Contractible edges of k-connected graphs for $k = 4, 5$ *

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Abstract

Dean ([3])show that if G be a $k-$ connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{2}$, then the subgraph $H = (V(G), E_k(G))$ formed by $V(G)$ and the k-contractible edges of G is 2-connected. In this paper we show that for $k = 4$, Dean's result holds when reduced $\frac{k}{2}$ to $\frac{k}{4}$. But for $k \geq 5$, we give a counterexample to show that it is false and give a low bound of the number of k-contractible edges for $k = 5$.

Keywords: k-connected graph; contractible edge; fragment

1 Introduction

We only consider finite simple undirected graph. Let k be a positive integer, G a k-connected graph. An edge of G is said to be a k-contractible edge if its contraction yields again a k-connected graph. By Tutte's famous result, any 3-connected graph with order at least 5 has a 3-contractible edge. But for $k \geq 4$, Thomassen ([10]) showed that there are infinitely many k-connected graph k-regular graphs which do not have any k-contractible

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edge. So, the contraction-critical k-connected graph for $k \geq 4$ was introduced, which is the non-complete k-connected graph without k-contractible edges. The contraction-critical 4-connected graphs are characterized, which are two special classes of 4-regular graphs. For $k \geq 5$, the characterization of contraction-critical k-connected graphs seems to be very hard. In general, Egawa ([4]) showed that every contraction critical k-connected graph has a vertex of degree at most $\lfloor \frac{5k}{4} \rfloor - 1$. Then, for $4 \leq k \leq 7$ every contraction-critical k-connected graph contains a vertex of degree k. It is very interesting to study the properties of contraction critical k-connected graphs and the distribution of k-contractible edges. N. Dean([3]) show that for every k-connected graph such that any fragments whose neighborhood contains an edge has cardinality exceeding $\frac{k}{2}$, the collection of k-contractible edges of G induced a 2-connected spanning subgraph of G. On the other hand, by Egawa's ([4]) results, every k-connected graph whose fragment has cardinality more than $\lfloor \frac{k}{4} \rfloor$ has contractible edge. So the following problem was given.

Problem 1 *Let G be a k-connected graph such that any fragments whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$. Is the subgraph $H = (V(G), E_k(G))$ form by $V(G)$ and the k-contractible edges of G 2-connected?*

In the following, we show that it is true for $k = 4$. But for $k \geq 5$ it is false.

In fact for $k = 2t + 1 (t \geq 2)$, we can constructs such a counterexample as following. Let $G_i \cong K_t$ be a complete graph for $i = 1, 2, \cdots, l$, where $l \geq 4$. Let G be a graph with $V(G) = \cup_{i \in \{1,2,\cdots,l\}} V(G_i) \cup \{x\}$, $E(G) = \cup_{i \in \{1,2,\cdots,l\}} E(G_i) \cup \{zy | z \in V(G_i), y \in V(G_{(i+1) mod(l)})\} \cup \{xy | y \in \cup_{i \in \{1,2,\cdots,l\}} V(G_i)\}$. Clearly, G is k-connected graph and the edges incident with x are not contractible. As $\delta(G) = 3t$ and $\kappa(G) = k$, every fragment of G has at least t vertices. Thus every fragment of G has cardinality greater than $\frac{k}{4} = \frac{2t+1}{4}$. But, on the other hand, the spanning graph induced by the set of contractible edges is not connected, as x is not incident to any contractible edge.

Further, we give a low bound for the number of contractible edges of 5-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{5}{4}$.

Theorem 1 *Let G be a 5-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{5}{4}$. Then, the number of 5-contractible edges of G is at least $|V(G)|$, that is $|E_5(G)| \geq |V(G)|$.*

Theorem 2 *Let G be a 4-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding 1. Then the subgraph $H = (V(G), E_4(G))$ is 2-connected.*

