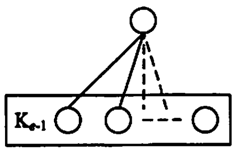
$V(H)$ denotes the nodes set of H (see Fig.1.(a)). Brown, Colbourn and Devitt[4] gave network transformations for computing lower and upper bounds on the all-terminal reliability. Petingi, Saccoman and Schoppmann [5] proved that for the class of simple graphs $\Omega(n,e)$ with $e \geq (n-1)(n-2)/2+1$ there exists a simple graph B_e in the same class, referred to as the balloon, such that B_e is uniformly least reliable in $\Omega(n,e)$. The balloon B_e is composed of a clique of size $n-1$ and a single cone point of degree r such $1 \leq r < n$ (see Fig.1.(b)). Schoppmann [6] enumerated the lower bound graph in each case from $e = n+2$ to $e = n+7$. In this paper, we prove that there exist the uniformly least reliable graphs in the class $\Omega(n,e)$ as $e \leq n+1$ and give the topology of the uniformly least reliable graphs.



(a) Uniformly least reliable graph L



(b) Balloon B_e

Figure 1: Two important graphs

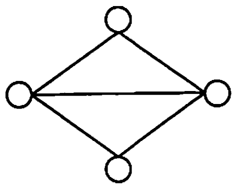


Figure 2: Kite $T(4,5)$

The remainder is organized as follows. Section 2 lists the definitions and known results used in this paper. Section 3 presents the uniformly least reliable graphs in the class $\Omega(n,n-1)$ and $\Omega(n,n)$, while Section 4 does the same for $\Omega(n,n+1)$.

2 Definitions and Known Results

To investigate conveniently, we list the following definitions directly from literature.

Definition 1. *A kite is a graph obtained by deleted an edge in graph K_4.*

For example, graph $T(4,5)$ in Fig. 2 is a *Kite*[7].

Definition 2. *A bridge of a graph is an edge whose deletion increases the number of components.*

Definition 3. *Suppose the nodes of graph $G(n,e)$ never fail and the edges fail independently with probability $1-p$. Then the all-terminal reliability of graph $G(n,e)$ is*

$$R(G,p) = \sum_{k=n-1}^{e} S_k(G)p^k(1-p)^{n-k},$$

where $S_k(G)$ is the number of spanning connected subgraphs of graph G that contain exactly k edges.

Definition 4. *A graph $G \in \Omega(n,e)$ is said to be uniformly least reliable if $R(G,p) \leq R(G',p)$ among all $G' \in \Omega(n,e)$, and for all edge failure probabilities $0 < 1-p < 1$.*

Lemma 5. *An edge is bridge if and only if it belongs to no cycle.*

Moskowitz[8] proposed *Factoring Theorem* as follows.

Theorem 6. *Suppose that $G(V,E)$ is an undirected graph and the edges of G fail with independent but known probabilities. If $e \in E$ and e fails with probability $1-p$, then $R(G,p) = pR(G/e) + (1-p)R(G-e)$.*

Let e be an edge of G with endpoint u and v, and the contraction of edge e means to coalesce nodes u and v into a single node, with the resulting self-loop disregarded as it does not impact reliability calculations. The graph obtained by contracting e in G is denoted by G/e. The deletion of an edge e from a graph G yields the spanning subgraph $G-e$ containing all edges of G except e.

3 Uniformly Least Reliable Graphs in $\Omega(n,e)$, while $e \leq n$

For the disconnected graph G, the all-terminal reliability of graph G is zero from Definition 3. To investigate conveniently, we focus on the case that the graph G in its class $\Omega(n,e)$ is connected.

Theorem 7. *Suppose the nodes of simple connected graph $G(n,n-1)$ never fail and the edges fail independently with probability $1-p$. Any graph G is the uniformly least reliable graph in the class $\Omega(n,n-1)$.*

The proof is obviously obtained, because any connected graph G is a tree in the class $\Omega(n, n-1)$ and $R(G,p) = p^{n-1}$ from Definition 3.

We know that every connected graph has a cycle in the class $\Omega(n,n)$. A 3-cycle is also referred to as a triangle. Let $\Omega_1(n,n)$ denote the class in which each graph contains a triangle. Let $\Omega_2(n,n)$ be the class in which each graph has a cycle whose length is more than 3. This is, $\Omega(n,n) = \Omega_1(n,n) \bigcup \Omega_2(n,n)$ and $\Omega_1(n,n) \bigcap \Omega_2(n,n) = \varnothing$.

Theorem 8. *Suppose the nodes of simple connected graph $G(n,n)$ never fail and the edges fail independently with probability $1-p$. Then $G \in \Omega_1(n,n)$ is the uniformly least reliable graph in the class $\Omega(n,n)$.*

Proof. Let graph G and H be the arbitrary simple graph in $\Omega_1(n,n)$ and $\Omega_2(n,n)$, respectively.

