Some Identities for Leibniz Numbers[1]

Feng-Zhen Zhao
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China

Wuyungaowa
School of Mathematical Sciences, Inner Mongolia University Huhehaote 010021, P. R. China

Abstract

In this paper, we are concerned in Leibniz numbers. We establish a series of identities involving Leibniz numbers, Stirling numbers, harmonic numbers, arctan numbers by making use of generating functions. In addition, we give the asymptotic expansion of certain sums related to Leibniz numbers by Laplace's method.

1. Introduction

It is well known that binomial coefficients $\binom{n}{k}$ are defined by

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!}, & n \geq k, \\ 0, & n < k, \end{cases}$$

where n and k are nonnegative integers, and $(n+1)^{-1}\binom{n}{k}^{-1}$ are called Leibniz numbers $\Re(n,k)$ (see [2]). The properties of Leibniz numbers are similar to that of binomial coefficients. For example, for $k \geq 1$,

$$\begin{aligned} \Re(n,k) + \Re(n,k-1) &= \Re(n-1,k-1), \\ \sum_{m=l}^{n} \Re(m,k) &= \Re(l-1,k-1) - \Re(n,k-1), \\ \sum_{i=0}^{k} (-1)^k \Re(n,k) &= \Re(n+1,0) + (-1)^k \Re(n+1,k+1). \end{aligned}$$

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The double generating function of $\Re(n,k)$ is

$$\sum_{n\geq k\geq 0} \Re(n,k)t^{n+1}u^k = \frac{-\ln[(1-t)(1-ut)]}{1+u(1-t)}.$$

It is clear that the vertical generating function of $\Re(n,k)$ is

$$\sum_{n=k}^{\infty} \Re(n,k)t^{n+1} = \sum_{i=1}^{k}(-1)^{k-i}i^{-1}t^i(1-t)^{k-i} - (-1)^k(1-t)^k\ln(1-t). \tag{1.1}$$

The vertical generating function of Leibniz numbers means that $\Re(n,k)$ are related to Stirling numbers, harmonic numbers, and Cauchy numbers. The purpose of this paper is to investigate the properties of Leibniz numbers. The paper is organized as follows. In Section 2, by means of generating functions, we establish a series of identities involving Leibniz numbers, Stirling numbers. By using an integral, we deduce some other relations between Leibniz and Stirling numbers. In Section 3, we derive some identities relating Leibniz and harmonic numbers. In Section 4, we obtain some identities involving Leibniz and arctan numbers. In Section 5, we give the asymptotic expansion of certain sums involving Leibniz numbers.

For convenience, we recall some definitions of combinatorial numbers involved in the paper. The first kind of Cauchy numbers a_n is given by

$$a_n = \int_0^1 (x)_n dx,$$

where n is a nonnegative integer and

$$(x)_n = \begin{cases} x(x-1)\cdots(x-n+1), & n\geq 1,\\ 1, & n=0,\\ \dfrac{1}{(x+1)\cdots(x-n)}, & n<0. \end{cases}$$

The generating function of a_n is

$$\sum_{n=0}^{\infty} a_n\frac{t^n}{n!} = \frac{t}{\ln(1+t)}.$$

Throughout this paper, we denote the first and second kinds of Stirling numbers by $s(n,k)$ and $S(n,k)$, respectively. Let H_n and $t(n,k)$ stand for harmonic numbers and arctan numbers, respectively. That is,

$$\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{\ln^k(1+t)}{k!}, \quad \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!},$$

$$\sum_{n=2}^{\infty} \frac{H_{n-1}}{n}t^n = \frac{1}{2}\ln^2(1-t), \quad \sum_{n=k}^{\infty} t(n,k)\frac{t^n}{n!} = \frac{(\arctan t)^k}{k!}.$$

2. Identities involving Leibniz and Stirling numbers

Stirling numbers are related to binomial coefficients. From [2], we know that

$$\begin{aligned} s(n,k) &= \sum_{h=0}^{n-k}(-1)^h\binom{n-1+h}{n-k+h}\binom{2n-k}{n-k-h}S(n-k+h,h), \\ S(n,k) &= \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n. \end{aligned}$$

