# Decompositions of Complete Graphs Into Paths and Cycles

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**Abstract.** Let $P_{k+1}$ denote a path of length $k$ and let $C_k$ denote a cycle of length $k$. As usual $K_n$ denotes the complete graph on $n$ vertices. In this paper we investigate decompositions of $K_n$ into paths and cycles, and give some necessary and/or sufficient conditions for such a decomposition to exist. Besides, we obtain a necessary and sufficient condition for decomposing $K_n$ into $p$ copies of $P_5$ and $q$ copies of $C_4$ for all possible values of $p \geq 0$ and $q \geq 0$.

Keywords: Graph decompositions, Path, Cycle, Complete graph.

## 1 Introduction

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [11].

As usual $K_n$ denotes the complete graph on $n$ vertices and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes $m$ and $n$. Let $P_{k+1}$ denote a path of length $k$ and let $C_k$ denote a cycle of length $k$. Let $L = \{H_1, H_2, \ldots, H_r\}$ be a family of subgraphs of $G$. An *L-decomposition* of $G$ is an edge-disjoint decomposition of $G$ into positive integer $\alpha_i$ copies of $H_i$, where $i \in \{1, 2, \ldots, r\}$. Furthermore, if each $H_i$ ($i \in \{1, 2, \ldots, r\}$) is isomorphic to a graph $H$, we say that $G$ has an *H-decomposition*. It is easily seen that $\sum_{i=1}^{r} \alpha_i e(H_i) = e(G)$ is one of the obvious necessary

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conditions for the existence of a $\{H_1, H_2, \ldots, H_r\}$-decomposition of $G$. For convenience, we call the equation, $\sum_{i=1}^{r} \alpha_i e(H_i) = e(G)$, a *necessary sum condition.* The problem of $L$-decompositions of $\lambda K_n$ is the well-known Alspach's conjecture [7] when $L$ is any set of cycles of length at most $n$ satisfying the necessary sum condition and $2|\lambda(n-1)$. For the case $\lambda = 1$, the Alspach conjecture is also stated for even values of $n$, where in this case the cycles should decompose $K_n$ minus a one-factor. There are many related results, but only special cases of this conjecture are solved completely (see e.q. [5, 6, 7, 8, 9, 12, 13, 14, 15, 16]). Recent results of Alspach, Gavlas, and Šajna settle Alspach's problem in the case where the cycle lengths are all the same [10, 19]. When $L$ is a set of paths, in this case the problem of $L$-decomposition has been investigated by Tarsi [17] who showed that if $(n-1)\lambda$ is even and $L$ is any set of paths of length at most $n-3$ satisfying the necessary sum condition, then $\lambda K_n$ has an $L$-decomposition. The problem of $L$-decomposition of $\lambda K_{m,n}$ has been investigated by M. Truszczyński [18] when $m$ and $n$ are even and $L$ is any set of paths with some constraints on length satisfying the necessary sum condition.

It is natural to consider the problem of $L$-decompositions of $K_n$, where $L$ is a combination of paths, cycles, and some other subgraphs. We will restrict our attention to $L$ which is any set of paths and cycles satisfying the necessary sum condition. There are several similarly known results as follows. A graph-pair of order $t$ consists of two non-isomorphic graphs $G$ and $H$ on $t$ non-isolated vertices for which $G \cup H$ is isomorphic to $K_t$. If $G$ and $H$ form a graph-pair of order $t$, then Abueida, Daven, and Roblee [1, 3] completely determine the values of $n$ for which $\lambda K_n$ admits a $\{G, H\}$-decomposition, when $\lambda \geq 1$ and $t = 4, 5$. In [2], Abueida and Daven proved that there exists a $\{K_k, K_{1,k}\}$-decomposition of $K_n$ for all $k \geq 3$ and $n \equiv 0, 1 \pmod{k}$. Abueida and O'Neil [4] proved that for $k = 3$, 4, and 5, there exists a $\{C_k, K_{1,k-1}\}$-decomposition of $\lambda K_n$ for any $n \geq k+1$ except when the ordered triple $(k, n, \lambda) \in \{(3,4,1), (4,5,1), (5,6,1), (5,6,2), (5,6,4), (5,7,1), (5,8,1)\}$.

In this paper we investigate decompositions of $K_n$ into paths and cycles, and give some necessary or sufficient conditions for such a decomposition to exist. Besides, we obtain a necessary and sufficient condition for decomposing $K_n$ into $p$ copies of $P_5$ and $q$ copies of $C_4$ for all possible values of $p \geq 0$ and $q \geq 0$.

# 2 Necessary conditions for decomposing $K_n$ into paths and cycles

For our discussion we need the following notations. Let $x_1 x_2 \ldots x_{k+1}$ denote the path $P_{k+1}$ with vertices $x_1, x_2, \ldots, x_{k+1}$ and edges $x_1x_2, x_2x_3, \ldots,$

$x_k x_{k+1}$ and let $(x_1, x_2, \ldots, x_k)$ denote the cycle $C_k$ with vertices $x_1, x_2, \ldots, x_k$ and edges $x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1$.

In the following lemma we will show a special case for decomposing complete graphs into paths and cycles.

**Lemma 2.1** *If $p$ and $q$ are nonnegative integers such that $p+q=9$ and $p \neq 1$, then $K_9$ can be decomposed into $p$ copies of $P_5$ and $q$ copies of $C_4$.*

