# DOMINATION IN THE CROSS PRODUCT OF DIGRAPHS

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ABSTRACT. In many papers, the relation between the domination number of a product of graphs and the product of domination numbers of factors is studied. Here we investigate this problem for domination and total domination numbers in the cross product of digraphs. We give analogues of known results for graphs, and we also present new results for digraphs with sources. Using these results we find domination (total domination) numbers for some classes of digraphs.

## 1. INTRODUCTION

Domination of the products of graphs attracted attention of graph theorists for more than forty years starting with the well known Vizing's conjecture [13]. This conjecture claims that for any two graphs, the product of domination numbers of these graphs is not greater than the domination number of cartesian product of these graphs. Many generalizations of the original problem and related problems were introduced [4], [5], [11]. In recent years the domination and total domination of cross (or direct) product of graphs have been studied for example in [2], [3], [12]. It was shown that

1991 *Mathematics Subject Classification.* 05C69, 05C20, 05C70, 05C76.

*Key words and phrases.* domination number, packing number, total domination number, directed graph, cross product.

[a] Supported by Kuwait University, Research Grant No. SM 01/08

[b] Supported by Slovak research grants APVV-0040-06, APVV-0104-07 and VEGA 1/0489/08.

for cross product the analogue of Vizing's conjecture is not valid [9]. Upper and lower bounds for domination number of the cross product of graphs in terms of domination numbers and packing numbers of factors can be found in [1]. Upper and lower bounds for total domination number of the cross product of graphs were found independently in [2], [3], [14]. In this note we turn our attention to the problem of domination of the cross product of digraphs. Domination and total domination numbers are generalized to digraphs in a natural way [4], [5], [6]. It was proved that Vizing's conjecture is not valid for the cartesian product of directed graphs [10]. In this paper we present lower and upper bounds for domination numbers of cross product of digraphs. These bounds are similar to those for graphs, mentioned in the above papers. We show that these bounds are attained by infinite classes of digraphs, and in all cases we choose our classes so that the digraphs are not symmetric.

## 2. Preliminaries

We consider only simple digraphs, i.e., having neither loops nor multiple arcs. Let $G = (V, E)$ be a digraph. By $V(G)$ and $E(G)$ we denote its vertex set and its arc set, respectively. We say that vertex $u$ *dominates* vertex $v$ if $uv \in E(G)$. The open in-neighborhood of a vertex $v$ is the set $N_o^+(v)$ of all vertices that are dominating vertex $v$. (In other words, $N_o^+(v)$ is the set of all $u$'s such that $uv \in E(G)$.) Similarly, the open out-neighborhood of a vertex $v$ is the set $N_o^-(v)$ of all vertices that are dominated by vertex $v$. The size of $N_o^+(v)$ ($N_o^-(v)$) is denoted by $deg^+(v)$ ($deg^-(v)$) and it is called the *in-degree* (*out-degree*) of $v$. The minimum $deg^+(v)$, taken through all $v \in V(G)$, is the minimum in-degree, $\delta^+(G)$, of $G$.

Closed neighborhoods of vertex $v \in V(G)$ are $N^+(v) = N_o^+(v) \cup \{v\}$ and $N^-(v) = N_o^-(v) \cup \{v\}$. A set $D \subseteq V(G)$ is a *dominating set* of $G$ if $\bigcup_{v \in D} N^-(v) = V(G)$, and $T \subseteq V(G)$ is a *total dominating set* of $G$ if $\bigcup_{v \in T} N_o^-(v) = V(G)$. The *domination number* of $G$, $\gamma(G)$, is the minimum cardinality of a dominating set of $G$, while its *total domination number*, $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. Note that total dominating sets exist in digraphs without sources (vertices of in-degree 0) only.

A set $P \subseteq V(G)$ is called *in-packing* if $N^+(v) \cap N^+(u) = \emptyset$ for any two distinct vertices $u, v \in P$. A set $P' \subseteq V(G)$ is called *open in-packing* if $N_o^+(v) \cap N_o^+(u) = \emptyset$ for any two distinct vertices $u, v \in P'$. The in-packing number $\rho^+(G)$ is the maximum cardinality of an in-packing set of $G$, while the open in-packing number $\rho_o^+(G)$ is the maximum cardinality of an open in-packing set of $G$.