For Leibniz numbers and Stirling numbers, we have

Theorem 2.1: For $n \geq k$ and $k \geq 2$, $\Re(n,k)$ and $s(n,k)$ satisfy

$$\begin{aligned} &\sum_{i_1+i_2+i_3=n-k}(-1)^{i_1+i_2+1}\Re(i_1+k,k)\binom{k+i_2-1}{i_2}\frac{s(i_3+k,k)}{(i_3+k)!} \\ &=\sum_{i=1}^{k}\frac{1}{i}\sum_{j_1+j_2=n-i+1}(-1)^{j_1}\binom{i+j_1-1}{j_1}\frac{s(j_2+k,k)}{(j_2+k)!} \\ &\quad-\frac{(k+1)s(n+k+1,k+1)}{(n+k+1)!}. \end{aligned} \tag{2.1}$$

Proof: From (1.1), we obtain

$$\begin{aligned} \sum_{p=0}^{\infty}\frac{\Re(p+k,k)(-t)^{p+k+1}\ln^k(1+t)}{(1+t)^k k!} &= \sum_{i=1}^{k}\frac{(-1)^k t^i\ln^k(1+t)}{ik!(1+t)^i} \\ &\quad -\frac{(-1)^k\ln^{k+1}(1+t)}{k!}. \end{aligned}$$

One can verify that

$$\sum_{p=0}^{\infty}\Re(p+k,k)(-1)^{p+k+1}t^{p+k+1}\sum_{q=0}^{\infty}\frac{(-k)_q t^q}{q!}\sum_{r=0}^{\infty}\frac{s(r+k,k)t^{r+k}}{(r+k)!}$$

$$= (-1)^k \sum_{i=1}^{k} \frac{t^i}{i} \sum_{j=0}^{\infty} \frac{(-i)_j}{j!} t^j \sum_{m=0}^{\infty} \frac{s(m+k,k)t^{m+k}}{(m+k)!}$$
$$-(-1)^k (k+1) \sum_{p=0}^{\infty} \frac{s(p+k+1,k+1)t^{p+k+1}}{(p+k+1)!}. \tag{2.2}$$

Comparing the coefficients of t^n on both sides of (2.2), we obtain

$$\sum_{i_1+i_2+i_3=n-k} (-1)^{i_1+k+1} \Re(i_1+k,k) \frac{(-k)_{i_2} s(i_3+k,k)}{i_2!(i_3+k)!}$$
$$= (-1)^k \sum_{i=1}^{k} \frac{1}{i} \sum_{j_1+j_2=n-i+1} \frac{(-i)_{j_1} s(j_2+k,k)}{j_1!(j_2+k)!}$$
$$-(-1)^k (k+1) \frac{s(n+k+1,k+1)}{(n+k+1)!}.$$

Then we have (2.1). This completes the proof.

In the rest of this section, we discuss the computation of other sums involving Leibniz and Stirling numbers. It is based on Euler's well-known Beta function defined by (see [9])

$$B(n,m) = \int_0^1 t^{n-1}(1-t)^{m-1} dt$$

for all positive integers n and m. Since

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \frac{(n-1)!(m-1)!}{(n+m-1)!},$$

$$\binom{n}{m}^{-1} = (n+1) \int_0^1 t^m (1-t)^{n-m} dt. \tag{2.3}$$

By means of (2.3), many identities involving inverses of binomial coefficients were obtained. See for instance [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. It is clear that

$$\Re(n,m) = \int_0^1 t^m (1-t)^{n-m} dt. \tag{2.4}$$

In terms of (2.4), we prove that

Theorem 2.2: For positive integers $n \geq 1$, $k \geq 1$ and $l \geq 1$, we have

$$\sum_{n=k}^{\infty} \frac{s(n,k)\Re(n+l,n)u^n}{n!}$$

$$= \frac{(1+u^{-1})^{l+1}}{(l+1)k!}\sum_{j=0}^{k}(-1)^j(k)_j \ln^{k-j}(1+u)h_{l+1}(j) + \frac{(1+u^{-1})^{l+1}h_{l+1}(k)}{l+1}$$

$$-\frac{1}{u^{l+1}}\sum_{i=0}^{l}\binom{l}{i}\frac{(-1)^i(1+u)^{l-i}}{(i+1)^{k+1}}, \quad -1 < u \le 1, \quad u \ne 0, \tag{2.5}$$

$$\sum_{n=k}^{\infty} S(n,k)\frac{\Re(n+l,n)u^n}{n!} = \frac{(-1)^k}{k!}\left[\sum_{j=1}^{l+1}\frac{(l)_{j-1}}{u^j}h_k(j) + \frac{l!}{u^{l+1}}h_k(l+1)\right.$$

$$\left. + \frac{l!}{u^{l+1}}\sum_{i=1}^{k}\binom{k}{i}\frac{(-1)^i e^{ui}}{i^{l+1}}\right] + \frac{(-1)^k}{(l+1)k!}, \quad u \ne 0, \tag{2.6}$$

where $h_k(m) = \sum_{i=1}^{k}\binom{k}{i}\frac{(-1)^{i+1}}{i^m}$.