For terms not defined here we refer the reader to [2]. Let $G = (V(G), E(G))$ be a graph, $V(G)$ denotes the vertex set and $E(G)$ the edge set. Let $|G| = |V(G)|$, $\kappa(G)$ denote the vertex connectivity of G. An edge joining the vertex x, y will be written as xy. By $E_k(G)$, we denote the collection of all k-contractible edges in k-connected graph G. For $x \in F \subseteq V(G)$, we define $N_G(x) = \{y | xy \in E(G)\}$. By $d_G(x) = |N_G(x)|$ we denote the degree of x. $N_G(F) = \cup_{x \in F} N_G(x) - F$. A set $T \subseteq V(G)$ is called a separating set of a connected graph G, if $G-T$ has at least two connected components. A separating set with $\kappa(G)$ vertices is called a smallest separating set. Let G be a non-complete graph, T a smallest separating set. The union of at least one but not of all the components of $G - T$ is called a $T-$fragment. A fragment of G is a T-fragment for some smallest separating set T. Let $F \subseteq V(G)$ be a T-fragment. Then, $\overline{F} = V(G) - (F \cup T) \neq \emptyset$, and $\overline{F}$ is also a T-fragment and $N_G(F) = T = N_G(\overline{F})$. The set of all smallest separating sets of G will be denote by $\mathcal{T}_G$. We often omit the index G if it is clear from the context.

We need more definitions introduced in [6]. For a graph G, let $\mathcal{S}$ be a non-empty set of subset of $V(G)$. An $\mathcal{S}$-fragment of G is a T-fragment of G for any $T \in \mathcal{T}_G$ such that there is an $S \in \mathcal{S}$ with $S \subseteq T$. An inclusion-minimal $\mathcal{S}$-fragment of G is called an $\mathcal{S}$-end and one of the least vertex numbers is an $\mathcal{S}$-atom. A graph G is called $\mathcal{S}$-critical if for each $S \in \mathcal{S}$ there is $T \in \mathcal{T}_G$ such that $S \subseteq T$, and for any $\mathcal{S}$-fragment F there is a $T' \in \mathcal{T}_G$ such that $T' \cap F \neq \emptyset$ and $T' \cap (F \cup N(F))$ contains an element of $\mathcal{S}$.

The following properties of fragments are folklore (for the proof see [6]), we will use them without any further reference.

Let $T, T' \in \mathcal{T}_G$, and F, F' be the T, T'-fragment of G, respectively. If $F \cap F' \neq \emptyset$, then

$$|F \cap T'| \geq |\overline{F'} \cap T|, |F' \cap T| \geq |\overline{F} \cap T'| \quad (1)$$

If $F \cap F' \neq \emptyset \neq \overline{F} \cap \overline{F'}$, then both $F \cap F'$ and $\overline{F} \cap \overline{F'}$ are fragments of G, and $N(F \cap F') = (F' \cap T) \cup (T' \cap T) \cup (F \cap T')$. If $F \cap F' \neq \emptyset$ and $F \cap F'$ is not a fragment of G, then $\overline{F} \cap \overline{F'} = \emptyset$ and

$$|F \cap T'| > |\overline{F'} \cap T|, |F' \cap T| > |\overline{F} \cap T'| \quad (2)$$

Also, by definition, the two endvertices of any edge which is not k-contractible is contained in some smallest separating set. For an edge e of G, a fragment A of G is said to be a fragment with respect to e if $V(e) \subseteq N(A)$.

2 Some Lemmas

Lemma 1 *([6]) Let G be a k-connected graph and let $\mathcal{S}$ be a non-empty set of subset of $V(G)$. Let A be an $\mathcal{S}$-atom of G. If $T \in \mathcal{T}_G$ such that $T \cap A \neq \emptyset$ and $T \cap (A \cap N(A))$ contains one element of $\mathcal{S}$, then $A \subsetneq T$ and $|A| \leq \frac{1}{2}|N(A) - T|$.*

The following Lemma was seen in ([8]). For the convenience of the reader, we rewrite it here.

Lemma 2 *([8]) Let G be a k-connected graph for $k \geq 4$. Let $V_1 \subsetneq V(G)$ be a nonempty set such that $V(G) - V_1 \neq \emptyset$ and all the edges between V_1 and $V(G) - V_1$ are not contractible, let $\mathcal{S} = \{\{x, y\} \mid x \in V_1, y \in V(G) - V_1, xy \in E(G)\}$. Then G is $\mathcal{S}$-critical.*