For digraphs $G$ and $H$, their *cross product* $G \times H$ is a digraph with vertex set $V(G \times H) = V(G) \times V(H)$ and with $(u, v)(u', v') \in E(G \times H)$

if and only if $uu' \in E(G)$ and $vv' \in E(H)$. For each vertex $v \in V(H)$ we denote $G_v = \{(u,v);\, u \in V(G)\}$. We call $G_v$ the column of $G \times H$ corresponding to vertex $v$. Similarly the rows of $G \times H$ are defined by $H_u = \{(u,v) : v \in V(H)\}$, where $u \in V(G)$.

For the notions and notation not mentioned here, see [4].

## 3. Lower and upper bounds for $\gamma_t(G \times H)$ and $\gamma(G \times H)$.

**Theorem 1.** *For any two digraphs $G$ and $H$ with $\delta^+(G) \geq 1$ and $\delta^+(H) \geq 1$, we have $\gamma_t(G \times H) \leq \gamma_t(G)\gamma_t(H)$.*

*Proof.* Since $\delta^+(G) \geq 1$ and $\delta^+(H) \geq 1$, there exist total dominating sets in both $G$ and $H$. Let $D_1$ and $D_2$ be minimum total dominating sets in $G$ and $H$, respectively, and let $D = D_1 \times D_2$. Then $D \subseteq V(G \times H)$. Let $(u,v)$ be an arbitrary vertex from $V(G \times H)$. Then there are vertices $u' \in D_1$ and $v' \in D_2$ such that $u'$ dominates $u$ and $v'$ dominates $v$. Therefore $(u',v')(u,v) \in E(G \times H)$, so that $D$ is a total dominating set in $G \times H$. Consequently, $\gamma_t(G \times H) \leq |D| = \gamma_t(G)\gamma_t(H)$. □

Let $G$ be a digraph with $\delta^+(G) \geq 1$. Suppose that $D$ is a total dominating set in $G$, and $S$ is an open in-packing in $G$. Let $v_i \in S$. Then $v_i$ is dominated by at least one vertex $u_i \in D$, where $u_i \neq v_i$ and $u_i \in N_o^+(v_i)$. For any two vertices $u_i, u_j \in S$, we have $N_o^+(u_i) \cap N_o^+(u_j) = \emptyset$, so that $u_i$ and $u_j$ are dominated by different vertices from $D$. This means that $\rho_o^+(G) \leq \gamma_t(G)$ if $\delta^+(G) \geq 1$.

**Theorem 2.** *For any two digraphs $G$ and $H$ with $\delta^+(G) \geq 1$ and $\delta^+(H) \geq 1$, we have $\gamma_t(G \times H) \geq \rho_o^+(G)\gamma_t(H)$.*

*Proof.* Since $\delta^+(G) \geq 1$ and $\delta^+(H) \geq 1$, we have $\delta^+(G \times H) \geq 1$. Let $D$ be a total dominating set in $G \times H$. Suppose that $S = \{s_1, s_2, \ldots s_k\}$ is an open in-packing in $G$. Denote $D_i = D \cap (N_o^+(s_i) \times V(H))$, $1 \leq i \leq k$. Each vertex in the row $H_{s_i}$ is dominated by a vertex of $D_i$. Let $p_i$ be a projection of the set $D_i$ to $V(H)$ defined by $p_i(u,v) = v$ for all $(u,v) \in D_i$. The set $p_i(D_i)$ is a total dominating set in $H$, so that $|D_i| \geq \gamma_t(H)$. As sets $D_i$ are disjoint, $|D| \geq \sum_i |D_i| \geq \rho_o^+(G)\gamma_t(H)$. □

Observe that Theorems 1 and 2 are best possible for digraphs $G$ with $\delta^+(G) \geq 1$ and $\gamma_t(G) = \rho_o^+(G)$. Simplest examples of such digraphs are directed cycles.

Let $\Delta^-(G)$ be the maximum out-degree in $G$. Modifying the proof of Theorem 2 slightly we are able to prove the following statement.

**Theorem 3.** *For any two digraphs $G$ and $H$ with $\delta^+(G) \geq 1$ and $\delta^+(H) \geq 1$, we have $\gamma_t(G \times H) \geq |V(G)|\gamma_t(H)/(\Delta^-(G))$.*

*Proof.* Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$, and denote $D_i = D \cap (N_o^+(v_i) \times V(H))$, $1 \leq i \leq n$, where $D$ is a total dominating set of $G \times H$.

