A Contribution to the characterization of Quasi-groups that are Isotopic to Abelian groups

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Abstract

In this paper, we characterize the variety of quasi-groups isotopic to abelian groups by four-variable identities.

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§1. Introduction

A common theme underlying many different areas of mathematics is the study of sets of objects with some operations for combining the objects into new ones. We define an algebra as a set of objects together with a list of one or more operations on the objects. We classify algebras according to the identities (the laws or axioms) that they satisfy. For any collection of algebras, we can form the collection of all identities that are satisfied by all those algebras. Conversely, for any collection of identities we can form the collection of all algebras that satisfy those identities. Such a collection is called a *variety*. This back-and-forth interplay between algebras and identities is called a *Galois connection*. This Galois connection means that the classes of algebras which are closed under the formation of homomorphic images, subalgebras and products are precisely the classes determined by some set of identities.

The main results of this paper are Theorems 2.1-2.4. Theorem 2.1 and Theorem 2.3 can be regarded as new contributions to knowledge relative to the "Loop's 99" open problem of Fedir Sokhatsky relative to abelian groups.

Let **H, S, P** denote closure (of a class of algebras in the same language) under homomorphic images, subalgebras and products respectively. Theorems 1.1 to 1.4 are the basic concerning these closure operators are.

Theorem 1.1. (R.Willard, [7], Theorem 1.1) *A class K of algebras in the same language is an equational class if and only if it is closed under* **H, S, P**.

Theorem 1.2. (R.Willard, [7], Theorem 1.2) *Suppose K is a class of algebras in the same language. Then the smallest equational class containing K is* $\mathbf{HSP}(K)$

An algebra $Q(\cdot)$ is called a quasi-group, if each of the equations $ax = b$ and $ya = b$ have only unique solution for any $a, b \in Q$. A quasi-group with an identity element is called a loop.

Theorem 1.3. (H.O.Pflugfelder, [6], Corollary II.3.3) *Reidemeister's condition is valid in quasi-group $Q(\cdot)$ iff the quasi-group is isotopic to a group.*

Theorem 1.4. (H.O.Pflugfelder, [6], Corollary II.3.7) *Thompson's condition is valid in quasi-group $Q(\cdot)$ iff the quasi-group is isotopic to an Abelian group.*

In the language of algebraic systems, Theorem 1.3 and 1.4 are respectively, interpreted as follows (See [3]).

1. The class of all quasi-groups which are isotopic to groups, forms a quasi-variety.

2. The class of all quasi-groups which are isotopic to Abelian groups, forms a quasi-variety.

In 1966, V. D. Belousov proved [1, 2] that the classes of quasi-groups isotopic to groups which forms a quasi-variety, are also varieties. That is, they are characterized by identities or are closed for homomorphisms, subalgebras, and direct products (See Theorem 1.1).

If $Q(\cdot)$ is a quasi-group then, by denoting the unique solutions of the equations $ax = b$ and $ya = b$ by $x = a \setminus b$ and $y = b/a$, respectively, we get an algebra $Q(\cdot, \setminus, /)$ with the following identities:

$$x(x \setminus y) = y, \quad x \setminus (x \cdot y) = y,$$
$$(y/x) \cdot x = y, \quad (y \cdot x)/x = y.$$

Evans [4] showed that a quasi-group $(G, \cdot)$ is isotopic to a group if and only if it satisfies the law

$$[(xP_1 \cdot yP_2)P_3 \cdot zP_4]P_5 = [xQ_1 \cdot (yQ_2 \cdot zQ_3)Q_4]Q_5,$$

where $P_i, Q_i, i = 1, 2, 3, 4, 5$ are permutations on the quasi-group.

V. D. Belusov [2] defined a balanced identity as an identity $\omega_1(x_1, x_2, ..., x_n) = \omega_2(x_1, x_2, ..., x_n)$ where each $x_i, i = 1, 2, ..., n$, occurs exactly once in ω_1 and once in ω_2, and ω_1 and ω_2 involve different grouping of some triples. He proved that a quasi-group is isotopic to a group if and only if it satisfies a balanced identitiy. Also he characterized quasi-groups isotopic to groups (and Abelian groups) by Theorems 1.5 to 1.7 .

Theorem 1.5. (Belousov, [2], Theorem 8) *The quasi-group $Q(\cdot)$ is isotopic to a group if and only if it satisfies the following identity:*

$$x(y \setminus ((z/u)v)) = ((x(y \setminus z))/u)v.$$

Theorem 1.6. (Belousov, [2], Theorem 9) *The quasi-group $Q(\cdot)$ is isotopic to an Abelian group if and only if it satisfies the following identity:*

$$x \setminus (y(u \setminus v)) = u \setminus (y(x \setminus v)).$$

Theorem 1.7. (Belousov's Theorem on four quasi – groups, [1], Theorem *If four quasi-groups A_i $(i = 1, 2, 3, 4)$ defined on a set Q, satisfy the associative law $A_1 [A_2(x, y), z] = A_3 [x, A_4(y, z)]$, then each of the quasi-groups $Q(A_i)$ is isotopic to a group.*

