Quasi-tree graphs with the largest number of maximal independent sets

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Abstract

A maximal independent set is an independent set that is not a proper subset of any other independent set. A connected graph (respectively, graph) G with vertex set $V(G)$ is called a quasi-tree graph (respectively, quasi-forest graph), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). In this paper, we determine the largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs. We also characterize those extremal graphs achieving these values.

1 Introduction

All the graphs considered in this paper are simple, finite, undirected, and without multiple edges or loops. In a graph $G = (V, E)$, an *independent set* is a subset S of V such that no two vertices in S are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of a graph G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$.

The problem of determining the largest value of $mi(G)$ in a general graph of order n and those graphs achieving the largest number was proposed by Erdös and Moser, and solved by Moon and Moser [5]. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, (k-)connected graphs, bipartite graphs; for a survey see [3]. A connected graph (respectively, graph) G with vertex set $V(G)$ is called a *quasi-tree graph* (respectively, *quasi-forest graph*), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). The concept of quasi-tree graphs was mentioned by H. Liu and M. Lu in [4].

The purpose of this paper is to determine the largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs of order n. Extremal graphs achieving these values are also given.

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2 Preliminary

For a vertex $x \in V(G)$, let $\mathrm{MI}_{-x}(G) = \{I \in \mathrm{MI}(G) : x \notin I\}$ and $\mathrm{MI}_{+x}(G) = \{I \in \mathrm{MI}(G) : x \in I\}$. Note that $mi(G) = |\mathrm{MI}_{-x}(G)| + |\mathrm{MI}_{+x}(G)|$. The *neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent to x in G and the *closed neighborhood* $N_G[x]$ is $\{x\} \cup N_G(x)$. The *degree* of x is the cardinality of $N_G(x)$, denoted by $\deg_G(x)$. A vertex x is called a *leaf* if $\deg_G(x) = 1$. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. nG is the short notation for the union of n copies of disjoint graphs isomorphic to G. Denote by C_n a *cycle* with n vertices and P_n a *path* with n vertices. Throughout this paper, for simplicity, let $r = \sqrt{2}$. We begin with the following useful lemmas which are needed in this paper.

Lemma 2.1. ([1]) *For any vertex x in a graph G, the following hold.*
(1) $mi(G) \le mi(G - x) + mi(G - N_G[x])$.
(2) *If x is a leaf adjacent to y, then* $mi(G) = mi(G - N_G[x]) + mi(G - N_G[y])$.

Lemma 2.2. *Let x be the vertex in a graph G such that $mi(G) = mi(G - x) + mi(G - N_G[x])$, the following hold.*
(1) $mi(G - x) = |\mathrm{MI}_{-x}(G)|$.
(2) *For a maximal independent set $I \in \mathrm{MI}(G - x)$, $I \cap N_G(x) \neq \emptyset$.*

Proof. The results follow from the fact that $mi(G - N_G[x]) = |\mathrm{MI}_{+x}(G)|$ and $mi(G) = |\mathrm{MI}_{-x}(G)| + |\mathrm{MI}_{+x}(G)|$. □

Lemma 2.3. ([1]) *If G is the union of two disjoint graphs G_1 and G_2, then $mi(G) = mi(G_1)mi(G_2)$.*

The results of the largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.4 and 2.5, respectively.

Theorem 2.4. ([1], [2]) *If T is a tree with $n \ge 1$ vertices, then $mi(T) \le t(n)$, where*

$$t(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even;} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t(n)$ if and only if $T \cong T(n)$, where

$$T(n) = \begin{cases} B(2, \frac{n-2}{2}) \text{ or } B(4, \frac{n-4}{2}), & \text{if } n \text{ is even;} \\ B(1, \frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

where $B(i,j)$ is the set of batons, which are the graphs obtained from a path P of $i \geq 1$ vertices by attaching $j \geq 0$ paths of length two to the endpoints of P in all possible ways (see Figure 1).