Define $p_i$ analogously as in the proof of Theorem 2. Then $p_i(D_i)$ is a total dominating set in $H$, so that $|D_i| \geq \gamma_t(H)$. Now summing for all vertices of $V(G)$ we get

$$\sum_{v_i \in V(G)} |D_i| \geq |V(G)|\gamma_t(G).$$

Since every vertex of $D$ is counted in the sum exactly $deg^-(v)$ times, we get $D \geq |V(G)|\gamma_t(H)/(\Delta^-(G))$. □

Analogously as Theorems 1 and 2, Theorem 3 is best possible for directed cycles.

A digraph is $d$-regular if the in-degree and out-degree of every vertex is $d$. Obviously, a $d$-regular digraph has $\rho_o^+(G) \leq |V(G)|/d$. Hence, Theorem 3 is better than Theorem 2 for $d$-regular digraphs $G$ such that $d \nmid |V(G)|$. On the other hand, denote by $W_n$ a digraph obtained from a wheel on $n+1$ vertices in which we direct all the spokes into the center and in which the "non-spoke" arcs form a directed cycle. Then $\rho_o^+(W_n) = n$ while $\Delta^-(W_n) = 2$, so that $|V(W_n)|/\Delta^-(W_n) = \frac{n+1}{2}$. Thus, if we choose $W_n$ for $G$, then Theorem 2 is better than Theorem 3.

Now we find upper and lower bounds for $\gamma(G \times H)$. Since $\cup_{v \in V(G)} N^-(v) \supseteq \cup_{v \in V(G)}\{v\} = V(G)$, each digraph has a dominating set. Thus, $\gamma(G \times H)$ is defined for all pairs of digraphs $G$ and $H$. Let $G$ be a digraph and let $V_0(G)$ be the set of all sources in $G$. Obviously, every dominating set $D$ of $G$ contains $V_0(G)$, since each vertex $v \in V_0(G)$ is not dominated by any vertex of $G$. In what follows, we shall consider two cases, namely $V_0(G) = \emptyset$ and $V_0(G) \neq \emptyset$.

**Theorem 4.** *For any two digraphs $G$ and $H$ with $\delta^+(G) \geq 1$ and $\delta^+(H) \geq 1$, we have $\gamma(G \times H) \leq 3\gamma(G)\gamma(H)$.*

*Proof.* Let $D_1$ and $D_2$ be minimum dominating sets in $G$ and $H$, respectively. We construct sets $D_1^+$ and $D_2^+$, such that $D_1^+ \subseteq V(G)$ and $D_2^+ \subseteq V(H)$. For each vertex $v \in D_1$ we add to $D_1^+$ one vertex from $N_o^+(v)$. Then $|D_1^+| \leq |D_1|$ and all vertices from $D_1$ are dominated by $D_1^+$. Let $D_2^+$ be defined analogously. We show that $D = (D_1 \times D_2) \cup (D_1^+ \times D_2) \cup (D_1 \times D_2^+)$ is a dominating set in $G \times H$. Let $(u, v) \in V(G \times H)$. There are 4 cases to consider:

1. $u \in D_1$ and $v \in D_2$. Then $(u, v) \in D$.
2. $u \in D_1$ and $v \in V(H) - D_2$. Then $(u, v)$ is dominated by a vertex of $D_1^+ \times D_2$.
3. $u \in V(G) - D_1$ and $v \in D_2$. Then $(u, v)$ is dominated by a vertex of $D_1 \times D_2^+$.
4. $u \in V(G) - D_1$ and $v \in V(H) - D_2$. Then $(u, v)$ is dominated by a vertex of $D_1 \times D_2$.

Since $|D| \le |D_1 \times D_2| + |D_1^+ \times D_2| + |D_1 \times D_2^+| \le 3\gamma(G)\gamma(H)$, the result follows. $\square$