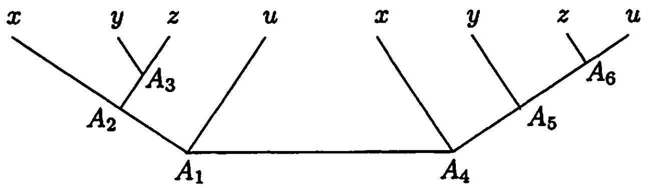
In general, we can consider algebras with at least two operation by the following theorem:

Theorem 1.8. ([2], *Theorem* 2) *All operations in equivalence class K of non-cancelable balanced identities $w_1 = w_2$ or $\Phi_1(w) = \Phi_2(w)$ are isotopic to one and the same group, if K contains more than two operation.*

Falconer [5] generalized the notion of balanced to what she called generalized group identity so that a balanced identity is a special case of a generalized group identity.

Theorem 1.9. (Falconer, [5], Theorem 2.10) *A necessary and sufficient condition for a quasi-group to be isotopic to a group is that it satisfies a generalized group identity.*

Similarly, there is also a generalization of the Evans Identity, called the generalized associative laws by Falconer [5].
In summary, there are three existing results on the identities that characterize varieties of quasi-groups that are isotopic to a group.
(1) A quasi-group that obeys the Evans identity.
(2) A quasi-group that obeys a balanced identity.
(3) A quasi-group that obeys a generalized group identity.
In this paper we will study identities that characterize the variety of quasi-groups isotopic to abelian groups. As well as four-length balance quasi-group identities, which are defined by the following graph:



The results can be used in the theory of quasi-groups, in the studies of the Latin squares, and in the theory of nets, as well as in the invertible universal algebras.

§2. Characterization of quasi-groups

The earliest studies of quasi-groups concern the varieties of quasi-groups, which are defined by identities of Moufang, mediality, distributivity. The existence of all four-variable balanced identities that characterize quasi-groups isotopic to groups is one of important problems in quasi-group theory. Such identities are called linear equations or balanced identities (See [1] and [2]). In these identities every variable occurs only once in each side and is arranged in the same order.

DEFINITION 2.1. Let $A = \langle Q; \Sigma \rangle$ and $A' = \langle Q'; \Sigma' \rangle$ are binary algebras (algebras with binary operations). An algebra A is isotopic to algebra A', if there exist permutations $\alpha, \beta, \gamma : Q \longrightarrow Q'$ and a bijection $\tilde{\psi} : \Sigma \to \Sigma'$ such that for any operator $A \in \Sigma$ and any $x, y \in Q$ we have

$$\alpha A(x, y) = \left[\tilde{\psi} A\right] (\beta x, \gamma y). \tag{1}$$

In addition, if

$$x \cdot y = \gamma^{-1}(\alpha x \circ \beta y) \tag{2}$$

and $x \cdot y = z$, then one can obtain $x = z/y$ and $y = x \setminus z$. Therefore, from (2) we can write $\gamma z = \alpha x \circ \beta y$ or $\alpha x = \gamma z \circ \overline{\beta y}$. Hence we have $x = \alpha^{-1} \left(\gamma z \circ \overline{\beta y}\right)$. Also from $\beta y = \overline{\alpha x} \circ \gamma z$ we have $y = \beta^{-1} \left(\overline{\alpha x} \circ \gamma z\right)$, where $\alpha x \circ \overline{\alpha x} = \beta x \circ \overline{\beta x} = e$ is the identity element of group. In this paper we consider $\left(\overline{\alpha x} = (\alpha x)^{-1}\right)$.

Note. For simplicity, we let $\lambda x = \overline{\beta x}$ and $\delta x = \overline{\alpha x}$. Therefore, we use the following notations:

$x \cdot y = \gamma^{-1}(\alpha x \circ \beta y)$,
$x/y = \alpha^{-1}(\gamma x \circ \lambda y)$,

$x \setminus y = \beta^{-1}(\delta x \circ \gamma y)$.

Also, we note that if a quasi-group $Q(\cdot, \setminus, /)$ is isotopic to a group $Q(\circ)$, then each of the quasi-groups $Q(\cdot)$, $Q(\setminus)$ and $Q(/)$ are isotopic to the same group $Q(\circ)$.

As a generalization of Theorem 1.6, we present Theorems 2.1 to 2.4 for quasi-groups isotopic to some abelian groups.

Theorem 2.1. *The quasi-group $Q(\cdot, \setminus, /)$ is isotopic to an abelian group if and only if in $Q(\cdot, \setminus, /)$ each of the following identities holds:*

(a) $x / [(y/z) \setminus u] = y / [(x/z) \setminus u]$,

(b) $x \cdot [y \setminus (z \cdot u)] = z \cdot [y \setminus (x \cdot u)]$,

(c) $[(x/y) \cdot z] / u = [(x/u) \cdot z] / y$.