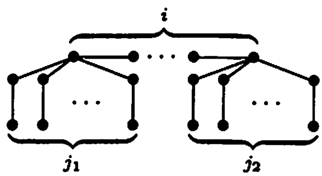

Figure 1: The baton $B(i,j)$ with $j = j_1 + j_2$

Theorem 2.5. ([1], [2]) *If F is a forest with $n \geq 1$ vertices, then $mi(F) \leq f(n)$, where*

$$f(n) = \begin{cases} r^n, & \text{if } n \text{ is even;} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f(n)$ if and only if $F \cong F(n)$, where

$$F(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even;} \\ B(1, \frac{n-1-2s}{2}) \cup sP_2 \text{ for some } s \text{ with } 0 \leq s \leq \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

3 Main results

This section gives the solution to the problem of determining the largest values of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order n. Extremal graphs achieving these values are also given.

Theorem 3.1. *If Q is a quasi-tree graph with $n \geq 5$ vertices, then $mi(Q) \leq q(n)$, where*

$$q(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even;} \\ r^{n-1} + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = q(n)$ if and only if $Q \cong Q(n)$ or $Q \cong C_5$, where $Q(n)$ is shown in Figure 2.

Proof. By repeatedlty applying Lemma 2.1 (2) to the leaves of $Q(n)$, we have $mi(Q(n)) = q(n)$. Let x be a vertex of Q such that $Q - x$ is a tree. Since $t(n) < q(n)$ for $n \geq 5$, we may suppose that Q is a quasi-tree graph of order $n \geq 5$ with at least one cycle. Then x is on some cycle of Q, it follows

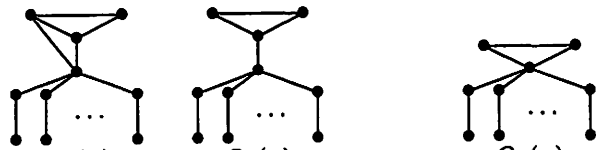

Figure 2: The graph $Q(n)$

that $\deg_Q x \geq 2$. By Theorem 2.4, $mi(Q-x) \leq t(n-1)$. On the other hand, $Q - N_Q[x]$ is a forest with at most $n-3$ vertices, by Theorem 2.5, $mi(Q - N_Q[x]) \leq f(n-3)$. Thus, by Lemma 2.1 (1), we have

$$\begin{aligned} mi(Q) &\leq mi(Q-x) + mi(Q - N_Q[x]) \\ &\leq t(n-1) + f(n-3) \\ &= \begin{cases} r^{n-2} + r^{n-4}, & \text{if } n \text{ is even;} \\ (r^{n-3}+1) + r^{n-3}, & \text{if } n \text{ is odd.} \end{cases} \\ &= q(n). \end{aligned}$$

Furthermore, the equalities holding imply that $|\mathrm{MI}_{-x}(Q)| = mi(Q-x) = t(n-1)$ and $|\mathrm{MI}_{+x}(Q)| = mi(Q - N_Q[x]) = f(n-3)$.

We will characterize the quasi-tree graph Q of order $n \geq 5$ for which $mi(Q) = q(n)$. Since $|\mathrm{MI}_{+x}(Q)| = mi(Q - N_Q[x]) = f(n-3)$, by Theorem 2.5, we have that $Q - N_Q[x] \cong F(n-4)$ or $Q - N_Q[x] \cong F(n-3)$. We consider two cases.

Case 1. $\deg_Q(x) = 3$. Then n is even. In addition, by Theorems 2.4 and 2.5, we have that $Q - x \cong T(n-1) \cong B(1, \frac{n-2}{2})$ and $Q - N_Q[x] \cong F(n-4) \cong \frac{n-4}{2}P_2$. Hence we obtain that $Q \cong Q_e^*(n)$.

Case 2. $\deg_Q(x) = 2$. Since $Q-x \cong T(n-1)$ and $Q - N_Q[x] \cong F(n-3)$, by Theorems 2.4 and 2.5, we have that

$$Q - x \cong \begin{cases} B(1, \frac{n-2}{2}), & \text{if } n \text{ is even;} \\ B(2, \frac{n-3}{2}) \text{ or } B(4, \frac{n-5}{2}), & \text{if } n \text{ is odd.} \end{cases} \qquad (\dagger)$$

and

$$Q - N_Q[x] \cong \begin{cases} B(1,0) \cup \frac{n-4}{2}P_2 \text{ or } B(1, \frac{n-4}{2}), & \text{if } n \text{ is even;} \\ \frac{n-3}{2}P_2, & \text{if } n \text{ is odd.} \end{cases} \qquad (\ddagger)$$

Hence there are six possibilities for graphs Q meeting the requirements $(\dagger)$ and $(\ddagger)$. See Figure 3.

