THE DYNAMICS OF THE DIFFERENCE EQUATION $x_{n+1} = \frac{\alpha x_{n-k}}{\beta+\gamma x^{p}_{n-(k+1)}}$

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Abstract. In this paper, we investigate the global behavior of the diœerence equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x^{p}_{n-(k+1)}}, \quad n = 0, 1, ...$$

with non-negative parameters and non-negative initial conditions where k is an odd number .

1. INTRODUCTION

Consider the $(k+2)$ order diœerence equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x^{p}_{n-(k+1)}}, \; n = 0, 1, ..., \tag{1.1}$$

where the parameters α, β, γ and ρ are non-negative real numbers, k is an odd number and the initial conditions $x_0, x_{-1}, x_{-2}, ..., x_{-(k+1)}$ are non-negative real numbers such that

$$\beta + \gamma x^{p}_{n-(k+1)} > 0, \; n = 0, 1, ...,$$

We investigate the global asymptotic behaviour and the periodic character of the solutions of the diœerence Eq.(1.1), by generalizing the results due to El-Owaidy et al.[2] corresponding to the diœerence equation

$$y_{n+1} = \frac{r y_{n-1}}{1 + y^{p}_{n-2}}, \; n = 0, 1, ...,$$

where $r \geq 0$ and the initial conditions y_0, y_{-1}, y_{-2} are arbitrary non-negative real numbers. Similar recursive sequences are studied. See for example $[1-7]$. We need the following deønitions:

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Deønition 1. Let I be an interval of real numbers and let $f : I^{k+2} \to I$ be a continuously diœerentiable function. Consider the diœerence equation

(1.2) $$x_{n+1} = f\left(x_n, x_{n-1}, ..., x_{n-k}, x_{n-(k+1)}\right), \quad n = 0, 1, ...,$$

with $x_{-(k+1)}, x_{-k,...,} x_0 \in I$. Let $\overline{x}$ be the equilibrium point of Eq.(1.2). The linearized equation of Eq.(1.2) about the equilibrium point $\overline{x}$ is

(1.3) $$y_{n+1} = c_1 y_n + c_2 y_{n-1} + \dots + c_{(k+1)} y_{n-k} + c_{(k+2)} y_{n-(k+1)}, \quad n = 0, 1, ...$$

Where

$$c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, ..., \overline{x}), c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, ..., \overline{x}), ..., c_{(k+2)} = \frac{\partial f}{\partial x_{n-(k+1)}}(\overline{x}, ..., \overline{x})$$

The characteristic equation of Eq.(1.3) is

(1.4) $$\lambda^{(k+2)} - c_1 \lambda^{(k+1)} - c_2 \lambda^k - ... - c_{(k+1)} \lambda - c_{(k+2)} = 0$$

Deønition 2. A positive semicycle of a solution $\{y_n\}_{n=-(k+1)}^{\infty}$of Eq.(1.2) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to equilibrium $\overline{x}$ with $l \geq -(k+1)$ and $m \leq \infty$ such that either $l = -(k+1)$ or $l > -(k+1)$ and $x_{l-1} < \overline{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} < \overline{x}$.

A negative semicycle of a solution $\{y_n\}_{n=-(k+1)}^{\infty}$of Eq.(1.2) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$ all less than $\overline{x}$ with $l \geq -(k+1)$ and $m \leq \infty$ such that either $l = -(k+1)$ or $l > -(k+1)$ and $x_{l-1} \geq \overline{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} \geq \overline{x}$.

Deønition 3. A solution $\{y_n\}_{n=-(k+1)}^{\infty}$ of Eq.(1.2) is called nonoscillatory if there exists $N \geq -(k+1)$ such that either

$$x_n > \overline{x} \text{ for } \forall n \geq N \quad \text{or} \quad x_n < \overline{x} \text{ for } \forall n \geq N,$$

and it is called oscillatory if it is not nonoscillatory.

We need the following theorem.

Theorem 1. (i) If all roots of Eq.(1.4) have absolute values less than one, then the equilibrium point $\overline{x}$of Eq.(1.2) is locally asymptotically stable.

(ii)If at least one of the roots of Eq.(1.4) has absolute value greater than one, then the equilibrium point $\overline{x}$of Eq.(1.2) is unstable.

2. THE SPECIAL CASE $\alpha\beta\gamma p = 0$

In this section, we examine the character of solutions of Eq.(1.1) when one or more of the parameters of Eq.(1.1) are zero. There are øve such equations for $k = 1, 3, ...$ and $n = 0, 1, ...,$ namely

$\alpha = 0:$

(2.1) $$x_{n+1} = 0$$

$\beta = 0$:

(2.2) $$x_{n+1} = \frac{\alpha x_{n-k}}{\gamma x^p_{n-(k+1)}}$$

$p = 0$:

(2.3) $$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma}$$

$\gamma = 0$:

(2.4) $$x_{n+1} = \frac{\alpha x_{n-k}}{\beta}$$

$\beta = p = 0$:

(2.5) $$x_{n+1} = \frac{\alpha x_{n-k}}{\gamma}$$

In each of the above øve equations, we assume that all parameters in the equations are positive. Eq.(2.1) is trivial and Eqs. (2.3)-(2.5) are linear. Eq.(2.2) can be also reduced to a linear diœerence equation by the chance of variables $x_n = e^{y_n}$.