**Proof.** Let $V(K_9) = \{x_1, x_2, \ldots, x_9\}$. We exhibit that $K_9$ can be decomposed into $p$ copies of $P_5$ and $q$ copies of $C_4$, for each pair $p$, $q$ of nonnegative integers such that $4(p+q) = \binom{9}{2}$ (i.e., $p+q=9$) and $p \neq 1$ as follows:
(1) $p=0$ and $q=9$.
$(x_1, x_5, x_2, x_3)$, $(x_2, x_6, x_3, x_4)$, $(x_3, x_7, x_4, x_5)$, $(x_4, x_8, x_5, x_6)$, $(x_5, x_9, x_6, x_7)$, $(x_6, x_1, x_7, x_8)$, $(x_7, x_2, x_8, x_9)$, $(x_8, x_3, x_9, x_1)$, $(x_9, x_4, x_1, x_2)$.
(2) $p=2$ and $q=7$.
$x_8x_1x_9x_2x_4$, $x_8x_2x_3x_1x_4$, $(x_1, x_2, x_7, x_5)$, $(x_1, x_6, x_3, x_7)$, $(x_2, x_5, x_9, x_6)$, $(x_3, x_5, x_6, x_4)$, $(x_6, x_8, x_9, x_7)$, $(x_4, x_5, x_8, x_7)$, $(x_3, x_8, x_4, x_9)$.
(3) $p=3$ and $q=6$.
$x_4x_1x_2x_9x_7$, $x_4x_9x_3x_8x_6$, $x_6x_1x_9x_8x_7$, $(x_1, x_5, x_2, x_3)$, $(x_2, x_6, x_3, x_4)$, $(x_3, x_7, x_4, x_5)$, $(x_4, x_8, x_5, x_6)$, $(x_5, x_9, x_6, x_7)$, $(x_1, x_8, x_2, x_7)$.
(4) $p=4$ and $q=5$.
$x_2x_1x_9x_8x_6$, $x_2x_8x_1x_7x_9$, $x_3x_9x_4x_1x_6$, $x_3x_8x_7x_2x_9$, $(x_1, x_5, x_2, x_3)$, $(x_2, x_6, x_3, x_4)$, $(x_3, x_7, x_4, x_5)$, $(x_4, x_8, x_5, x_6)$, $(x_5, x_9, x_6, x_7)$.
(5) $p=5$ and $q=4$.
$x_2x_1x_9x_7x_5$, $x_2x_9x_8x_7x_6$, $x_2x_8x_3x_9x_4$, $x_2x_7x_1x_8x_6$, $x_4x_1x_6x_9x_5$, $(x_1, x_5, x_2, x_3)$, $(x_2, x_6, x_3, x_4)$, $(x_3, x_7, x_4, x_5)$, $(x_4, x_8, x_5, x_6)$.
(6) $p=6$ and $q=3$.
$x_2x_1x_4x_8x_3$, $x_2x_9x_8x_7x_6$, $x_2x_8x_5x_9x_4$, $x_2x_7x_1x_6x_5$, $x_3x_9x_1x_8x_6$, $x_4x_6x_9x_7x_5$, $(x_1, x_5, x_2, x_3)$, $(x_2, x_6, x_3, x_4)$, $(x_3, x_7, x_4, x_5)$.
(7) $p=7$ and $q=2$.
$x_1x_9x_8x_7x_5$, $x_2x_1x_8x_6x_4$, $x_2x_9x_4x_8x_3$, $x_3x_7x_2x_8x_5$, $x_4x_1x_6x_9x_5$, $x_4x_5x_6x_7x_1$, $x_4x_7x_9x_3x_5$, $(x_1, x_5, x_2, x_3)$, $(x_2, x_6, x_3, x_4)$.
(8) $p=8$ and $q=1$.
$x_1x_3x_9x_7x_5$, $x_2x_4x_9x_8x_5$, $x_1x_9x_5x_6x_8$, $x_2x_9x_6x_7x_8$, $x_1x_8x_2x_7x_3$, $x_2x_5x_3x_8x_4$, $x_3x_6x_4x_5x_1$, $x_4x_7x_1x_6x_2$, $(x_1, x_2, x_3, x_4)$.
(9) $p=9$ and $q=0$.
$x_1x_7x_2x_6x_3$, $x_3x_5x_4x_9x_8$, $x_2x_8x_3x_7x_4$, $x_4x_6x_5x_9x_1$, $x_3x_1x_4x_8x_5$, $x_5x_7x_6x_9x_2$, $x_4x_2x_5x_1x_6$, $x_6x_8x_7x_9x_3$, $x_8x_1x_2x_3x_4$. □

The following theorem gives a necessary condition for decomposing complete graphs $K_n$ into paths and cycles when $n$ is odd.

**Theorem 2.2** *Let $n$, $l$, and $k$ be positive integers such that $n$ is odd and $n \geq max\{l, k+1\}$. If $K_n$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$*

*copies of $C_l$ for nonnegative integers $p$ and $q$, then $pk + ql = e(K_n)$ and $p \neq 1$.*

**Proof.** Condition $pl + qk = e(K_n)$ is trivial. On the contrary, suppose that $p = 1$. Let $P$ denote the only path of length $k$ in the decomposition. It follows that the end vertices of $P$ have odd degree $n - 2$ in $K_n - E(P)$. Therefore, $K_n - E(P)$ can not be decomposed into cycles. We obtained a contradiction. □

In the following lemma we will show another special case for decomposing complete graphs into paths and cycles.

**Lemma 2.3** *If $p$ and $q$ are nonnegative integers such that $p + q = 7$ and $p \geq 4$, then $K_8$ can be decomposed into $p$ copies of $P_5$ and $q$ copies of $C_4$.*

**Proof.** Let $V(K_8) = \{x_1, x_2, \ldots, x_8\}$. We exhibit that $K_8$ can be decomposed into $p$ copies of $P_5$ and $q$ copies of $C_4$, for each pair $p$, $q$ of nonnegative integers such that $4(p+q) = \binom{8}{2}$ (i.e., $p + q = 7$) and $p \geq 4$ as follows:

(1) $p = 4$ and $q = 3$.
$x_1x_6x_2x_4x_5$, $x_2x_5x_1x_3x_6$, $x_3x_5x_7x_1x_8$, $x_4x_6x_8x_2x_7$, $(x_1, x_2, x_3, x_4)$, $(x_5, x_6, x_7, x_8)$, $(x_3, x_8, x_4, x_7)$.

(2) $p = 5$ and $q = 2$.
$x_1x_3x_5x_8x_7$, $x_2x_7x_5x_4x_6$, $x_3x_6x_8x_2x_5$, $x_4x_2x_6x_1x_5$, $x_5x_6x_7x_1x_8$, $(x_1, x_2, x_3, x_4)$, $(x_3, x_8, x_4, x_7)$.

(3) $p = 6$ and $q = 1$.
$x_1x_3x_7x_8x_6$, $x_1x_8x_2x_7x_5$, $x_2x_6x_1x_7x_4$, $x_3x_6x_4x_5x_1$, $x_4x_2x_5x_3x_8$, $x_4x_8x_5x_6x_7$, $(x_1, x_2, x_3, x_4)$.

(4) $p = 7$ and $q = 0$.
$x_1x_8x_2x_7x_5$, $x_2x_1x_5x_6x_7$, $x_2x_6x_1x_7x_4$, $x_2x_3x_4x_8x_6$, $x_3x_6x_4x_5x_8$, $x_4x_2x_5x_3x_8$, $x_4x_1x_3x_7x_8$. □

The following theorem gives a necessary condition for decomposing complete graphs $K_n$ into paths and cycles when $n$ is even.