Proof: It follows from (2.4) and the definition of $s(n,k)$ that

$$\sum_{n=k}^{\infty} s(n,k)\frac{\Re(n+l,n)u^n}{n!} = \sum_{n=k}^{\infty}\frac{s(n,k)}{n!}\int_0^1 (1-t)^l t^n u^n dt$$

$$= \frac{1}{k!}\int_0^1 (1-t)^l \ln^k(1+tu)dt.$$

Let $y = tu$. Then

$$\sum_{n=k}^{\infty} s(n,k)\frac{\Re(n+l,n)u^n}{n!}$$

$$= \frac{1}{k!u^{l+1}}\int_0^u (u-y)^l \ln^k(1+y)dy$$

$$= \frac{1}{k!u^{l+1}}\sum_{i=0}^{l}\binom{l}{i}(u+1)^{l-i}(-1)^i\int_0^u (1+y)^i \ln^k(1+y)dy$$

$$= \frac{1}{k!u^{l+1}}\sum_{i=0}^{l}\binom{l}{i}(u+1)^{l-i}(-1)^i d_{k,i},$$

where $d_{k,i} = \int_0^u (1+y)^i \ln^k(1+y)dy$.
Owing to

$$d_{k,i} = (1+u)^i\sum_{j=0}^{k}\frac{(-1)^j(k)_j \ln^{k-j}(1+u)}{(i+1)^{j+1}} + \frac{k![(1+u)^{i+1}-1]}{(i+1)^{k+1}},$$

we have

$$\sum_{n=k}^{\infty} s(n,k)\frac{\Re(n+l,n)u^n}{n!}$$
$$=\frac{(1+u)^{l+1}}{k!u^{l+1}}\sum_{j=0}^{k}(-1)^j(k)_j\ln^{k-j}(1+u)\sum_{i=0}^{l}\binom{l}{i}\frac{(-1)^i}{(i+1)^{j+1}}$$
$$+\frac{(1+u)^{l+1}}{u^{l+1}}\sum_{i=0}^{l}\binom{l}{i}\frac{(-1)^i}{(i+1)^{k+1}}-\frac{1}{u^{l+1}}\sum_{i=0}^{l}\binom{l}{i}\frac{(-1)^i(1+u)^{l-i}}{(i+1)^{k+1}}.$$

Then (2.5) holds.

Using the similar method, we have

$$\begin{aligned}\sum_{n=k}^{\infty} S(n,k)\frac{\Re(n+l,n)u^n}{n!} &= \frac{1}{k!}\int_0^1 (1-t)^l(e^{tu}-1)^k dt\\ &= \frac{(-1)^k}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^i\int_0^1(1-t)^l e^{itu}dt\\ &\quad +\frac{(-1)^k}{k!}\int_0^1(1-t)^l dt\\ &= \frac{(-1)^k}{k!}\Big[\sum_{i=1}^{k}\binom{k}{i}(-1)^{i+1}\sum_{j=1}^{l+1}\frac{(l)_{j-1}}{i^j u^j}\\ &\quad +\frac{l!}{u^{l+1}}\sum_{i=1}^{k}\binom{k}{i}(-1)^i\frac{e^{ui}-1}{i^{l+1}}\Big]\\ &\quad +\frac{(-1)^k}{k!}\int_0^1(1-t)^l dt\\ &= \frac{(-1)^k}{k!}\Big[\sum_{j=1}^{l+1}\frac{(l)_{j-1}}{u^j}\sum_{i=1}^{k}\binom{k}{i}\frac{(-1)^{i+1}}{i^j}\\ &\quad +\frac{l!}{u^{l+1}}\sum_{i=1}^{k}\binom{k}{i}(-1)^i\frac{e^{ui}-1}{i^{l+1}}\Big]+\frac{(-1)^k}{k!(l+1)}.\end{aligned}$$

Hence (2.6) holds. This completes the proof.