3. DYNAMICS OF EQ.(1.1)

In this section, we investigate the dynamics of Eq.(1.1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = (\frac{\beta}{\gamma})^{\frac{1}{p}} y_n$ reduces Eq.(1.1) to the diœerence equation

(3.1) $$y_{n+1} = \frac{r y_{n-k}}{1 + y^p_{n-(k+1)}} \quad \text{for } k = 1, 3, \ldots \text{ and } n = 0, 1, \ldots,$$

where $r = \frac{\alpha}{\beta} > 0$. Note that $\overline{y_1} = 0$ is always an equilibrium point of Eq.(3.1). When $r > 1$, Eq.(3.1) also possesses the unique positive equilibrium $\overline{y_2} = (r-1)^{\frac{1}{p}}$.

Theorem 2. The following statements are true:

(i) If $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(3.1) is locally asymptotically stable,

(ii) If $r > 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(3.1) is unstable,

(iii) If $r > 1$, then the positive equilibrium point $\overline{y_2} = (r-1)^{\frac{1}{p}}$ of Eq.(3.1) is unstable.

Proof. The linearized equation of Eq.(3.1) about the equilibrium point $\overline{y_1} = 0$ is

$$z_{n+1} = rz_{n-k} \text{ for } k = 1, 3, ... \text{ and } n = 0, 1, ...,$$

so the characteristic equation of Eq.(3.1) about the equilibrium point $\overline{y_1} = 0$ is

$$\lambda^{k+2} - r\lambda = 0$$

and hence the proof of (i) and (ii) follows from Theorem 1.

For (iii), we assume that $r > 1$, then the linearized equation of Eq.(3.1) about the equilibrium point $\overline{y_2} = (r-1)^{\frac{1}{p}}$ has the form

$$z_{n+1} = z_{n-k} - \frac{p(r-1)}{r} z_{n-(k+1)} \text{ for } k = 1, 3, ... \text{ and } n = 0, 1, ...,$$

so the characteristic equation of Eq.(3.1) about the equilibrium point $\overline{y_2} = (r-1)^{\frac{1}{p}}$ is

$$\lambda^{k+2} - \lambda + \frac{p(r-1)}{r} = 0 \tag{3.2}$$

it is clear that Eq.(3.2) has a root in the interval $(-\infty, -1)$ and so $\overline{y_2} = (r-1)^{\frac{1}{p}}$ is an unstable equilibrium point from Theorem 1. This completes the proof. □

Theorem 3. Assume that $r > 1$ and let $\{y_n\}_{n=-(k+1)}^{\infty}$ be a solution of Eq.(3.1) such that

$$y_{-(k+1)}, y_{-(k-1)}, ..., y_0 \geq \overline{y_2} \text{ and } y_{-k}, y_{-(k-2)}, ..., y_{-1} < \overline{y_2} \tag{3.3}$$

or

$$y_{-(k+1)}, y_{-(k-1)}, ..., y_0 < \overline{y_2} \text{ and } y_{-k}, y_{-(k-2)}, ..., y_{-1} \geq \overline{y_2} \tag{3.4}$$

Then, $\{y_n\}_{n=-(k+1)}^{\infty}$ oscillates about $\overline{y_2} = (r-1)^{\frac{1}{p}}$ with semicycles of length one.

Proof. Assume that (3.3) holds. Then,

$$y_1 = \frac{ry_{-k}}{1+y^p_{-(k+1)}} < \overline{y_2}$$

and

$$y_2 = \frac{ry_{-(k-1)}}{1+y^p_{-k}} > \overline{y_2}$$

then, the proof follows by induction. The case where (3.4) holds is similar and will be omitted. □

Theorem 4. Assume that $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(3.1) is globally asymptotically stable.