**Theorem 2.4** *Let $n$, $l$, and $k$ be positive integers such that $n$ is even and $n \geq max\{l, k+1\}$. If $K_n$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_l$ for nonnegative integers $p$ and $q$, then $pk + ql = e(K_n)$ and $p \geq \frac{n}{2}$.*

**Proof.** Condition $pl + qk = e(K_n)$ is trivial. Let $D$ be an arbitrary decomposition of $K_n$ into $p$ copies of $P_{k+1}$ and $q$ copies of $C_l$; let $P^{(1)}, P^{(2)}, \ldots, P^{(p)}$ denote those $p$ copies of $P_{k+1}$ in $D$. By assumption, $K_n - E(P^{(1)} \cup P^{(2)} \cup \cdots \cup P^{(p)})$ has a $C_l$-decomposition. It follows that each vertex of $K_n - E(P^{(1)} \cup P^{(2)} \cup \cdots \cup P^{(p)})$ has even degree. Since $n$ is even, each vertex of $K_n$ must be an end vertex of at least one $P^{(i)}$ $(1 \leq i \leq p)$. It implies that $2p \geq n$. □

## 3 Decompositions of $K_n$ into $P_{k+1}$'s and $C_k$'s

In this section we investigate the problem of decomposing the complete graph $K_n$ into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$ for all possible values of $p \geq 0$ and $q \geq 0$, and obtain some sufficient conditions for such a decomposition to exist when $k$ is even. Besides, we establish a necessary and sufficient condition for decomposing $K_n$ into $p$ copies of $P_5$ and $q$ copies of $C_4$ for all possible values of $p \geq 0$ and $q \geq 0$. Let us first introduce four results on $P_{k+1}$-decomposition and $C_k$-decomposition.

**Theorem 3.1** (Tarsi [17]) *Let $k$ and $n$ be positive integers. $K_n$ has a $P_{k+1}$-decomposition if and only if $n \geq k+1$ and $n(n-1) \equiv 0 \pmod{2k}$.* □

**Theorem 3.2** (Truszczyński [18]) *Let $k$ be a positive integer and let $m$ and $n$ be positive even integers such that $m \geq n$. $K_{m,n}$ has a $P_{k+1}$-decomposition if and only if $m \geq \lceil \frac{k+1}{2} \rceil$, $n \geq \lceil \frac{k}{2} \rceil$, and $mn \equiv 0 \pmod{k}$.* □

**Theorem 3.3** (Alspach and Šajna [10, 19]) *Let $n$ and $k$ be positive integers. $K_n$ has a $C_k$-decomposition if and only if $n$ is odd, $3 \leq k \leq n$, and $n(n-1) \equiv 0 \pmod{2k}$.* □

**Theorem 3.4** (Sotteau [20]) *Let $m$, $n$, and $k$ be positive integers. $K_{m,n}$ has a $C_{2k}$-decomposition if and only if $m$ and $n$ are even, $k \geq 2$, $m \geq k$, $n \geq k$, and $mn \equiv 0 \pmod{2k}$.* □

By Theorem 3.2 and Theorem 3.4, we obtain a theorem below.

**Theorem 3.5** *Let $k$, $s$, and $t$ be positive even integers such that $k \geq 4$ and $t > s$. If $k \leq 2(t-s)$ and $k \leq 2s$, then there exists a decomposition of $K_{k,t}$ into $s$ copies of $P_{k+1}$ and $t-s$ copies of $C_k$.*

**Proof.** It is easily seen that $K_{k,t}$ can be decomposed into $K_{k,s}$ and $K_{k,t-s}$. Since $k \leq 2s$ and both $k$ and $s$ are even, by Theorem 3.2, $K_{k,s}$ can be decomposed into $s$ copies of $P_{k+1}$. On the other hand, since $k \leq 2(t-s)$ and both $k$ and $t-s$ are even, by Theorem 3.4, $K_{k,t-s}$ can be decomposed into $t-s$ copies of $C_k$. □

In the following theorems we will obtain some sufficient conditions for decomposing $K_n$ into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$ for all possible values of $p \geq 0$ and $q \geq 0$ when $n$ is odd and $k$ is even. We need the following lemma for our discussion.

**Lemma 3.6** *Let $r$ be a nonnegative integer and let $k$, $s$, and $t$ be positive integers such that $0 \leq r \leq k-1$, $2 \leq t < s$, $k \geq 4$, and $k$, $r$, and $t$ are* *all even. If $K_{tk+1}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q) = e(K_{tk+1})$ and $p \neq 1$, then $K_{sk+r+1} - E(K_{(s-t)k+r+1})$ can be decomposed into $p$ copies of $P_{k+1}$ and $\frac{t}{2}(tk+1) + t(s-t)k + tr - p$ copies of $C_k$ for $0 \leq p \leq \frac{t}{2}(tk+1) + t(s-t)k + tr$ and $p \neq 1$.*

**Proof.** It is easily seen that $K_{sk+r+1} - E(K_{(s-t)k+r+1})$ can be viewed as an edge-disjoint union of $K_{tk+1}$ and $K_{tk,(s-t)k+r}$. Since $e(K_{tk+1} \cup K_{tk,(s-t)k+r}) = \frac{(tk+1)tk}{2} + tk[(s-t)k+r]$, we get $0 \leq p \leq \frac{t}{2}(tk+1) + t(s-t)k + tr$ and $p \neq 1$. Now we consider three cases below.

*Case 1.* $0 \leq p \leq \frac{t}{2}(tk+1)$ and $p \neq 1$.
By assumption, we can obtain $p$ copies of $P_{k+1}$ and $\frac{t}{2}(tk+1) - p$ copies of $C_k$ from $K_{tk+1}$. As to the remaining graph, by Theorem 3.4, $K_{tk,(s-t)k+r}$ has a $C_k$-decomposition.