From Definition 3, we have

$$R(G,p) = \sum_{k=n-1}^{n} S_k(G)p^k(1-p)^{n-k}, \tag{1}$$

$$R(H,p) = \sum_{k=n-1}^{n} S_k(H)p^k(1-p)^{n-k}. \tag{2}$$

Since the edges of graph $G(n,n)$ consist of three edges of a triangle and $(n-3)$ bridges. From Definition 3, we obtain $S_{n-1}(G) = 3$ and $S_n(G) = 1$. And any graph has the same all-terminal reliability value with the graph G in the class $\Omega_1(n,n)$.

Let m be the length of cycle in the graph H. Since graph $H \in \Omega_2(n,n)$, we have $m > 3$. Since graph H contains $(n-m)$ bridges and a cycle whose length is m. From Definition 3, we have $S_{n-1}(H) = m > 3$, and $S_n(H) = 1$.

Thus we can get $S_{n-1}(H) > S_{n-1}(G)$ and $S_n(H) = S_n(G)$.

Then $R(H,p) > R(G,p)$.

Thus graph $G \in \Omega_1(n,n)$ is the uniformly least reliable graph in class $\Omega(n,n)$.

□

Example 9. *Let graph $G_1(6,6)$ and $G_2(6,6)$ be in the class $\Omega_1(6,6)$ and $H_1(6,6)$ and $H_2(6,6)$ in the class $\Omega_2(6,6)$, as illustrated in Fig.3.*

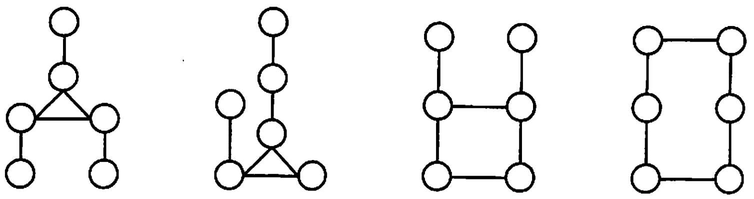
From Definition 3, we have

$$R(G_1,p) = R(G_2,p) = 3p^5(1-p) + p^6, \tag{3}$$

$$R(H_1,p) = 4p^5(1-p) + p^6, \tag{4}$$

$$R(H_2,p) = 6p^5(1-p) + p^6. \tag{5}$$

Then $G_1(6,6)$ and $G_2(6,6)$ are the uniformly least reliable graphs in the class $\Omega(6,6)$, which is consistent with Theorem 8.



(a) Graph G_1 (b) Graph G_2 (c) Graph H_1 (d) Graph H_2

Figure 3: Different structure graphs in $\Omega(6,6)$

4 Uniformly Least Reliable Graphs in $\Omega(n,e)$, while $e=n+1$

Let $\Omega_3(n,n+1)$ be the class in which every graph has a kite. Then class $\Omega_3(n,n+1)$ is in the class $\Omega(n,n+1)$.

Theorem 10. *Suppose the nodes of simple connected graph $G(n,n+1)$ never fail and the edges fail independently with probability $1-p$. Graph $G \in \Omega_3(n,n+1)$ is the uniformly least reliable graph in class $\Omega(n,n+1)$.*

Proof. Let graph G be an arbitrary simple graph in the class $\Omega_3(n,n+1)$, the edges of graph G consist of the edges of a kite and $(n-5)$ bridges. Let graph T be a kite. From Definition 3, we have

$$\begin{aligned} R(T,p) &= \sum_{k=3}^{5} S_k(T)p^k(1-p)^{5-k} \\ &= 8p^3(1-p)^2 + 5p^4(1-p) + p^5. \end{aligned} \tag{6}$$

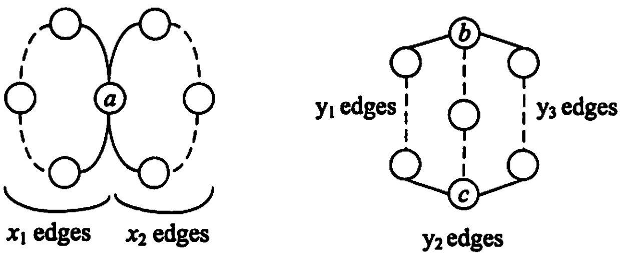
Then we can obtain $R(G,p)$ by using the Definition 3 and Theorem 6.

$$\begin{aligned} R(G,p) &= \sum_{k=n-1}^{n+1} S_k(G)p^k(1-p)^{n-k} \\ &= p^{n-5}R(T,p) \\ &= 8p^{n-1}(1-p)^2 + 5p^n(1-p) + p^{n+1}. \end{aligned} \tag{7}$$

Thus we obtain $S_{n-1}(G)=8$, $S_n(G)=5$ and $S_{n+1}(G)=1$.

Let graph H be an arbitrary simple graph in the class $\Omega(n,n+1)$. Let r be the number of bridges of graph H. From Lemma 5, we know any bridge belongs to no cycle. Then we can obtain the $R(H,p)$ by using the Factoring Theorem to every bridge.