Theorem 2.3: For integers $n \geq 0$, and $l \geq 0$, we have

$$\sum_{k=0}^{n} s(n,k)\Re(k+l,k) = A_{l,n}, \qquad (2.7)$$

where $A_{l,n}=\int_0^1(1-t)^l(t)_n dt$, and they satisfy that

$$A_{l,n}=l\sum_{i=0}^{n}\binom{n}{i}\frac{1}{i+1}a_{n-i}A_{l-1,i+1}, \tag{2.8}$$

where a_{n-i} is the first kind of Cauchy numbers.
Proof: We first give the proof of (2.7). It is evident that

$$\begin{aligned}\sum_{k=0}^{n}s(n,k)\Re(k+l,k) &= \sum_{k=0}^{n}s(n,k)\int_0^1(1-t)^l t^k dt\\ &= \int_0^1(1-t)^l\sum_{k=0}^{n}s(n,k)t^k dt\\ &= \int_0^1(1-t)^l(t)_n dt\\ &= A_{l,n}.\end{aligned}$$

Now we consider the exponential generating function of $\{A_{l,n}\}$:

$$f(u,l)=\sum_{n=0}^{\infty}A_{l,n}\frac{u^n}{n!}.$$

We have

$$\begin{aligned}f(u,l) &= \int_0^1(1-t)^l\sum_{n=0}^{\infty}\frac{(t)_n u^n}{n!}dt\\ &= \int_0^1(1-t)^l(1+u)^t dt\\ &= \int_0^1(1-t)^l e^{t\ln(1+u)}dt\\ &= -\frac{1}{\ln(1+u)}+\frac{l}{\ln(1+u)}\int_0^1(1-t)^{l-1}e^{t\ln(1+u)}dt\\ &= -\frac{1}{\ln(1+u)}+\frac{l}{\ln(1+u)}\sum_{n=0}^{\infty}\frac{A_{l-1,n}u^n}{n!}.\end{aligned}$$

By

$$\begin{aligned}\frac{u}{\ln(1+u)} &= \sum_{n=0}^{\infty}\frac{a_n}{n!}u^n,\\ A_{l-1,0} &= \frac{1}{l},\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} \frac{A_{l,n}u^n}{n!} = l \sum_{n=0}^{\infty} \frac{a_n u^n}{n!} \sum_{n=0}^{\infty} \frac{A_{l-1,n+1}u^n}{(n+1)!}. \qquad (2.9)$$

Comparing the coefficients of $u^n/n!$ on both sides of (2.9), we prove that (2.8) holds. This completes the proof.

Remark: It is well known that for the sequences $\{f_n\}$ and $\{g_n\}$

$$f_n = \sum_{k=0}^{n} S(n,k) g_k \Leftrightarrow g_n = \sum_{k=0}^{n} s(n,k) f_k. \qquad (2.10)$$

From (2.10), we have $\sum_{k=0}^{n} S(n,k) A_{l,k} = \Re(n+l,l)$.

3. Identities involving Leibniz and harmonic numbers

For Leibniz and harmonic numbers, we have

Theorem 3.1: For integers $n \geq 0$, and $l \geq 0$, we have,

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{n+1} \Re(n+l+2, n+2)}{n+2} = \frac{2^l \ln 2}{l+1} \left[\ln 2 - 2H_{l+1} + \frac{1}{\ln 2} \sum_{i=1}^{l+1} \binom{l+1}{i} \frac{(-1)^i (2^{-i} - 1)}{i^2} \right], \qquad (3.1)$$

$$\sum_{n=0}^{\infty} (-1)^n H_{n+1} \Re(n+l+1, n+1) = 2^{l-1} \ln^2 2 - 2^l \ln 2 H_l - \sum_{i=1}^{l} \binom{l}{i} \frac{(-1)^i (2^l - 2^{l-i})}{i^2}, \qquad (3.2)$$

$$\sum_{n=0}^{\infty} (-1)^n H_{n+1} \Re(n+l, n) = \sum_{i=1}^{l} \binom{l}{i} \frac{H_i[(-1)^i - 2]}{i} + 2^l H_l \ln 2 - 2^{l-1} \ln 2 + 2^l \sum_{i=1}^{l} \binom{l}{i} \frac{(-1)^i (1 - 2^{-i})}{i^2} - \ln 2 \sum_{i=1}^{l} \frac{2^i}{i} + \frac{\pi^2}{12}. \qquad (3.3)$$