*Case 2.* $\frac{t}{2}(tk+1)+1 \leq p \leq t(s-t)k+tr$. (When $\frac{t}{2}(tk+1) \geq t(s-t)k+tr$, we skip this case.)
Let $l = \lfloor \frac{p-\frac{tk}{2}}{t} \rfloor$ if $\lfloor \frac{p-\frac{tk}{2}}{t} \rfloor$ is even and $l = \lfloor \frac{p-\frac{tk}{2}}{t} \rfloor - 1$ if $\lfloor \frac{p-\frac{tk}{2}}{t} \rfloor$ is odd. It is easily seen that $K_{tk,(s-t)k+r}$ can be decomposed into $t$ copies of $K_{k,(s-t)k+r}$. Since $l \geq \lfloor \frac{\frac{t}{2}(tk+1)+1-\frac{tk}{2}}{t} \rfloor - 1 = \lfloor (t-1)\frac{k}{2} + \frac{1}{2} + \frac{1}{t} \rfloor - 1 \geq \frac{k}{2}$ (note that $t \geq 2$ is even), $(s-t)k + r - l \geq (s-t)k + r - \frac{t(s-t)k+tr-\frac{tk}{2}}{t} = \frac{k}{2}$, and both $l$ and $(s-t)k+r-l$ are even, by Theorem 3.5, we can obtain $l$ copies of $P_{k+1}$ and $(s-t)k+r-l$ copies of $C_k$ from each copy of $K_{k,(s-t)k+r}$. Therefore, we can obtain $tl$ copies of $P_{k+1}$ and $t(s-t)k+tr-tl$ copies of $C_k$ from $K_{tk,(s-t)k+r}$. Since $p - \frac{tk}{2} - (2t-1) \leq tl \leq p - \frac{tk}{2}$, we have $\frac{tk}{2} \leq p - tl \leq \frac{tk}{2} + 2t - 1 \leq \frac{t(k+4)}{2} - 1 \leq \frac{t}{2}(tk-1) \leq \frac{t}{2}(tk+1)$ (note that $k \geq 4$ and $t \geq 2$). By assumption, $K_{tk+1}$ can be decomposed into $p - tl$ copies of $P_{k+1}$ and $\frac{t}{2}(tk+1) - (p-tl)$ copies of $C_k$.

*Case 3.* $t(s-t)k + tr + 1 \leq p \leq \frac{t}{2}(tk+1) + t(s-t)k + tr$.
When $p \neq t(s-t)k+tr+1$, we can obtain $t(s-t)k+tr$ copies of $P_{k+1}$ from $K_{tk,(s-t)k+r}$ first. Since $2 \leq p-[t(s-t)k+tr] \leq \frac{t}{2}(tk+1)$, by assumption, we can obtain $p-[t(s-t)k+tr]$ copies of $P_{k+1}$ and $\frac{t}{2}(tk+1)-[p-(t(s-t)k+tr)]$ copies of $C_k$ from $K_{tk+1}$. When $p = t(s-t)k+tr+1$, since $K_{tk,(s-t)k+r}$ can be decomposed into $K_{(t-1)k,(s-t)k+r}$ and $K_{k,(s-t)k+r}$, by Theorem 3.5, we can obtain $k+r$ copies of $C_k$ and $(s-t-1)k$ copies of $P_{k+1}$ from $K_{k,(s-t)k+r}$ (note that when $s = t+1$, $(s-t-1)k = 0$); by Theorem 3.2, we can obtain $(t-1)(s-t)k+(t-1)r$ copies of $P_{k+1}$ from $K_{(t-1)k,(s-t)k+r}$. Therefore, we can obtain $t(s-t)k+tr-k-r$ copies of $P_{k+1}$ and $k+r$ copies of $C_k$ from $K_{tk,(s-t)k+r}$. Since $p - [t(s-t)k+tr-k-r] = k+r+1 \leq \frac{t}{2}(tk+1)$, by assumption, we can obtain $k+r+1$ copies of $P_{k+1}$ and $\frac{t}{2}(tk+1)-(k+r+1)$ copies of $C_k$ from $K_{tk+1}$. □

**Theorem 3.7** *Let $n$ be a positive odd integer and let $k$ and $t$ be positive even integers such that $k \geq 4$, $t \geq 2$, $n \geq (t+1)k+1$, and $n(n-1) \equiv 0 \pmod{2k}$. If $K_{tk+1}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q) = e(K_{tk+1})$ and $p \neq 1$, then $K_n$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q) = e(K_n)$ and $p \neq 1$.*

**Proof.** Write $n = sk+r+1$, where $s$ and $r$ are nonnegative integers such that $s \geq t+1$ and $0 \leq r \leq k-1$. Since $n$ is odd and $k$ is even, $r$ is even. The proof is by induction on $s$. It is easily seen that $K_{sk+r+1}$ can be viewed as an edge-disjoint union of $K_{tk+1}$, $K_{tk,(s-t)k+r}$, and $K_{(s-t)k+r+1}$. Since $p+q = \frac{n(n-1)}{2k} = \frac{(sk+r+1)(sk+r)}{2k}$, we get $p \leq \frac{[(s-t)k+r+1][(s-t)k+r]}{2k} + \frac{t}{2}(tk+1)+t(s-t)k+tr$ and $p \neq 1$. Note also that $2k|(sk+r+1)(sk+r)$ and $t$ is even implies $2k|[(s-t)k+r+1][(s-t)k+r]$, and so $\frac{[(s-t)k+r+1][(s-t)k+r]}{2k}$ is an integer.

When $t+1 \leq s \leq 2t$, we consider two cases below. When $0 \leq p \leq \frac{t}{2}(tk+1)+t(s-t)k+tr$ and $p \neq 1$, by Lemma 3.6, an edge-disjoint union of $K_{tk+1}$ and $K_{tk,(s-t)k+r}$ can be decomposed into $p$ copies of $P_{k+1}$ and $\frac{t}{2}(tk+1)+t(s-t)k+tr-p$ copies of $C_k$. By Theorem 3.3, the remaining graph $K_{(s-t)k+r+1}$ has a $C_k$-decomposition.

When $\frac{t}{2}(tk+1)+t(s-t)k+tr+1 \leq p \leq \frac{[(s-t)k+r+1][(s-t)k+r]}{2k} + \frac{t}{2}(tk+1)+t(s-t)k+tr$, by Theorem 3.1, we can obtain $\frac{[(s-t)k+r+1][(s-t)k+r]}{2k}$ copies of $P_{k+1}$ from $K_{(s-t)k+r+1}$ first. On the other hand, since $\frac{t}{2}(tk+1)+t(s-t)k+tr-\frac{[(s-t)k+r+1][(s-t)k+r]}{2k} > 1$, we have $2 \leq p-\frac{[(s-t)k+r+1][(s-t)k+r]}{2k} \leq \frac{t}{2}(tk+1)+t(s-t)k+tr$. By Lemma 3.6 again, we can obtain $p-\frac{[(s-t)k+r+1][(s-t)k+r]}{2k}$ copies of $P_{k+1}$ and $\frac{t}{2}(tk+1)+t(s-t)k+tr-[p-\frac{[(s-t)k+r+1][(s-t)k+r]}{2k}]$ copies of $C_k$ from an edge-disjoint union of $K_{tk+1}$ and $K_{tk,(s-t)k+r}$.