Proof As G is k-connected($k \geq 4$), there are some edges joining the vertices of V_1 and $V(G) - V_1$, so $\mathcal{S}$ consists of the end-vertices of such edges. Let F be an $\mathcal{S}$-fragment of G such that $\{x, y\} \in \mathcal{S}$ and $\{x, y\} \subseteq N(F)$. We may assume that $x \in V_1$, $y \in V(G) - V_1$. Let $C \subseteq F$ be a connected component of F. As $N(F)$ is a smallest separating set of G, $N(x) \cap C \neq \emptyset$ and $N(y) \cap C \neq \emptyset$. Pick $x' \in N(x) \cap C$ and $y' \in N(y) \cap C$. Then there is a path P' connecting x', y' in C. Let $P = P' \cup \{xx', yy'\}$. Then P is a path connected x and y without using the edge xy. As $x \in V_1$, $y \in V(G) - V_1$, P contains one edge $f = uv$ which join one vertex of V_1 and one vertex of

$V(G) - V_1$. Clearly, $u, v \in F \cup N(F)$ and $\{u, v\} \cap F \neq \emptyset$. As the edges between V_1 and $V(G) - V_1$ is non-contractible , there is a $T' \in \mathcal{T}_G$ such that $T' \supseteq \{u, v\}$. So $T' \cap F \neq \emptyset$ and $T' \cap (F \cup N(F)) \supseteq \{u, v\}$. This implies that G is $\mathcal{S}$-critical. ■

By Lemma 1 and Lemma 2 we can deduce the following Lemma

Lemma 3 *Let G be a k-connected graph and $x \in V(G)$ is not incident to any contractible edges. Let $\mathcal{S} = \{\{x, y\} \mid y \in N(x)\}$, A be a $\mathcal{S}-$ atom. Then $|A| \leq \frac{k-1}{2}$.*

Notice that if G be a k-connected graph($k \geq 4$) such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$, then every vertex of G with degree k is not contain in any triangle.

Lemma 4 *Let G be a k-connected ($k \geq 4$)graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$, then every vertex of G with degree k is not contained in any triangle.*

Proof Let $x \in V(G)$ with $d(x) = k$. Let $A = \{x\}$, then $N(A)$ be a smallest separator. If x is contained in some triangles, then there would be an edge in $N(A)$. So A is a fragment whose neighborhood contains an edge. But $|A| = 1 \leq \frac{k}{4}$, a contradiction. ■

If A is a connected fragment with cardinality 2, then, clearly, the vertices of A has at least one common neighbor in $N(A)$. Thus they are contained in some triangles. So, by Lemma 4, we have the following lemmas.

Lemma 5 *Let G be a k-connected ($k \geq 4$)graph such that any fragment whose neighbor contains an edge has cardinality exceeding $\frac{k}{4}$, A be a fragment such that $|A| = 2$. Then every vertex of A has degree $k + 1$.*

Lemma 6 *Let G be a k-connected graph, $A = \{x, y\}$ is a fragment of G, $z \in N(A) \cap N(x)$ is contained in some smallest separating set. Let B be a fragment with respect to xz. If $N(A) - \{z\} \subseteq N(y)$, then $A \subseteq N(B)$.*

Proof Assume that $A \nsubseteq N(B)$, we may let $y \in A \cap B$. Then, as $N(A) - \{z\} \subseteq N(y)$, we have $\overline{B} \cap N(A) = \emptyset$. Further, as $|A \cap N(B)| = 1$ and $\overline{B} \cap N(A) = \emptyset$, $\overline{B} \cap \overline{A} = \emptyset$. Thus we have $\overline{B} = \emptyset$, for $\overline{B} \cap N(A) = \emptyset$, $\overline{B} \cap \overline{A} = \emptyset$ and $\overline{B} \cap A = \emptyset$. This is a contradiction, thus $A \subseteq N(B)$. ■

3 Proof of Theorem 1

Lemma 7 *Let G be a 5-connected graph such that any fragment whose neighbor contains an edge has cardinality exceeding $\frac{5}{4}$ and $x \in V(G)$ is not incident to any contractible edge. Let $\mathcal{S} = \{\{x,y\}|y \in N(x)\}$, then*

(1). every $\mathcal{S}-$ end has cardinality two.

(2). there are at least three $\mathcal{S}-$ ends.

Proof By Lemma 2, G is $\mathcal{S}$-critical.

(1). Let A be an $\mathcal{S}$-end of G and $\{x,y\} \subseteq N(A)$ such that$\{x,y\} \in \mathcal{S}$. By Lemma 4, we have $|A| \geq 2$, $|\overline{A}| \geq 2$.

Pick $y' \in N(x) \cap A$, let B be a fragment with respect to xy'.