Proof: It follows from (2.4) and the definition of harmonic numbers that

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{n+1} \Re(n+l+2, n+2)}{n+2}$$
$$= \int_0^1 (1-t)^l \sum_{n=0}^{\infty} \frac{(-1)^n H_{n+1} t^{n+2}}{n+2} dt$$
$$= \frac{1}{2} \int_0^1 (1-t)^l \ln^2(1+t) dt$$
$$= \frac{1}{l+1} \int_0^1 \frac{(1-t)^{l+1}}{t+1} \ln(1+t) dt$$
$$= \frac{1}{l+1} \int_0^1 \sum_{i=0}^{l+1} \binom{l+1}{i} \frac{2^{l+1-i}(-1)^i (t+1)^i \ln(1+t)}{t+1} dt$$
$$= \frac{2^{l+1}}{l+1} \int_0^1 \frac{\ln(1+t)}{1+t} dt$$
$$+ \frac{2^{l+1}}{l+1} \sum_{i=1}^{l+1} \binom{l+1}{i} \frac{(-1)^i}{2^i} \int_0^1 (t+1)^{i-1} \ln(1+t) dt,$$
$$= \frac{2^l \ln^2 2}{l+1} + \frac{2^{l+1} \ln 2}{l+1} \sum_{i=1}^{l+1} \binom{l+1}{i} \frac{(-1)^i}{i}$$
$$+ \frac{2^{l+1}}{l+1} \sum_{i=1}^{l+1} \binom{l+1}{i} \frac{(-1)^i (2^{-i} - 1)}{i^2}.$$

Noting that

$$\sum_{i=1}^{l+1} \binom{l+1}{i} \frac{(-1)^i}{i} = -H_{l+1},$$

we have (3.1).

Because of

$$\frac{1}{n+2} \Re(n+l+2, n+2) = \frac{1}{l+1} \Re(n+1+l+1, n+1),$$

(3.1) is equivalent to (3.2).

Now, we prove that (3.3) holds. We know that

$$\sum_{n=0}^{\infty} (-1)^n H_{n+1} \Re(n+l, n)$$

$$= \int_0^1 (1-t)^l \sum_{n=0}^{\infty} (-1)^n H_{n+1} t^n dt$$
$$= \int_0^1 (1-t)^l \frac{\ln(1+t)}{t(1+t)} dt$$
$$= \int_0^1 \frac{\ln(1+t)}{t} dt + \sum_{i=1}^{l} \binom{l}{i} (-1)^i \int_0^1 t^{i-1} \ln(1+t) dt$$
$$- \int_0^1 (1-t)^l \frac{\ln(1+t)}{1+t} dt$$
$$= \int_0^1 \frac{\ln(1+t)}{t} dt + \sum_{i=1}^{l} \binom{l}{i} \frac{(-1)^i \ln 2}{i} - \sum_{i=1}^{l} \binom{l}{i} \frac{(-1)^i}{i} \int_0^1 \frac{t^i}{1+t} dt$$
$$- \int_0^1 (1-t)^l \frac{\ln(1+t)}{1+t} dt.$$

Noting that (3.2),

$$\int_0^1 t^i (1+t)^{-1} dt = (-1)^i \ln 2 + \sum_{j=1}^{i} \binom{i}{j} \frac{(-1)^{i-j}(2^j - 1)}{j},$$

and

$$\int_0^1 \frac{\ln(1+t)}{t} dt = \frac{\pi^2}{12},$$

we have (3.3). This completes the proof.

4. Identities involving Leibniz and arctan numbers

From the generating function of arctan numbers, we establish some identities involving Leibniz and arctan numbers.
Theorem 4.1: For integers $n \geq 0$, and $l \geq 0$, we have,

$$\sum_{n=1}^{\infty} t(n,1) \frac{\Re(n+l,n)}{n!} = \frac{\pi}{4(l+1)} - \sum_{i=0}^{l} \binom{l}{i} \frac{(-1)^i}{i+1} u_{i+1}, \tag{4.1}$$

where

$$u_i = \int_0^1 t^i (1+t^2)^{-1} dt, \quad u_0 = \frac{\pi}{4}, \quad u_1 = \frac{\ln 2}{2},$$
$$u_i + u_{i+2} = \frac{1}{i+1} \quad (i \geq 0).$$