Now we suppose that $s \geq 2t+1$. By induction hypothesis, $K_{(s-t)k+r+1}$ can be decomposed into $u$ copies of $P_{k+1}$ and $\frac{[(s-t)k+r+1][(s-t)k+r]}{2k} - u$ copies of $C_k$ for $0 \leq u \leq \frac{[(s-t)k+r+1][(s-t)k+r]}{2k}$ and $u \neq 1$. On the other hand, by Lemma 3.6, an edge-disjoint union of $K_{tk+1}$, $K_{tk,(s-t)k+r}$ can be decomposed into $v$ copies of $P_{k+1}$, with the remaining edges decomposed into copies of $C_k$ for $0 \leq v \leq \frac{t}{2}(tk+1)+t(s-t)k+tr$ and $v \neq 1$. Therefore, $K_{sk+r+1}$ can be decomposed into $p$ copies of $P_{k+1}$, with the remaining edges decomposed into copies of $C_k$ for $0 \leq p \leq \frac{[(s-t)k+r+1][(s-t)k+r]}{2k} + \frac{t}{2}(tk+1)+t(s-t)k+tr$ and $p \neq 1$. This completes the proof. □

Next theorem we will prove that for any positive even integer $k \geq 4$, $K_{4k+1}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$ for all possible values of $p \geq 0$ and $q \geq 0$. We need the following lemma for our discussion.

**Lemma 3.8** *Let $m$ and $n$ be positive integers such that $m \geq 3$ and $n \geq 2$. Suppose that for $i \in \{1, 2, \ldots, n\}$, $C_i$ denotes the cycle $(x_{(i,1)}, x_{(i,2)}, \ldots, x_{(i,m)})$ of length $m$. If $x_{(1,1)} = x_{(2,1)} = \cdots = x_{(n,1)}$, $x_{(i+1,2)} \notin \{x_{(i,1)}, x_{(i,2)}, \ldots, x_{(i,m)}\}$ for $i \in \{1, 2, \ldots, n-1\}$, and $x_{(1,2)} \notin \{x_{(n,1)}, x_{(n,2)}, \ldots, x_{(n,m)}\}$, then $\bigcup_{i=1}^{n} C_i$ can be decomposed into $n$ paths of length $m$.*

**Proof.** By assumption, $\bigcup_{i=1}^{n} C_i$ can be decomposed into $n$ paths of length $m$ below: $x_{(1,2)}x_{(1,3)} \cdots x_{(1,m)}x_{(1,1)}x_{(2,2)}$, $x_{(2,2)}x_{(2,3)} \cdots x_{(2,m)}x_{(2,1)}x_{(3,2)}$, $\ldots$, $x_{(n,2)}x_{(n,3)} \cdots x_{(n,m)}x_{(n,1)}x_{(1,2)}$. □

The *label* of an edge $x_i x_j$ of $K_n$ with vertex set $\{x_0, x_1, \ldots, x_{n-1}\}$ is the number $\min\{|j-i|, n-|j-i|\}$. The label of any edge is thus one of the numbers $1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. If $n$ is odd, then there are $n$ edges of label $i$ for $i \in \{1, 2, \ldots, \frac{n-1}{2}\}$. Suppose that $C$ is the cycle $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ in $K_n$. For an integer $t$, we use $C+t$ to denote the cycle $(x_{i_1+t}, x_{i_2+t}, \ldots, x_{i_k+t})$, where the subscripts of $x_i$'s are taken modulo $n$. It is easily seen that the labels of $(C+t)$'s edges and $C$'s are the same.

**Theorem 3.9** *If $k$ is an even integer such that $k \geq 4$, then $K_{4k+1}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q) = e(K_{4k+1})$ and $p \neq 1$.*

**Proof.** Let the vertices of $K_{4k+1}$ be labelled with $x_0, x_1, \ldots, x_{4k}$. We first construct two cycles $R$ and $Q$ of length $k$ such that $Q \cup R$ consists of edges with the labels $1, 2, \ldots, 2k$, and both $Q \cup R$ and $Q \cup R \cup (Q+j)$ can be decomposed into $P_{k+1}$'s, for some $j \in \{1, 2, \ldots, 4k\}$. We consider two cases.

*Case 1.* $k \equiv 0 \pmod 4$.
Let $k = 4m$, where $m$ is a positive integer. When $m = 1$, let $Q$ and $R$ denote the cycles $(x_0, x_4, x_1, x_2)$ and $(x_0, x_8, x_1, x_6)$, respectively. It is easily seen that both $Q$ and $R$ are cycles of length 4, where $Q$ consists of edges with the following labels in order of 4, 3, 1, 2, and $R$, in order of 8, 7, 5, 6. It is easily seen that $Q \cup R$ can be decomposed into paths $x_8x_1x_6x_0x_4$ and $x_4x_1x_2x_0x_8$; $Q \cup R \cup (Q+1)$ can be decomposed into paths $x_8x_1x_6x_0x_4$, $x_8x_0x_2x_1x_3$, and $x_3x_2x_5x_1x_4$.

When $m \geq 2$, let $Q$ and $R$ denote the cycles $(x_0, x_{4m}, x_1, x_{4m-1}, \ldots, x_{3m+1}, x_m, x_{m+1}, x_{3m}, x_{m+2}, \ldots, x_{2m+2}, x_{2m})$ and $(x_{4m}, x_{12m}, x_{4m+1}, x_{12m-1}, \ldots, x_{11m+1}, x_{5m}, x_{9m+1}, x_{15m}, x_{9m+2}, \ldots, x_{14m+2}, x_{10m})$, respectively. It is not difficult to see that both $Q$ and $R$ are cycles of length $4m$, where $Q$ consists of edges with the following labels in order of $4m$, $4m-1, \ldots, 2m+1$, $1$, $2m-1$, $2m-2, \ldots, 2$, $2m$, and $R$, in order of $8m$, $8m-1, \ldots, 6m+1$, $4m+1$, $6m-1$, $6m-2, \ldots, 4m+2$, $6m$.