First we show that $A \subseteq N(B)$. For otherwise, we may assume that $A \cap B \neq \emptyset$. Then, as A is a $\mathcal{S}$-end, we have $A \cap B$ is not a fragment. It follow that $\overline{B} \cap \overline{A} = \emptyset$ and $|(A \cap N(B)) \cup (N(A) \cap N(B)) \cup (N(A) \cap B)| \geq 6$. Now we must have $A \cap \overline{B} = \emptyset$. For otherwise, we have $B \cap \overline{A} = \emptyset$ and $|(A \cap N(B)) \cup (N(A) \cap N(B)) \cup (N(A) \cap \overline{B})| \geq 6$. By a simple calculation, we have $|\overline{A}| = 1$, contradicts to Lemma 4. So $A \cap \overline{B} = \emptyset$ and, also by Lemma 4, $|\overline{B}| = |\overline{B} \cap N(A)| \geq 2$. It follow that $|A \cap N(B)| \geq 3$. Thus, $|\overline{A} \cap N(B)| \leq 1$ and, further, $|\overline{A}| \leq 1$. Again, this is a contradiction.

So $A \cap B = \emptyset$. Similarly, $A \cap \overline{B} = \emptyset$ and $A \subseteq N(B)$.

If $|A| \geq 3$, then $|\overline{A} \cap N(B)| \leq 1$. As $|\overline{A}| \geq 2$, we may assume that $B \cap \overline{A} \neq \emptyset$. We have $|B \cap N(A)| \geq 3$, $|\overline{B} \cap N(A)| \leq 1$, thus $|\overline{B}| = |\overline{B} \cap N(A)| = 1$, a contradiction. So we have $|A| = 2$.

(2). First we can assertion that for any two $\mathcal{S}$-ends A, C, we have $A \cap C = \emptyset$. For otherwise, let $A \cap C \neq \emptyset$, then $|A \cap C| = 1$. Let $A = \{y,z\}, C = \{w,z\}$. Thus, a simple calculation show that $|N(C) \cap \overline{A}| = |N(A) \cap \overline{C}| = 1$. Thus $A \cap C$ is a fragment and z has degree 5. On the other hand, by Lemma 5, z has degree 6, a contradiction. Thus for any two $\mathcal{S}$-end A, C, we have $A \cap C = \emptyset$. As there is an other $\mathcal{S}-$ end in B, the third in $\overline{B}$ and the fourth in $\overline{A}$, so we can say that there are three $\mathcal{S}-$ ends by the above assertion. ■

Lemma 8 *Let G be a 5-connected graph such that any fragment whose neighbor contains an edge has cardinality exceeding $\frac{5}{4}$ and $x \in V(G)$ is not incident to any contractible edge. Let $\mathcal{S} = \{\{x,y\}|y \in N(x)\}$, A be an $\mathcal{S}-$*

end, then every vertex of A has degree 6 and is incident to four contractible edges.

Proof Let A be an $\mathcal{S}$-end of G with $\{x, y\} \in \mathcal{S}$. Let $N(A) = \{x, y, t_1, t_2, t_3\}$. By Lemma 7, we have $|A| = 2$, let $A = \{y'', y'\}$. As A is end, so A is connected. By Lemma 5, we have $d(y') = d(y'') = 6$. It implies that both y'' and y' are adjacent to every vertices of $N(A)$.

Let B be a fragment with respect to xy'. By Lemma 6, we have $A \subseteq N(B)$. Further, by Lemma 4, we have $|B \cap N(A)| = |\overline{B} \cap N(A)| = 2$. Without loss of generality, let $B \cap N(A) = \{y, t_1\}$ and $\overline{B} \cap N(A) = \{t_2, t_3\}$. Notice that both y'' and y' are adjacent to any other vertices of $A \cup N(A)$. Now we show that all the edges between A and $B \cap N(A)$ and the edges between A and $\overline{B} \cap N(A)$ are contractible.

We only show that $y't_2$ is contractible, the others can be deduced similarly. For otherwise, assume that $y't_2$ is not contractible, let B' be a fragment with respect to $y't_2$. Now by Lemma 4 and Lemma 6, $A \subseteq N(B')$ and $|B' \cap N(A)| = |\overline{B'} \cap N(A)| = 2$. We may let $B' \cap N(A) = \{y, x\}$, and thus $\overline{B'} \cap N(A) = \{t_1, t_3\}$.