Proof : Let the identity (a) be satisfied in the quasi-group $Q(\cdot, \setminus, /)$. By $x = a$ (a is an arbitrary fixed element), identity (a) is of the form:

$$B[A(y,z), u] = A[y, c(z,u)] , \tag{3}$$

where we consider $A(x,y) = x/y$, $B(x,y) = a/(x \setminus y)$ and $C(x,y) = (a/x) \setminus y$. Hence, by Theorem 1.7, it is obvious that $Q(A)$, $Q(B)$, $Q(C)$ are quasi-groups isotopic to groups. Therefore from $A(x,y) = x/y$ we can consider the isotopy $(/) \sim_{iso.} (\circ)$ such that

$$\gamma(x/y) = \alpha x \circ \beta y. \tag{4}$$

Hence we obtain

$$x/y = \gamma^{-1}(\alpha x \circ \beta y) \quad \text{and} \quad x \setminus y = \beta^{-1}\left(\overline{\alpha y} \circ \gamma x\right) , \tag{5}$$

where $\overline{\alpha x}$ denotes the inverse of αx in the group $Q(\circ)$. Therefore, substituting (4), (5) in (a) and calculating two sides, we obtain:

$$\gamma^{-1}\left(\alpha x \circ \overline{\alpha u} \circ \alpha y \circ \beta z\right) = \gamma^{-1}\left(\alpha y \circ \overline{\alpha u} \circ \alpha x \circ \beta z\right) \tag{6}$$

or equivalently

$$\alpha x \circ \overline{\alpha u} \circ \alpha y = \alpha y \circ \overline{\alpha u} \circ \alpha x. \tag{7}$$

By replacing αx, αy and $\overline{\alpha u}$ by x, y, u respectively, we will have

$$x \circ u \circ y = y \circ u \circ x, \tag{8}$$

Hence it follows that group $Q(\circ)$ is abelian.

Conversely :

Let the quasi-group $Q(\cdot, \setminus, /)$ be isotopic to abelian group $Q(\circ)$, through isotopy $\gamma(x/y) = \alpha x \circ \beta y$. Then, we have $x/y = \gamma^{-1}(\alpha x \circ \beta y)$ and

$x \setminus y = \beta^{-1}(\overline{\alpha y} \circ \gamma x)$. Now, since the group $Q(\circ)$ is abelian, for each of $x, y, z, u \in Q$, we have

$$x \circ u \circ y \circ z = y \circ u \circ x \circ z. \qquad (I)$$

By substituting αx, αy, βz and $\overline{\alpha u}$ for x, y, z, u respectively, and acting by γ^{-1} on the two sides of (I), we obtain

$$\gamma^{-1}(\alpha x \circ \overline{\alpha u} \circ \alpha y \circ \beta z) = \gamma^{-1}(\alpha y \circ \overline{\alpha u} \circ \alpha x \circ \beta z). \qquad (II)$$

Therefore we can obtain the identity (a) from (II) .

Formulas (b) and (c), by isotopy $x/y = \gamma^{-1}(\alpha x \circ \beta y)$, have similar proofs.

Theorem 2.2. *The quasi-group $Q(\cdot, \setminus, /)$ is isotopic a group of exponent 2 if and only if in $Q(\cdot, \setminus, /)$ the following identity holds:*
$x \cdot [y \setminus (z \cdot u)] = x \cdot [z \setminus (y \cdot u)]$,

Proof : By the proof of Theorem 2.1, it is enough to consider isotopy $(/) \sim_{iso.} (\circ)$, such that $\gamma(x \cdot y) = \alpha x \circ \beta y$.

Theorem 2.3. *Quasi-group $Q(\cdot, \setminus, /)$ is isotopic to an Abelian group if and only if in $Q(\cdot, \setminus, /)$ any one of the following identities holds:*

(a) $x \cdot [y \setminus (z \cdot u)] = z \cdot [y \setminus (x \cdot u)]$.

(b) $[(x \cdot y)/z] \cdot u = [(x \cdot u)/z] \cdot y$.

(c) $(x/y) \cdot (z \setminus u) = (u/y) \cdot (z \setminus x)$

(d) $(x \cdot y)/(z \setminus u) = (z \cdot y)/(x \setminus u)$.

Proof: By the proof of Theorem 2.1, it is enough to consider isotopy $(\cdot) \sim_{iso.} (\circ)$, such that $\gamma(x/y) = \alpha x \circ \beta y$.

Theorem 2.4. *Quasi-group $Q(\cdot, \setminus, /)$ is isotopic to an Abelian group if and only if in $Q(\cdot, \setminus, /)$ any one of the following identities holds:*

(a) $(x \cdot y)/(z \setminus u) = (z \cdot y)/(x \setminus u)$.

(b) $x \setminus [y \cdot (z \setminus u)] = z \setminus [y \cdot (x \setminus u)]$.

Proof: By the proof of Theorem 2.1, it is enough to consider isotopy $(\setminus) \sim_{iso.} (\circ)$, such that $\gamma(x \setminus y) = \alpha x \circ \beta y$.

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