Since $V(Q) \cap V(R) = \{x_{4m}\}$, by Lemma 3.8, $Q \cup R$ can be decomposed into two copies of $P_{k+1}$. On the other hand, since $x_{4m} \in [V(Q) \cap V(R) \cap V(Q+4m)]$, $x_{12m} \notin V(Q)$, $x_{8m} \notin V(R)$ and $x_1 \notin V(Q+4m)$, by Lemma 3.8 again, $Q \cup R \cup (Q+4m)$ can be decomposed into three copies of $P_{k+1}$.

*Case 2.* $k \equiv 2 \pmod 4$.
Let $k = 4m+2$, where $m$ is a positive integer. When $m = 1$, let $Q$ and $R$ denote the cycles $(x_0, x_7, x_2, x_5, x_3, x_4)$ and $(x_1, x_{13}, x_2, x_{12}, x_3, x_9)$, respectively. It is easily seen that both $Q$ and $R$ are cycles of length 6, where $Q$ consists of edges with the following labels in order of 7, 5, 3, 2, 1, 4, and $R$, in order of 12, 11, 10, 9, 6, 8. It is easily seen that $Q \cup R$ can be decomposed into paths $x_5x_3x_4x_0x_7x_2x_{12}$ and $x_{12}x_3x_9x_1x_{13}x_2x_5$; $Q \cup R \cup (Q+2)$ can be decomposed into paths $x_5x_3x_4x_0x_7x_2x_9$, $x_{12}x_3x_9x_1x_{13}x_2x_5$, and $x_9x_4x_7x_5x_6x_2x_{12}$.

When $m \geq 2$, let $Q$ and $R$ denote the cycles $(x_0, x_{4m+3}, x_2, x_{4m+2}, \ldots, x_{3m+4}, x_{m+1}, x_{3m+2}, x_{m+2}, x_{3m+1}, \ldots, x_{2m+1}, x_{2m+2})$ and $(x_{4m+3}, x_{12m+7}, x_{4m+4}, x_{12m+6}, \ldots, x_{11m+7}, x_{5m+4}, x_{11m+5}, x_{5m+5}, \ldots, x_{10m+7}, x_{6m+3}, x_{10m+5})$, respectively. It is not difficult to see that both $Q$ and $R$ are cycles of length $4m+2$, where $Q$ consists of edges with the following labels in order of $4m+3$, $4m+1$, $4m, \ldots$, $2m+3$, $2m+1$, $2m, \ldots$, 1, $2m+2$, and $R$, in order of $8m+4$, $8m+3, \ldots$, $6m+3$, $6m+1$, $6m, \ldots$, $4m+4$, $4m+2$, $6m+2$. Since $V(Q) \cap V(R) = \{x_{4m+3}\}$, by Lemma 3.8, $Q \cup R$ can be decomposed into two copies of $P_{k+1}$. On the other hand, since $x_{4m+3} \in [V(Q) \cap V(R) \cap V(Q+(4m+1))]$, $x_{12m+7} \notin V(Q)$, $x_{8m+3} \notin V(R)$ and $x_2 \notin V(Q+(4m+1))$, by Lemma 3.8 again, $Q \cup R \cup (Q+(4m+1))$ can be decomposed into three copies of $P_{k+1}$.

In each case mentioned above, we have that $Q \cup R$ consists of edges with the labels $1, 2, \ldots, 2k$. It implies that $K_{4k+1}$ can be decomposed into $8k+2$ copies of $C_k$ as follows: $Q$, $Q+1, \ldots$, $Q+4k$, $R$, $R+1, \ldots$, $R+4k$. Since $Q \cup R$ can be decomposed into two copies of $P_{k+1}$, $(Q+i) \cup (R+i)$ can be decomposed into two copies of $P_{k+1}$ for $i \in \{1, 2, \ldots, 4k\}$. Therefore, $K_{4k+1}$ can be decomposed into $p$ copies of $P_{k+1}$ and $8k+2-p$ copies of $C_k$ for nonnegative even integer $p$ such that $0 \leq p \leq 8k+2$. On the other hand, since $Q \cup R \cup (Q+j)$ can be decomposed into three copies of $P_{k+1}$ for some $j \in \{1, 2, \ldots, 4k\}$, we have that $Q \cup R \cup (Q+j) \cup (R+j)$ can be decomposed into three copies of $P_{k+1}$ and one copy of $C_k$. It implies that $K_{4k+1}$ can be decomposed into $p$ copies of $P_{k+1}$ and $8k+2-p$ copies of $C_k$ for positive odd integer $p$ such that $3 \leq p \leq 8k+2$. This completes the proof. □

The following theorem follows immediately from Theorem 2.2, Theorem 3.7 and Theorem 3.9.

**Theorem 3.10** *Let $p$ and $q$ be nonnegative integers and let $n$ and $k$ be* *positive integers such that $n$ is odd, $k \geq 4$ is even, and $n \geq 5k+1$. There exists a decomposition of $K_n$ into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$ if and only if $k(p+q) = e(K_n)$ and $p \neq 1$.* □

In the following theorem we will give a sufficient condition for decomposing $K_n$ into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$ for all possible values of $p \geq 0$ and $q \geq 0$ when $n$ and $k$ are even. We need the following lemma for our discussion.

**Lemma 3.11** *Let $r$ be a nonnegative integer and let $k$, $s$, and $t$ be positive integers such that $0 \leq r \leq k-1$, $4 \leq t+2 \leq s$, $k \geq 4$, and $k$, $r$, and $t$ are all even. If $K_{tk}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q) = e(K_{tk})$ and $p \geq \frac{tk}{2}$, then $K_{sk+r} - E(K_{(s-t)k+r})$ can be decomposed into $p$ copies of $P_{k+1}$ and $\frac{t}{2}(tk-1) + t(s-t)k + tr - p$ copies of $C_k$ for $\frac{tk}{2} \leq p \leq \frac{t}{2}(tk-1) + t(s-t)k + tr$.*

**Proof.** The procedure of proof is similar to the proof of Lemma 3.6. It is easily seen that $K_{sk+r} - E(K_{(s-t)k+r})$ can be viewed as an edge-disjoint union of $K_{tk}$ and $K_{tk,(s-t)k+r}$. Since $e(K_{tk} \cup K_{tk,(s-t)k+r}) = \frac{tk(tk-1)}{2} + tk[(s-t)k+r]$, we get $\frac{tk}{2} \leq p \leq \frac{t}{2}(tk-1) + t(s-t)k + tr$. Now we consider four cases below.