Let us focus on $B, N(B), \overline{B}$ and $B', N(B'), \overline{B'}$. We know that $\{y'', y'\} \subseteq N(B') \cap N(B)$, $x \in B' \cap N(B)$, $t_2 \in N(B') \cap \overline{B}$, $t_1 \in \overline{B'} \cap N(B)$, $y \in B' \cap B$ and $t_3 \in \overline{B'} \cap \overline{B}$. So we have both $B' \cap B$ and $\overline{B'} \cap \overline{B}$ are not empty and thus they both are fragments of G. It is obviously that $N(y') \cap (B' \cap B) = N(y'') \cap (B' \cap B) = \{y\}$. On the other hand, as $yy'y''$ is a triangle, we have $d(y) \geq 6$ and thus $|B' \cap B| \geq 2$. So we have $((B \cap N(B')) \cup (N(B) \cap N(B')) \cup (N(B) \cap B') - \{y', y''\}) \cup \{y\}$ is a separating set with cardinality 4, a contradiction. ■

Lemma 9 *Let G be a 5-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{5}{4}$ and $x \in V(G)$ is incident to exactly one contractible edge. Let $\mathcal{S} = \{\{x, y\} | y \in N(x)$ and xy is not contractible $\}$, then there are two $\mathcal{S}$-ends such that both of them have cardinality two and every vertex of the end is incident to four contractible edges.*

Proof Let xy_0 be the only contractible that incident to x. Let A' be an $\mathcal{S}$-fragment of G. We know that either $y_0 \notin A'$ or $y_0 \notin \overline{A'}$. We may assume that $y_0 \notin A'$ and pick the $\mathcal{S}$-end, say A, in A'. Let $\{x, y\} \subseteq N(A)$

with $\{x, y\} \in \mathcal{S}$ and $N(A) = \{x, y, t_1, t_2, t_3\}$. Pick $y' \in N(x) \cap A$, let B be a fragment with respect to xy'. Similar to the proof of Lemma 7, we have $A \subseteq N(B)$, $|A| = 2$ and, further, $|B \cap N(A)| = |\overline{B} \cap N(A)| = 2$. Let $A = \{y'', y'\}$ and without loss of generality, let $B \cap N(A) = \{y, t_1\}$ and $\overline{B} \cap N(A) = \{t_2, t_3\}$. By Lemma 5, both y'' and y' are adjacent to any other vertices of $A \cup N(A)$. Now we know, similar to Lemma 8, the edges between A and $B \cap N(A)$ and the edges between A and $\overline{B} \cap N(A)$ are contractible.

Further we notice that either $y_0 \notin B$ or $y_0 \notin \overline{B}$. Without lose of generality, we may let $y_0 \notin B$. It follow that there is a $\mathcal{S}$-end B' in B. Similarly, we can show that $|B'| = 2$ and every vertex of B' is incident to four contractible edges. ■

Now we are ready to prove Theorem 1.

By Lemma 7 and Lemma 8, we know that if x is not incident to any contractible edge, then there are six vertices which adjacent to x. Further every neighbor of these vertices, except x, is incident to at least two contractible edges. By Lemma 9, we know that if x is incident to exactly one contractible edge, then there are four vertices which adjacent to x. Further every neighbor of these vertices, except x, is incident to at least two contractible edges. Now let $U_1 = \{v | v$ is not incident to any contractible edge $\}$ and $U_2 = \{v | v$ is adjacent one member of U_1 and except the one in U_1, all neighbor of v is incident to at least two contractible edges $\}$. Let $T_1 = \{v | v$ is incident to exactly one contractible edges $\}$ and $T_2 = \{v | v$ is adjacent one member of T_1 and except the one in T_1, all neighbor of v is incident to at least two contractible edges $\}$ and, by Lemma 7 and Lemma 8, we have $T_i \cap U_j = \emptyset$, for $i, j = 1, 2$. Let $T = V(G) - U_1 - U_2 - T_1 - T_2$. Now, by Lemma 7 and Lemma 8, we have $|U_2| \geq 6|U_1|$. By Lemma 9, $|T_2| \geq 4|T_1|$.