Let $S_k$ be a digraph consisting of a directed cycle $(v_0, v_1, \dots v_{2k-1})$ in which at every vertex of even index $v_{2i}$ there are attached at least 2 pending arcs $v_{2i}u_{2i,1}$ and $v_{2i}u_{2i,2}$, $0 \le i \le k$. I.e., the in-degree of $u_{2i,j}$ is 1 and its out-degree is 0 for $j = 1, 2$. The unique minimum dominating set in $S_k$ is $\{v_0, v_2, \dots, v_{2k-2}\}$, so that $\gamma(S_k) = k$. Now denote by $S_l'$ a copy of $S_l$ with vertices $v_i'$ and $u_{i,j}'$, and consider a minimum dominating set $D$ in $S_k \times S_l'$. Since both $(u_{2i,1}, v_j')$ and $(u_{2i,2}, v_j')$ have in-degree 1 and out-degree 0 and both of them are dominated by $(v_{2i}, v_{j-1}')$, the vertex $(v_{2i}, v_{j-1}')$ must be in $D$. Hence, $D_1 = \{(v_{2i}, v_{j-1}');\ 0 \le i < k,\ 0 \le j < 2l\}$ is a subset of $D$. (The arithmetics at indices of $v$ is considered in $\mathbb{Z}_{2k}$, while the arithmetics at indices of $v'$ is considered in $\mathbb{Z}_{2l}$.) Analogously, $D_2 = \{(v_{i-1}, v_{2j}'); 0\ \le i < 2k,\ 0 \le j < l\}$ is a subset of $D$. As $D_1 \cup D_2 = \{(v_{2i}, v_{2j}'), (v_{2i}, v_{2j-1}'), (v_{2i-1}, v_{2j}'); 0 \le i < k,\ 0 \le j < l\}$, the set $D_1 \cup D_2$ is a dominating set in $S_k \times S_l'$, by the proof of Theorem 4. Hence, $D = D_1 \cup D_2$. Since $|D| = 3kl$, we have $\gamma(S_k \times S_l') = 3\gamma(S_k)\gamma(S_l')$. Hence, Theorem 4 is tight in this case.

Now consider digraphs with sources. If $u \in V_0(G)$, then for every $v \in V(H)$ the vertex $(u, v)$ is included in any dominating set of $G \times H$. Similarly if $v \in V_0(H)$, then for every $u \in V(G)$ the vertex $(u, v)$ is included in any dominating set of $G \times H$. Denote by $g_0$ and $h_0$ the cardinality of $V_0(G)$ and $V_0(H)$, respectively. Then the number of sources in $G \times H$ is $h_0|V(G)| + g_0|V(H)| - g_0h_0$.

**Theorem 5.** *Let $G$ and $H$ be digraphs, $g_0 = |V_0(G)|$ and $h_0 = |V_0(H)|$. Then $\gamma(G \times H) \le h_0|V(G)| + g_0|V(H)| - g_0h_0 + 3(\gamma(G) - g_0)(\gamma(H) - h_0)$.*

*Proof.* Let $D_1$ and $D_2$ be minimum dominating sets in $G$ and $H$, respectively. We construct a dominating set $D$ in $G \times H$ in the following way. Let $D = D_s \cup D_o$, where

$$\begin{aligned} D_s =& (V(G) \times V_0(H)) \cup (V_0(G) \times V(H)) \\ D_o =& ((D_1 - V_0(G)) \times (D_2 - V_0(H))) \\ & \cup ((D_1 - V_0(G))^+ \times (D_2 - V_0(H))) \\ & \cup ((D_1 - V_0(G)) \times (D_2 - V_0(H))^+). \end{aligned}$$

The sets $(D_1 - V_0(G))^+$ and $(D_2 - V_0(H))^+$ are defined in the same way as in the proof of Theorem 4. The sources of $G \times H$ are exactly the vertices of $D_s$. Now analogously as in the proof of Theorem 4 one can show that all other vertices of $G \times H$ are included in or dominated by $D_o$, and the result follows. $\square$

Denote by $K_n$ a complete digraph on $n$ vertices. Then $uv \in E(K_n)$ for every $u, v \in V(K_n)$, $u \neq v$. Although this digraph is symmetric, we determine the domination number of $K_{m\times n} = K_m \times K_n$, where $m, n \geq 3$. Suppose that there is a dominating set $D = \{(u_1, v_1), (u_2, v_2)\}$ in $K_{m\times n}$. Since $n \geq 3$, there is a vertex $v$ in $K_n$ such that $v \notin \{v_1, v_2\}$. Therefore, if $u_1 = u_2$ then $D$ does not dominate $(u_1, v)$. Consequently $u_1 \neq u_2$, and analogously one can show that $v_1 \neq v_2$. But then $D$ does not dominate $(u_1, v_2)$, a contradiction. This means that $\gamma(K_{m\times n}) \geq 3$. Since $\gamma(K_m) = \gamma(K_n) = 1$, we have $\gamma(K_{m\times n}) \leq 3$ by Theorem 4, so that $\gamma(K_{m\times n}) = 3$.