*Case 1.* $\frac{tk}{2} \leq p \leq \frac{t}{2}(tk-1)$.
By assumption, we can obtain $p$ copies of $P_{k+1}$ and $\frac{t}{2}(tk-1) - p$ copies of $C_k$ from $K_{tk}$. As to the remaining graph, by Theorem 3.4, $K_{tk,(s-t)k+r}$ has a $C_k$-decomposition.

*Case 2.* $\frac{t}{2}(tk-1) + 1 \leq p \leq t(s-t)k + tr$. (When $\frac{t}{2}(tk-1) \geq t(s-t)k + tr$, we skip this case.)
When $\frac{t}{2}(tk-1) + 1 \leq p \leq \frac{t}{2}(tk+1)$, since $K_{tk,(s-t)k+r}$ can be decomposed into two copies of $K_{\frac{t}{2}k,(s-t)k+r}$ and $K_{\frac{t}{2}k,(s-t)k+r}$ can be decomposed into $K_{\frac{t}{2}k,k}$ and $K_{\frac{t}{2}k,(s-t-1)k+r}$ (note that $s \geq t+2$), by Theorem 3.2 and Theorem 3.4, we can obtain $\frac{tk}{2}$ copies of $P_{k+1}$ and $\frac{t}{2}[(s-t-1)k+r]$ copies of $C_k$ from one copy of $K_{\frac{t}{2}k,(s-t)k+r}$, and obtain $\frac{t}{2}[(s-t)k+r]$ copies of $C_k$ from the other copy of $K_{\frac{t}{2}k,(s-t)k+r}$. Therefore, we can obtain $\frac{tk}{2}$ copies of $P_{k+1}$ and $t(s-t)k + tr - \frac{tk}{2}$ copies of $C_k$ from $K_{tk,(s-t)k+r}$. Since $t \geq 2$ and $k \geq 4$, we have that $\frac{tk}{2} \leq p - \frac{tk}{2} \leq \frac{t}{2}(tk-1)$. By assumption, we can obtain $p - \frac{tk}{2}$ copies of $P_{k+1}$ and $\frac{t}{2}(tk-1) - (p - \frac{tk}{2})$ copies of $C_k$ from $K_{tk}$.

When $\frac{t}{2}(tk+1) + 1 \leq p \leq t(s-t)k + tr$, by the same method in the Case 2 of Lemma 3.6's proof, it can be proved easily. The details are left to the reader.

*Case 3.* $t(s-t)k+tr+1 \le p \le t(s-t)k+tr+\frac{tk}{2}-1$ (When $\frac{t}{2}(tk-1) \ge t(s-t)k+tr+\frac{tk}{2}-1$, we skip this case.)
Since $K_{tk,(s-t)k+r}$ can be decomposed into two copies of $K_{\frac{t}{2}k,(s-t)k+r}$ and $K_{\frac{t}{2}k,(s-t)k+r}$ can be decomposed into $K_{\frac{t}{2}k,(s-t-1)k+r}$ and $K_{\frac{t}{2}k,k}$ (note that $s \ge t+2$), by Theorem 3.2 and Theorem 3.4, we can obtain $\frac{tk}{2}$ copies of $C_k$ and $\frac{t}{2}[(s-t-1)k+r]$ copies of $P_{k+1}$ from one copy of $K_{\frac{t}{2}k,(s-t)k+r}$, and obtain $\frac{t}{2}[(s-t)k+r]$ copies of $P_{k+1}$ from the other copy of $K_{\frac{t}{2}k,(s-t)k+r}$. Therefore, we can obtain $t(s-t)k+tr-\frac{tk}{2}$ copies of $P_{k+1}$ and $\frac{tk}{2}$ copies of $C_k$ from $K_{tk,(s-t)k+r}$. Since $\frac{tk}{2}+1 \le p-[t(s-t)k+tr-\frac{tk}{2}] \le \frac{tk}{2}+\frac{tk}{2}-1 = \frac{t}{2}(2k)-1 \le \frac{t}{2}(tk-1)$ (note that $t \ge 2$), by assumption, we can obtain $p-[t(s-t)k+tr-\frac{tk}{2}]$ copies of $P_{k+1}$ and $\frac{t}{2}(tk-1)-[p-[t(s-t)k+tr-\frac{tk}{2}]]$ copies of $C_k$ from $K_{tk}$.

*Case 4.* $t(s-t)k+tr+\frac{tk}{2} \le p \le \frac{t}{2}(tk-1)+t(s-t)k+tr$.
We can obtain $t(s-t)k+tr$ copies of $P_{k+1}$ from $K_{tk,(s-t)k+r}$ first. Since $\frac{tk}{2} \le p-[t(s-t)k+tr] \le \frac{t}{2}(tk-1)$, by assumption, we can obtain $p-[t(s-t)k+tr]$ copies of $P_{k+1}$ and $\frac{t}{2}(tk-1)-[p-(t(s-t)k+tr)]$ copies of $C_k$ from $K_{tk}$.
□

**Theorem 3.12** *Let $k$, $m$, $n$, and $t$ be positive integers and let $r$ be a nonnegative integer such that $k \ge 4$, $t \ge 2$, $2 \le m \le t+1$, $[(nt+m)k+r][(nt+m)k+r-1] \equiv 0 \pmod{2k}$, and $k$, $r$, and $t$ are all even. If $K_{tk}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q) = e(K_{tk})$ and $p \ge \frac{tk}{2}$; $K_{mk+r}$ can be decomposed into $\frac{mk+r}{2}$ copies of $P_{k+1}$ and $\frac{e(K_{mk+r})}{k} - \frac{mk+r}{2}$ copies of $C_k$, then $K_{(nt+m)k+r}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q) = e(K_{(nt+m)k+r})$ and $p \ge \frac{(nt+m)k+r}{2}$.*

**Proof.** The procedure of proof is similar to the proof of Theorem 3.7. The proof is by induction on $n$. Let $nt+m=s$. It is easily seen that $K_{sk+r}$ can be viewed as an edge-disjoint union of $K_{tk}$, $K_{tk,(s-t)k+r}$, and $K_{(s-t)k+r}$. Since $p+q = \frac{(sk+r)(sk+r-1)}{2k}$, we get $\frac{sk+r}{2} \le p \le \frac{[(s-t)k+r][(s-t)k+r-1]}{2k} + \frac{t}{2}(tk-1)+t(s-t)k+tr$. Note also that $2k|(sk+r)(sk+r-1)$ and $t$ is even implies $2k|[(s-t)k+r][(s-t)k+r-1]$, and so $\frac{[(s-t)k+r][(s-t)k+r-1]}{2k}$ is an integer.