Let $H = (V(G), E_5(G))$, then the degree sum of H would be

$$\sum_{v \in V(H)} d_H(v) \geq 4|U_2| + 4|T_2| + 2|T| + |T_1|$$

$$\geq (2|U_2| + 2|T_2| + 2|T| + |T_1|) + (2|U_2| + 2|T_2|)$$

$$\geq 2(|U_2| + |T_2| + |T| + |U_1| + |T_1|) + 10|U_1| + 7|T_1|$$

$$\geq 2(|U_2| + |T_2| + |T| + |U_1| + |T_1|) = 2|V(G)|$$

So $|E_5(G)| \geq |V(G)|$. ■

4 Proof of Theorem 2

Lemma 10 *Let G be a 4-connected graph such that any fragment whose neighbor contains an edge has cardinality exceeding* 1, *then any $x \in V(G)$ is incident to at least two noncontractible edges.*

Proof Let $x \in V(G)$ is incident to at most one noncontractible edge. Let $\mathcal{S} = \{\{x, y\} | y \in N(x)$ and xy is not contractible $\}$, then there is an $\mathcal{S}$-end A such that the edges between x and A are noncontractible. As A is an $\mathcal{S}$-fragment, there is an edge in $N(A)$. So, by Lemma 4, we have $|A| \geq 2$, $|\overline{A}| \geq 2$. Let $\{x, y\} \subseteq N(A)$ and $\{x, y\} \in \mathcal{S}$. Pick $y' \in N(x) \cap A$, let B be a fragment with respect to xy'. Similar to Lemma 7, we have $|A| = 2$ and $A \subseteq N(B)$. As $|N(A)| = 4$ and $|T \cap N(A)| \geq 1$, then either $|B \cap N(A)| = 1$ or $|\overline{B} \cap N(A)| = 1$. We may assume that $|B \cap N(A)| = 1$, then have $|B| = 1$. But there is an edge in $N(B)$, a contradiction. ■

Lemma 11 *Let G be a 4-connected graph such that any fragment whose neighbor contains an edge has cardinality exceeding* 1, *then $H = (V(G), E_4(G))$ is connected.*

Proof For otherwise let H_1 be a component of H, $\mathcal{S} = \{\{x, y\} \mid x \in H_1, y \in V(G) - H_1, xy \in E(G)\}$. Then, by Lemma 2, G is $\mathcal{S}$-critical. Let A be an $\mathcal{S}$-atom, $\{x, y\} \subseteq N(A)$ and $\{x, y\} \in \mathcal{S}$ with $x \in H_1, y \in V(G) - H_1$. Similar to Lemma 2, we can find $\{x', y'\} \in \mathcal{S}$, $\{x', y'\} \subseteq A \cup N(A)$ and $\{x', y'\} \neq \{x, y\}$. Further, if there are some member of $\mathcal{S}$ which intersection with $N(A)$, we would pick $\{x', y'\} \cap N(A) \neq \emptyset$ priority.

Let B be a fragment with respect to $x'y'$. By Lemma 1, we have $|A| \leq \frac{1}{2}|N(A) - N(B)| \leq 2$. If $N(A) \cap N(B) \neq \emptyset$, then $|A| = 1$. But there is an edge in $N(A)$, a contradiction. So assume that $N(A) \cap N(B) = \emptyset$, hence we have $\{x', y'\} = A$. Let $x' \in H_1$, then $y' \in V(G) - H_1$. But now, by Lemma 5, we have $x'y \in E(G)$. So $\{x', y\} \in \mathcal{S}$ and $\{x', y\} \cap N(A) \neq \emptyset$, this is contradicts to the choices of $\{x', y'\}$. ■

Now we are ready to prove Theorem 2.

By Lemma 11, we have $H = (V(G), E_4(G))$ is 1-connected. Suppose that H have a cut point w. Let H_1 be a component of $H - w$, $H_2 = H - w - H_1$, $\mathcal{S} = \{\{x, y\} \mid x \in H_1,\ y \in H_2,\ xy \in E(G)\}$. As G is 4