Denote by $Y_n$ a digraph consisting of complete digraph on $n$ vertices and two extra vertices $a$ and $b$. The arc set of $Y_n$ contains all the arcs of complete digraph $K_n$, together with arc $ab$ and all arcs $ub$, where $u \in V(K_n)$. Let $Y_{m\times n} = Y_m \times Y_n$, where $m, n \geq 3$. Then $Y_{m\times n}$ contains $K_{m\times n}$, and no vertex of $K_{m\times n}$ in $Y_{m\times n}$ is dominated by a vertex of $V_0(Y_{m\times n})$. Thus, $\gamma(Y_{m\times n}) \geq |V_0(Y_{m\times n})| + \gamma(K_{m\times n})$ as shown before Theorem 5. Since $\gamma(K_{m\times n}) = 3$ and $\gamma(Y_m) = |V_0(Y_m)| + 1 = 2 = \gamma(Y_n)$, Theorem 5 is tight in this case.

Now we make an observation, similar to that stated before Theorem 2. Let $G$ be a digraph. Suppose that $D$ is a dominating set in $G$ and $S$ is an in-packing in $G$. Then distinct vertices of $S$ are dominated by distinct vertices of $D$, so that $\rho^+(G) \leq \gamma(G)$.

**Theorem 6.** *For any two digraphs $G$ and $H$, with $\delta^+(H) \geq 1$, we have $\gamma(G \times H) \geq \rho^+(G)\gamma_t(H)$.*

*Proof.* Let $S = \{s_1, s_2, \ldots, s_k\}$ be an in-packing with cardinality $\rho^+(G)$, and let $D$ be a minimum dominating set in $G \times H$. Denote by $H_i$ the subgraph of $G \times H$ generated by vertices $N^+(s_i) \times V(H)$, and denote by $D_i$ the intersection $D \cap V(H_i)$. We distinguish two cases:

(i) $deg^+(s_i) = 0$. Then there is no vertex $u \in V(G)$ such that $us_i \in E(G)$. Thus, all vertices $(s_i, v)$ of the row $H_{s_i} = H_i$ are in $D$, so that $D_i = V(H_i)$. Since $\delta^+(H) \geq 1$, there is a total dominating set in $H$. As $|V(H)| \geq \gamma_t(H)$, we have $|D_i| \geq \gamma_t(H)$.

(ii) $deg^+(s_i) \geq 1$. Each vertex $(s_i, v)$ of the row $H_{s_i}$ is either dominated by a vertex of $D_i$ or it is included in $D_i$. In the later case we replace the vertex $(s_i, v)$ in $D_i$ by a vertex $(s_i', v')$ that dominates $(s_i, v)$. Resulting set $D_i'$ dominates all vertices in the row $H_{s_i}$, so that the projection of $D_i'$ on $H$ is a total dominating set. Thus, $|D_i| \geq |D_i'| \geq \gamma_t(H)$.

Since the subgraphs $H_i$ are pairwise disjoint and in every case $|D_i| \geq \gamma_t(H)$, the result follows. □

Denote by $C_{2n}$ a directed cycle of even length $2n$. Let $C_{2m\times 2n} = C_{2m} \times C_{2n}$. Assume that $C_{2m} = (u_0, u_1, \ldots, u_{2m-1})$ and $C_{2n} = (v_0, v_1, \ldots, v_{2n-1})$.

As $deg^-(u_i) = deg^-(v_j) = 1$, we have $deg^-((u_i, v_j)) = 1$, where $0 \leq i \leq 2m-1$ and $0 \leq j \leq 2n-1$. Analogously $deg^+((u_i, v_j)) = 1$, so that $C_{2m\times 2n}$ is a collection of vertex-disjoint cycles. Denote by $t$ the least common multiple of $2m$ and $2n$. The cycle starting at $(u_0, v_0)$ is $((u_0, v_0), (u_1, v_1), \ldots, (u_{2m-1}, v_{2n-1}))$ and its length is $t$. Since both $C_{2m}$ and $C_{2n}$ are vertex-transitive, so is $C_{2m\times 2n}$. Hence, $C_{2m\times 2n}$ is a collection of $(2m \cdot 2n)/t$ cycles, all of even length $t$. Thus, $\gamma(C_{2m\times 2n}) = 2mn$. As $\rho^+(C_{2m}) = m$ and $\gamma_t(C_{2n}) = 2n$, we have $\gamma(C_{2m\times 2n}) = \rho^+(C_{2m})\gamma_t(C_{2n})$, so that Theorem 6 is tight in this case.