Let M be the graph obtained by contracting the r bridges in graph H. Then $n+1-r$ is the number of edges in graph M. By Definition 3 and Theorem 6, we



(a) Two cycles (b) Three paths

Figure 4: Two topology structures of graph M

get

$$\begin{aligned} R(H,p) &= \sum_{k=n-1}^{n+1} S_k(H)p^k(1-p)^{n-k} \\ &= p^r R(M,p) \\ &= p^r[S_{n-r-1}(M)p^{n-r-1}(1-p)^2 \\ &+ S_{n-r}(M)p^{n-r}(1-p) + S_{n+1-r}(M)p^{n+1-r}] \\ &= S_{n-r-1}(M)p^{n-1}(1-p)^2 \\ &+ S_{n-r}(M)p^n(1-p) + S_{n+1-r}(M)p^{n+1}. \end{aligned} \tag{8}$$

The graph M can have one of two different topologies, as illustrated in Fig. 4.

(1) The graph M consists of two cycles connected by an node a. (see Fig. 4.(a))

(2) The graph M consists of three paths form node b to c. (see Fig. 4.(b))

Consider these two cases separately:

Case (1): Two cycles has x_1 and x_2 edges, respectively.

Then we have $n+1-r = x_1 + x_2$, $x_1 \geq 3$ and $x_2 \geq 3$. From Definition 3, we obtain that

$$S_{n-r-1}(M) = \binom{x_1}{1}\binom{x_2}{1} \geq \binom{3}{1}\binom{3}{1} = 9 > 8 = S_{n-1}(G), \tag{9}$$

$$S_{n-r}(M) = \binom{x_1}{1} + \binom{x_2}{1} \geq \binom{3}{1} + \binom{3}{1} = 6 > 5 = S_n(G), \tag{10}$$

$$S_{n+1-r}(M) = 1 = S_{n+1}(G). \tag{11}$$

From Eq.(7), (8), (9), (10) and (11), then we have $R(H,p) \geq R(G,p)$.

Case (2): The graph M consists of three paths form node b to c.

Let y_1, y_2 and y_3 be the number of edges in the three paths, respectively. Since graph M is simple graph, we have $n+1-r = y_1+y_2+y_3$, $y_1 \geq 2$, $y_2 \geq 1$ and $y_3 \geq 2$. By Definition 3, we get that

$$\begin{aligned} S_{n-r-1}(M) &= \binom{y_1}{1}\binom{y_2}{1} + \binom{y_1}{1}\binom{y_3}{1} + \binom{y_2}{1}\binom{y_3}{1} \\ &\geq 2\binom{2}{1}\binom{1}{1} + \binom{2}{1}\binom{2}{1} = 8 = S_{n-1}(G), \end{aligned} \tag{12}$$

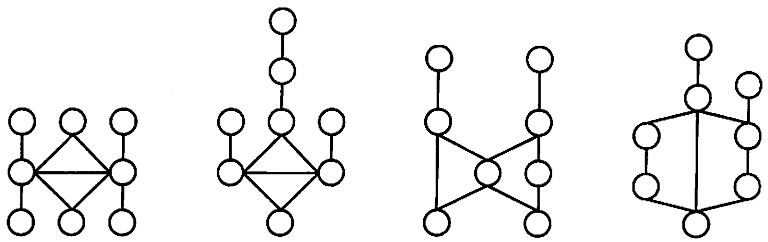
$$\begin{aligned} S_{n-r}(M) &= \binom{y_1}{1} + \binom{y_2}{1} + \binom{y_3}{1} \\ &\geq \binom{2}{1} + \binom{1}{1} + \binom{2}{1} = 5 = S_n(G), \end{aligned} \tag{13}$$

$$S_{n+1-r}(M) = 1 = S_{n+1}(G). \tag{14}$$

From Eq.(7), (8), (12), (13) and (14), we have $R(H,p) \geq R(G,p)$.

Then graph $G \in \Omega_3(n, n+1)$ is the uniformly least reliable graph in class $\Omega(n, n+1)$. □

Example 11. *Let graph $G_3(8,9)$, $G_4(8,9)$, $H_3(8,9)$ and $H_4(8,9)$ be in the class $\Omega(8,9)$, as illustrated in Fig.5.*



(a) Graph G_3 (b) Graph G_4 (c) Graph H_3 (d) Graph H_4

Figure 5: Different structure graphs in $\Omega(8,9)$

Since $G_3(8,9)$ and $G_4(8,9)$ have a kite, we know $G_3(8,9)$ and $G_4(8,9)$ are in the class $\Omega_3(8,9)$. From Theorem 10, we obtain that $G_3(8,9)$ and $G_4(8,9)$ are the uniformly least reliable graphs in the class $\Omega(8,9)$.

From Definition 3, we have

$$R(G_3,p) = R(G_4,p) = 8p^7(1-p)^2 + 5p^8(1-p) + p^9, \tag{15}$$

$$R(H_3,p) = 12p^7(1-p)^2 + 7p^8(1-p) + p^9, \tag{16}$$

$$R(H_4,p) = 15p^7(1-p)^2 + 7p^8(1-p) + p^9. \tag{17}$$

Then the all-terminal reliability values of $G_3(8,9)$, $G_4(8,9)$, $H_3(8,9)$ and $H_4(8,9)$ are consistent with Theorem 10.

Acknowledgements
The authors would like to thank anonymous referee for their professional comments and suggestions regarding the original version of this paper.

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