-connected, we know that $\mathcal{S} \neq \emptyset$. Then there is an $\mathcal{S}$- end A such that $w \notin A$. By Lemma 4, we have $|A| \geq 2$ and $|\overline{A}| \geq 2$. Let $\{x, y\} \subseteq N(A)$ with $\{x, y\} \in \mathcal{S}$ and $x \in H_1$, $y \in H_2$. Let $x' \in N(x) \cap A$, $y' \in N(y) \cap A$. As A is connected, we know that there is a $x - y$ path in A which does not use w and the edge xy. So there must be a member of $\mathcal{S}$ which is contained in A or intersection both A and $N(A)$. If it intersection both A and $N(A)$, then, similar to the proof of Lemma 7, we can easily deduce that there is a vertex of degree 4 which is contained in a triangle, a contraction. So we may assume that there no member of $\mathcal{S}$ which intersection both A and $N(A)$. Let $\{z, t\} \in \mathcal{S}$ which was contained in A. Further, let $z \in H_1, t \in H_2$. Let $T \supseteq \{z, t\}$ be a smallest separator, let B be a $T-$ fragment. Then, similar to the proof of Lemma 7, we can deduce that $A \subseteq T$, $A = \{z, t\}$ and $T \cap N(A) = \emptyset$. Thus it follow that $|B \cap N(A)| = 2 = |\overline{B} \cap N(A)|$. Let, without lose of generality, $B \cap N(A) = \{x, y\}$. By Lemma 5, both vertices of A has degree 5. Thus $xt \in E(G), yz \in E(G)$ and $\{x, t\}, \{y, z\}$ are contained in $\mathcal{S}$, a contradiction. ■

5 Some Remarks

In [5]K. kriesell given the following problem .

Problem 2 *Is any vertex of contraction critical 5-connected graph contained in a triangle?*

In [7] we show that there are contraction critical 5-connected graphs which has some vertices that dose not contained in any triangle. Especially there are contraction critical 5-connected graph which has some vertex of degree 5 that is not contained in any triangle. Also in [7] we have show that in any contraction critical 5-connected graph there must a vertex of degree 5 such that every edge incident with it is contained in some triangles. So for a 5-connected graph in which every vertex of degree 5 dose not contained in any triangle must have a contractible edge.

In fact, for $k = 4, 5$ the problem that any fragments whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$ is equivalent to which that every vertex of degree $k (k = 4, 5)$ is not contained in any triangle.

Proposition 1 *Let G be a k-connected graph for $k = 4, 5$ such that every vertex of degree k (if there are any)is not contained in any triangle. Then fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$.*

So, together with Lemma 4, the following result is holds.

Proposition 2 *Let G be a k-connected graph for $k = 4, 5$ which does have some vertices of degree k. Then every vertex of degree k is not contained in any triangle if and only if every fragment whose neighbor contains an edge has cardinality exceeding $\frac{k}{4}$.*

Thus we may rewrite Theorem 1 and Theorem 2 in an other way.

Theorem A *Let G be a 5-connected graph such that every vertex of degree 5 (if there are any) is not contained in any triangle. Then, the number of 5-contractible edges of G is at least $|V(G)|$, that is $|E_5(G)| \geq |V(G)|$.*

Theorem B *Let G be a 4-connected graph such that every vertex of degree 4 (if there are any) is not contained in any triangle. Then the subgraph $H = (V(G), E_4(G))$ is 2-connected.*

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References

[1] K.Ando, K.Kawarabayashi, A.kaneko, Contractible edges in minimally k-connected graphs, SUT J. Math. 36(1) (2000) 99-103.

[2] J.A.Bondy and U.S.R.Murty, Graph Theory with Applications, MacMillan (1976).

[3] Dean,N.,Distribution of contractible edges in k- connected graphs. J.Comb.Theorey, Ser.B 48, 1-5(1990).

[4] Y.Egawa, Contractible edges in n-connected graphs with minimum degree greater than or equal to $\lfloor \frac{5n}{4} \rfloor$, *Graphs Combin.*, 1991, (7):15-21.

[5] M.Kriesell, Triangle density and contractibility, Combinatorics , Probability and Computing (2005)14,133-146.

[6] W.Mader, Generalizations of critical connectivity of graphs, *Discrete Math.*, 1988, (72):267-283.

[7] C.Qin, Xudong Yuan and Jianji Su, Triangles in contraction critical 5-connected graphs, *Australasian Journal of Combinatorics*, Volume 33(2005), Page:139-146

[8] C.Qin, Xudong Yuan and Jianji Su, Some properties of contraction critical 5-connected graphs, *Discrete Mathmatics,* Volume 308(2008), Page: 5742-5756

[9] Su Jianji, Vertices of degree 5 in contraction critical 5-connected graphs, *J. Guangxi Normal University*, 1997, (3):12-16 (*in chinese*)

[10] C. Thomassen, Nonseparating cycles in k-connected graphs, *J. Graph Theory*, 1981, (5):351-354.

[11] Yuan Xudong, The contractible edges of 5-connected graphs, *J. Guangxi Normal University*, 1994, (3):30-32 (*in chinese*).