For digraphs containing sources we have a straightforward consequence of Theorem 6.

**Theorem 7.** *For any two digraphs $G$ and $H$, we have $\gamma(G \times H) \geq \rho^+(G)\gamma(H)$.*

*Proof.* The proof is similar to the case (ii) of previous one, just the dominating sets $D_i$ are not replaced by $D_i'$, and consequently, the projected dominating sets need not to be total. So we have $|D_i| \geq \gamma(H)$ and the inequality follows. □

At the moment we are not familiar with digraphs $G$ and $H$ (other than trivial ones) for which Theorem 7 is tight. We remark that Theorems 5 and 7 have no analogues for graphs.

## 4. Domination in $(C_n \times H)$.

We now consider a special case of cross products, when one of the factors is a directed cycle.

**Theorem 8.** *If $G$ is a digraph without sources then $\gamma(G \times H) \leq |V(G)|\gamma(H)$.*

*Proof.* Let $D$ be a minimum dominating set of $H$. Denote by $D^\times$ the set of all vertices $(u, v)$ of $G \times H$, such that $u \in V(G)$ and $v \in D$. We show that $D^\times$ is a dominating set of $G \times H$.

Let $(x, y) \in V(G \times H)$. If $y \in D$ then $(x, y) \in D^\times$. Therefore suppose that $y \notin D$. Then there is $v \in D$ such that $v$ dominates $y$ in $H$. Further, since $G$ contains no sources there is $u \in V(G)$ such that $u \in N_o^+(x)$. But then $(u, v) \in D^\times$ and $(u, v)$ dominates $(x, y)$. □

Since for directed cycles $C_n$ it holds $\gamma_t(C_n) = |V(C_n)|$, by Theorem 8 we have

**Corollary 9.** *Let $C_n$ be a directed cycle with $n$ vertices and let $H$ be an arbitrary digraph. Then $\gamma(C_n \times H) \leq \gamma_t(C_n)\gamma(H) = n\gamma(H)$.*

From Theorem 6 and Corollary 9 it follows that if $\rho^+(H) = \gamma(H)$ then $\gamma(C_n \times H) = n\gamma(H)$. Examples of digraphs with this property are for instance digraphs with efficient dominating sets. These digraphs can be found for example in [7],[8] and [10]. Another class of digraphs with $\rho^+(H) = \gamma(H)$ is formed by directed rooted trees.

**Lemma 10.** *For a directed rooted tree R we have $\rho^+(R) = \gamma(R)$.*

*Proof.* As $\rho^+(H) \leq \gamma(H)$ for any digraph, it is enough to construct an in-packing of R with $\gamma(R)$ vertices. Let $D$ be a minimum dominating set of $R$. It is clear that for the root $r$ we have $r \in D$, since $r$ is a source. Now we modify $D$ as follows. Take a vertex $v \in D$, $v \neq r$, such that $v$ has no children in $V - D$. Denote by $v'$ the parent of $v$. Then $(D - \{v\}) \cup \{v'\}$ is again a dominating set, so replace $D$ by $(D - \{v\}) \cup \{v'\}$. Proceed in this procedure until every vertex $v \in D$, $v \neq r$, has a child in $V - D$. (Observe that this procedure is finite as in every step we decrease $\sum_{v \in D} dist(r, v)$.) Denote by $D'$ the resulting minimum dominating set in $R$. Now for every vertex $v \in D'$, $v \neq r$, denote by $u_v$ one child of $v$, such that $u_v \notin D'$. Define $P = \{r\} \cup \{u_v;\ v \in D' - \{r\}\}$. Then $N^+(r) = \{r\}$ and for every other vertex $u_v \in P$, the set $N^+(u_v)$ consists of $u_v$ and its parent $v$, where $u_v \notin D'$ and $v \in D'$. Hence, $P$ is an in-packing and $|P| = |D'| = |D|$. □

As a direct consequence we have

**Theorem 11.** *Let R be a directed rooted tree and let $C_n$ be a directed cycle of length n. Then $\gamma(R \times C_n) = n\gamma(R)$.*

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