Proof: It follows from (2.4) and the definition of arctan numbers that

$$\begin{aligned}
\sum_{n=1}^{\infty} t(n,1)\frac{\Re(n+l,n)}{n!} &= \int_0^1 (1-t)^l \sum_{n=1}^{\infty} t(n,1)\frac{t^n}{n!}dt \\
&= \int_0^1 (1-t)^l \arctan t dt \\
&= \sum_{i=0}^{l} \binom{l}{i}(-1)^i \int_0^1 t^i \arctan t dt \\
&= \frac{\pi}{4}\sum_{i=0}^{l}\binom{l}{i}\frac{(-1)^i}{i+1} - \sum_{i=0}^{l}\binom{l}{i}\frac{(-1)^i}{i+1}\int_0^1 \frac{t^{i+1}}{1+t^2}dt.
\end{aligned}$$

Hence (4.1) holds. This completes the proof.

Theorem 4.2: For $t(n,k)$ and $\Re(2n,n)$, we have

$$\sum_{n=k}^{\infty} t(n,k)\frac{\Re(2n,n)}{n!} \sim \frac{e^k[\arctan(1/4)]^{k+1}}{4k^{k+1}}\sqrt{\frac{17}{2}}, \quad k\to\infty.$$

Proof: It follows from (2.4) and the definition of arctan numbers that

$$\begin{aligned}
\sum_{n=k}^{\infty} t(n,k)\frac{\Re(2n,n)}{n!} &= \int_0^1 \sum_{n=k}^{\infty} t(n,k)\frac{(1-t)^n t^n}{n!}dt \\
&= \int_0^1 \frac{[\arctan(t(1-t))]^k}{k!}dt.
\end{aligned}$$

Let

$$\begin{aligned}
B_k &= \int_0^1 \frac{[\arctan(t(1-t))]^k}{k!}dt, \\
h(t) &= \ln \arctan t(1-t), \quad t\in(0,1).
\end{aligned}$$

Then $h(t)$ reaches the maximum at $t = 1/2, h'(1/2) = 0$, and $h''(1/2) = -\frac{32}{17}\arctan\frac{1}{4}$. By Laplace's method, we have

$$B_k \sim \frac{1}{k!}\left(\arctan\frac{1}{4}\right)^{k+1/2}\sqrt{\frac{-2\pi}{kh''(1/2)}}, \quad k\to\infty.$$

On the other hand,

$$k! \sim \left(\frac{k}{e}\right)^k\sqrt{2\pi k}, \quad k\to\infty.$$

Hence B_k has the following asymptotic expansion:

$$B_k \sim \frac{e^k[\arctan(1/4)]^{k+1}}{4k^{k+1}}\sqrt{\frac{17}{2}}, \quad k \to \infty.$$

This completes the proof.

5. The asymptotic expansion of certain sums involving Leibniz numbers

In the final of Section 4, we discuss the asymptotic expansion of the sum involving Leibniz and arctan numbers. In this section, we give the the asymptotic expansion of other certain sums involving Leibniz numbers by Laplace's method.

Theorem 5.1: When $k \to \infty$, we have

$$\sum_{n=0}^{k} \Re(2n,n) - \frac{2\pi\sqrt{3}}{9} \sim -\frac{1}{3\times 4^{k+1}}\sqrt{\frac{\pi}{k+1}}, \tag{5.1}$$

$$\sum_{n=0}^{\infty} \frac{k^n \Re(2n,n)}{n!} \sim \sqrt{\frac{\pi}{k}}e^{\frac{k}{4}}. \tag{5.2}$$

Proof: It is clear that

$$\begin{aligned}
\sum_{n=0}^{k} \Re(2n,n) &= \sum_{n=0}^{k}\int_0^1 t^n(1-t)^n dt \\
&= \frac{2\pi\sqrt{3}}{9} - \int_0^1 \frac{t^{k+1}(1-t)^{k+1}}{1-t(1-t)}dt, \\
\sum_{n=0}^{\infty} \frac{k^n \Re(2n,n)}{n!} &= \sum_{n=0}^{\infty}\frac{k^n}{n!}\int_0^1 t^n(1-t)^n dt \\
&= \int_0^1 e^{kt(1-t)}dt.
\end{aligned}$$

Let $w(t) = \ln t(1-t)$ $(t \in (0,1))$. Then $w(t)$ reaches the maximum at $t = 1/2, h'(1/2) = 0$, and $h''(1/2) = -8$. By Laplace's method, we obtain

$$\int_0^1 \frac{t^{k+1}(1-t)^{k+1}}{1-t(1-t)}dt \sim \frac{1}{3\times 4^{k+1}}\sqrt{\frac{\pi}{k+1}}, \quad k \to \infty.$$

Then (5.1) holds.