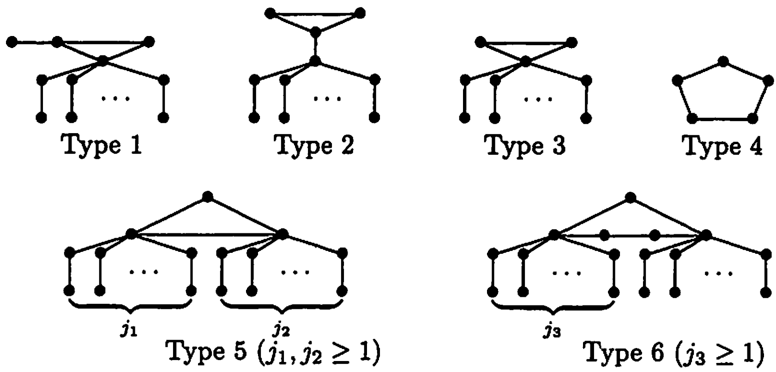

Figure 3: The six possibilities for graphs Q

Moreover, among these only those of Types 2, 3, and 4 satisfy Lemma 2.2 (2), hence we obtain that

$$Q \cong \begin{cases} Q_e(n), & \text{if } n \text{ is even;} \\ Q_o(n) \text{ or } C_5, & \text{if } n \text{ is odd.} \end{cases}$$

□

Theorem 3.2. *If Q is a quasi-forest graph with $n \geq 2$ vertices, then $mi(Q) \leq \overline{q}(n)$, where*

$$\overline{q}(n) = \begin{cases} r^n, & \text{if } n \text{ is even;} \\ 3r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = \overline{q}(n)$ if and only if $Q \cong \overline{Q}(n)$, where

$$\overline{Q}(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even;} \\ C_3 \cup \frac{n-3}{2}P_2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. It is clear that $mi(\overline{Q}(n)) = \overline{q}(n)$. Let x be a vertex of Q such that $Q - x$ is a forest. For the case when n is odd, since $f(n) < \overline{q}(n)$ for $n \geq 3$, we may suppose that Q is a quasi-forest graph of order $n \geq 3$ with at least one cycle. Then x is on some cycle of Q, it follows that $\deg_Q x \geq 2$. By Lemmas 2.1 (1),

$$\begin{aligned} mi(Q) &\leq mi(Q - x) + mi(Q - N_Q[x]) \\ &\leq f(n-1) + f(n-3) \\ &= r^{n-1} + r^{n-3} \\ &= 3r^{n-3} \\ &= \overline{q}(n); \end{aligned}$$

and the equalities holding imply that $|\mathrm{MI}_{-x}(Q)| = mi(Q-x) = f(n-1)$ and $|\mathrm{MI}_{+x}(Q)| = mi(Q-N_Q[x]) = f(n-3)$. By Lemma 2.5, $Q-x \cong \frac{n-1}{2}P_2$ and $Q-N_Q[x] \cong \frac{n-3}{2}P_2$. Hence we obtain that $Q \cong C_3 \cup \frac{n-3}{2}P_2$.

For the case when n is even, it is true for $n=2$. Let Q is a quasi-forest of order $n \geq 4$ such that $mi(Q)$ is as large as possible. By Theorem 2.5 , we have $mi(Q) \geq r^n$. Suppose that there exist some cycles in Q, then x is on some cycle of Q. Thus, by Lemma 2.1 and Theorem 2.5, we have that

$$\begin{aligned} r^n \leq mi(Q) &\leq mi(Q-x) + mi(Q-N_Q[x]) \\ &\leq f(n-1) + f(n-3) \\ &= r^{n-2} + r^{n-4} \\ &= 3r^{n-4}. \end{aligned}$$

This is a contradiction. Hence we obtain that $Q \cong \frac{n}{2}P_2$. □

Since every graph with at most one cycle is a quasi-forest graph, Theorem 3.2 gives an alternative proof for the solution to the problem in graphs with at most one cycle.

Corollary 3.3. ([2]) *If G is a graph with $n \geq 2$ vertices such that G contains at most one cycle, then $mi(G) \leq \overline{q}(n)$. Furthermore, $mi(G) = \overline{q}(n)$ if and only if $G \cong \overline{Q}(n)$.*

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