Proof. We know by Theorem 2 that the equilibrium point $\overline{y_1} = 0$ of Eq.(3.1) is locally asymptotically stable. So, let $\{y_n\}_{n=-(k+1)}^{\infty}$ be a solution of Eq.(3.1). It suﬃces to show that

$$\lim_{n\to\infty} y_n = 0$$

Since

$$0 \le y_{n+1} = \frac{ry_{n-k}}{1+y^p_{n-(k+1)}} \le ry_{n-k}$$

We obtain

$$y_{n+1} \le ry_{n-k}$$

Then, we can write,
$y_{t(k+1)+1} \le r^{(t+1)} y_{-k}$
$y_{t(k+1)+2} \le r^{(t+1)} y_{-(k-1)}$
...
$y_{t(k+1)+(k+1)} \le r^{(t+1)} y_0$ for $t = 0, 1, ...$
If $r < 1$, then $\lim_{t\to\infty} r^{(t+1)} = 0$.So,

$$\lim_{n\to\infty} y_n = 0$$

This completes the proof. □

Theorem 5. If Eq. (3.1) possesses the prime period $(k+1)$ solution, all of which aren't equal with each other at the same time, then both $r = 1$ and these solutions are at least in number $\frac{k+1}{2}$ equal to 0 and at least in number 1 greater than 0.

Proof. Let $a_0, a_1, ..., a_k$, all of which aren't equal with each other at the same time, be the solutions of Eq.(3.1)'s prime period. That's to say,

$$..., a_0, a_1, ..., a_k, a_0, a_1, ..., a_k, ...$$

be a period (k+1) solution of Eq.(3.1). Then,

$a_k = \frac{ra_k}{1+a_0^p}$

$a_{k-1} = \frac{ra_{k-1}}{1+a_k^p}$

...

$a_1 = \frac{ra_1}{1+a_2^p}$

$a_0 = \frac{ra_0}{1+a_1^p}$

So,

If $a_k = 0$ and $r \neq 1$ then,

$$a_0 = a_1 = ... = a_k = 0$$

which is impossible ($a_k = 0$ and $r \neq 1$ is a conradiction).

If $a_k \neq 0$ and $r \neq 1$ then,

$$a_0 = a_1 = ... = a_k = \overline{y_2}$$

which is impossible ($a_k \neq 0$ and $r \neq 1$ is a conradiction). This result in $r = 1$.

To complete the proof, we use $r = 1$ at above equalities

$a_k = \frac{a_k}{1+a_0^p}$

$a_{k-1} = \frac{a_{k-1}}{1+a_k^p}$

...

$a_1 = \frac{a_1}{1+a_2^p}$

$a_0 = \frac{a_0}{1+a_1^p}$

Let's do the proof with induction. Assume that $k = 1$,

$a_1 = \frac{ra_1}{1+a_0^p}$

$a_0 = \frac{ra_0}{1+a_1^p}$

So one of the solutions is certainly equal to 0.($a_1 = 0$ or $a_0 = 0$)

Assume that $k = t-2$ ($t \geqslant 5$ is an odd number),

$a_{t-2} = \frac{ra_{t-2}}{1+a_0^p}$

$a_{t-3} = \frac{ra_{t-3}}{1+a_{t-2}^p}$

...

$a_1 = \frac{ra_1}{1+a_2^p}$

$a_0 = \frac{ra_0}{1+a_1^p}$

these solutions must be at least in number $\frac{(t-2)+1}{2} = \frac{t-1}{2}$ equal to 0.

Assume that $k = t$,

$a_t = \frac{a_t}{1+a_0^p}$

$a_{t-1} = \frac{a_{t-1}}{1+a_t^p}$
...
$a_1 = \frac{a_1}{1+a_2^p}$
$a_0 = \frac{a_0}{1+a_1^p}$

We separete and then search the above equalities, with result of $k = t$ assumption. Hence, if $k = t$ we get these solutions are at least in number $\frac{t+1}{2}$ equal to 0. Now let's indicate that one of these solutions is greater than 0. All the solutions will be positive so it is equal or greater than 0. Let none of them be greater than 0. If they aren't greater than 0, then all the solutions equal to 0. This conrasts that all of the solutions which aren't equal to with each other at the same time hypothesis. Then at least one solution certainly greater than 0. This completes the proof. □

Theorem 6. Assume that $r > 1$, then Eq.(3.1) possesses an unbounded solution.

Proof. From Theorem 3, we can assume that (3.3) without loss of generality that the solution $\{y_n\}_{n=-(k+1)}^{\infty}$ of Eq.(3.1) is such that

$$y_{2n+1} < \overline{y_2} \text{ and } y_{2n+2} > \overline{y_2} \text{ for } n \geqslant 0.$$

Then

$$y_{2n+1} = \frac{ry_{2n-k}}{1+y_{2n-(k+1)}^p} < \frac{ry_{2n-k}}{1+(r-1)} = y_{2n-k}$$

and

$$y_{2n+2} = \frac{ry_{2n-(k-1)}}{1+y_{2n-k}^p} > \frac{ry_{2n-(k-1)}}{1+(r-1)} = y_{2n-(k-1)}$$

which it follows that

$$\lim_{n\to\infty} y_{2n} = \infty \text{ and } \lim_{n\to\infty} y_{2n+1} = 0$$

Then, the proof is complete. □

Remark 1. If $k = 1$, the results in [1] follow directly.

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