Let $g(t) = t(1-t)$ $(t \in (0,1))$. Then $g(t)$ reaches the maximum at $t = 1/2, g'(1/2) = 0$, and $g''(1/2) = -2$. By Laplace's method, we have

$$\int_0^1 e^{kt(1-t)} dt \sim e^{kg(1/2)} \sqrt{\frac{-2\pi}{kg''(1/2)}}, \quad k \to \infty.$$

Hence (5.2) holds. This completes the proof.

Theorem 5.2: When $l \to \infty$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{n+1} \Re(n+l+2, n+2)}{n+2} \sim \frac{2^l \ln 2}{l+1}[(\ln 2 - 2)\ln l + \ln 2 + (\ln 2 + 1)\gamma], \tag{5.3}$$

$$\sum_{n=0}^{\infty} (-1)^n H_{n+1} \Re(n+l+1, n+1) \sim 2^l \left[\frac{\ln 2}{2}(1-2\gamma) - \gamma \ln l\right], \tag{5.4}$$

$$\sum_{n=0}^{\infty} (-1)^n H_{n+1} \Re(n+l, n) \sim \frac{\pi^2}{12} - 2^{l-1}\ln 2 - \ln 2 \sum_{i=1}^{l} \frac{2^i}{i} + \sum_{i=1}^{l} \binom{l}{i} \frac{H_i[(-1)^i - 2]}{i}, \tag{5.5}$$

where γ is Euler's constant.

Proof: By Theorem 1 and Corollary 3 of [14], we have

$$\begin{aligned}
\sum_{i=1}^{l+1} \binom{l+1}{i} \frac{(-1)^i}{i^2 2^i} &= -[z^2] \frac{1}{2^z(1-z)(1-\frac{z}{2})\cdots(l-\frac{z}{l+1})} \\
&= -[z^2] \exp\left(-z\ln 2 - \sum_{j=1}^{l+1} \ln\left(1 - \frac{z}{j}\right)\right) \\
&= -[z^2] \sum_{i=0}^{\infty} (-1)^i z^i \ln^k 2 \exp\left(\sum_{k=1}^{\infty} \zeta_{l+1}(k) \frac{z^k}{k}\right) \\
&= -(\zeta_{l+1}(1)\ln 2 + \zeta_{l+1}(2)) \\
&\sim -\frac{1}{2}\ln^2 2 + (\ln 2 - \gamma)\ln l + \left(\ln 2 - \frac{1}{2}\right)\gamma - \frac{\pi^2}{12},
\end{aligned} \tag{5.6}$$

$[t^n]f(t)$ denotes the coefficient of t^n in $f(t)$, where

$$f(t) = \sum_{n=0}^{\infty} f_n t^n.$$

By Corollary 3 of [14], we have

$$\sum_{i=1}^{l+1} \binom{l+1}{i} \frac{(-1)^i}{i^2} \sim -\frac{1}{2}\ln^2 2 - \gamma \ln l - \frac{\gamma}{2} - \frac{\pi^2}{12}. \tag{5.7}$$

To (5.6), (5.7) into (3.1), (3.2), (3.3), available

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n H_{n+1} \Re(n+l+2, n+2)}{n+2} &\sim \frac{2^l \ln 2}{l+1}\Big[(\ln 2 - 2)\ln l \\
&\quad + \ln 2 + (\ln 2 + 1)\gamma\Big], \\
\sum_{n=0}^{\infty} (-1)^n H_{n+1} \Re(n+l+1, n+1) &\sim 2^l \left[\frac{\ln 2}{2}(1 - 2\gamma) - \gamma \ln l\right], \\
\sum_{n=0}^{\infty} (-1)^n H_{n+1} \Re(n+l, n) &\sim \frac{\pi^2}{12} - 2^{l-1}\ln 2 - \ln 2 \sum_{i=1}^{l} \frac{2^i}{i} \\
&\quad + \sum_{i=1}^{l} \binom{l}{i} \frac{H_i[(-1)^i - 2]}{i}.
\end{aligned}$$

Hence (5.3)-(5.5) hold. This completes the proof.

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