When $n=1$, we consider two cases below. When $\frac{sk+r}{2} \le p \le \frac{t}{2}(tk-1)+t(s-t)k+tr+\frac{(s-t)k+r}{2}$, since $s-t=m$, by assumption, the graph $K_{(s-t)k+r}$ can be decomposed into $\frac{(s-t)k+r}{2}$ copies of $P_{k+1}$ and $\frac{e(K_{(s-t)k+r})}{k} - \frac{(s-t)k+r}{2}$ copies of $C_k$. On the other hand, since $\frac{tk}{2} \le p-\frac{(s-t)k+r}{2} \le \frac{t}{2}(tk-1)+t(s-t)k+tr$, by Lemma 3.11, we can obtain $p-\frac{(s-t)k+r}{2}$ copies of $P_{k+1}$ and

$\frac{t}{2}(tk-1)+t(s-t)k+tr-(p-\frac{(s-t)k+r}{2})$ copies of $C_k$ from an edge-disjoint union of $K_{tk}$ and $K_{tk,(s-t)k+r}$.

When $\frac{t}{2}(tk-1)+t(s-t)k+tr+\frac{(s-t)k+r}{2}+1 \leq p \leq \frac{[(s-t)k+r][(s-t)k+r-1]}{2k}+\frac{t}{2}(tk-1)+t(s-t)k+tr$, by Theorem 3.1, we can obtain $\frac{[(s-t)k+r][(s-t)k+r-1]}{2k}$ copies of $P_{k+1}$ from $K_{(s-t)k+r}$ first. On the other hand, since $\frac{t}{2}(tk-1)+t(s-t)k+tr+\frac{(s-t)k+r}{2}-\frac{[(s-t)k+r][(s-t)k+r-1]}{2k} \geq \frac{tk}{2}$ (note that $s-t=m$), we have $\frac{tk}{2}+1 \leq p-\frac{[(s-t)k+r][(s-t)k+r-1]}{2k} \leq \frac{t}{2}(tk-1)+t(s-t)k+tr$. By Lemma 3.11 again, we can obtain $p-\frac{[(s-t)k+r][(s-t)k+r-1]}{2k}$ copies of $P_{k+1}$ and $\frac{t}{2}(tk-1)+t(s-t)k+tr-[p-\frac{[(s-t)k+r][(s-t)k+r-1]}{2k}]$ copies of $C_k$ from an edge-disjoint union of $K_{tk}$ and $K_{tk,(s-t)k+r}$.

Now we suppose that $n \geq 2$. Since $s-t=(n-1)t+m$, by induction hypothesis, $K_{(s-t)k+r}$ can be decomposed into $u$ copies of $P_{k+1}$ and $\frac{[(s-t)k+r][(s-t)k+r-1]}{2k}-u$ copies of $C_k$ for $\frac{(s-t)k+r}{2} \leq u \leq \frac{[(s-t)k+r][(s-t)k+r-1]}{2k}$. On the other hand, by Lemma 3.11, an edge-disjoint union of $K_{tk}$, $K_{tk,(s-t)k+r}$ can be decomposed into $v$ copies of $P_{k+1}$, with the remaining edges decomposed into copies of $C_k$ for $\frac{tk}{2} \leq v \leq \frac{t}{2}(tk-1)+t(s-t)k+tr$. Therefore, $K_{sk+r}$ can be decomposed into $p$ copies of $P_{k+1}$, with the remaining edges decomposed into copies of $C_k$ for $\frac{sk+r}{2} \leq p \leq \frac{[(s-t)k+r][(s-t)k+r-1]}{2k}+\frac{t}{2}(tk-1)+t(s-t)k+tr$. This completes the proof. □

The following corollary follows immediately from Theorem 3.12.

**Corollary 3.13** *Let $k$, $m$, and $t$ be positive integers such that $k \geq 4$, $t \geq 2$, and both $k$ and $t$ are even. If $K_{tk}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q)=e(K_{tk})$ and $p \geq \frac{tk}{2}$, then $K_{mtk}$ can be decomposed into $p$ copies of $P_{k+1}$ and $q$ copies of $C_k$, for each pair $p$, $q$ of nonnegative integers such that $k(p+q)=e(K_{mtk})$ and $p \geq \frac{mtk}{2}$.* □

Next theorem we will obtain a necessary and sufficient condition for decomposing $K_n$ into $p$ copies of $P_5$ and $q$ copies of $C_4$ for all possible values of $p \geq 0$ and $q \geq 0$.

**Theorem 3.14** *Let $p$ and $q$ be nonnegative integers and let $n$ be a positive integer. There exists a decomposition of $K_n$ into $p$ copies of $P_5$ and $q$ copies of $C_4$ if and only if $4(p+q)=\binom{n}{2}$, $p \neq 1$ if $n$ is odd, and $p \geq \frac{n}{2}$ if $n$ is even.*

**Proof.** (Necessity) Condition $4(p+q)=\binom{n}{2}$ is trivial. By Theorem 2.2 and Theorem 2.4, we have that $p \neq 1$ if $n$ is odd and $p \geq \frac{n}{2}$ if $n$ is even.

(Sufficiency) Observe that $4|\frac{n(n-1)}{2}$ implies either $8|n$ or $8|(n-1)$. It follows that $n$ will be either $8m$ or $8m+1$, where $m$ is a positive integer. By Lemma 2.1 and Theorem 3.7, $K_{8m+1}$ can be decomposed into $p$ copies of $P_5$ and $q$ copies of $C_4$, for each pair $p$, $q$ of nonnegative integers such that $4(p+q)=\binom{8m+1}{2}$ and $p\neq 1$. On the other hand, by Lemma 2.3 and Corollary 3.13, $K_{8m}$ can be decomposed into $p$ copies of $P_5$ and $q$ copies of $C_4$, for each pair $p$, $q$ of nonnegative integers such that $4(p+q)=\binom{8m}{2}$ and $p\geq 4m$. This completes the proof. □

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