On Vertex-Distinguishing Proper Edge Colorings of Graphs Satisfying the Ore Condition *

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Abstract An edge coloring is *proper* if no two adjacent edges are assigned the same color and *vertex-distinguishing proper coloring* if it is proper and incident edge sets of every two distinct vertices are assigned different sets of colors. The minimum number of colors required for a vertex-distinguishing proper edge coloring of a simple graph G is denoted by $\tilde{\chi}'(G)$. In this paper, we prove that $\tilde{\chi}'(G) \leq \Delta(G) + 4$ if $G = (V, E)$ is a connected graph of order $n \geq 3$ and $\sigma_2(G) \geq n$, where $\sigma_2(G) = \min\{d(x) + d(y) | \ \forall \ xy \ \notin \ E(G)\}$.

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1 Introduction

All graphs considered in this paper are finite and simple, and we use the standard notation of graph theory. Definitions not given here can be found in [5]. Let $G = (V, E)$ be a graph of order n with the vertex set $V = V(G)$ and the edge set $E = E(G)$. We denote by $V_d(G)$ the set of the vertices of degree d in G and $n_d(G) = |V_d(G)|$. The degree of a vertex v in G is denoted by $d_G(v)$ or simply $d(v)$, and the maximum and minimum degree of G by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. A *k-edge coloring* $f : E \to [k]$ of a graph G is an assignment of k colors to the edges of G. Let $f(e)$ be the color of the edge e. Denote by $F(v) = \{f(e)|\ e = uv \in E(G)\}$ the multiset of colors assigned to the set of edges incident to v. The coloring f is *proper* if no two adjacent edges are assigned the same color and *vertex-distinguishing proper coloring* (abbreviated *VDP-coloring*) if it is proper and $F(u) \neq F(v)$ for any two distinct vertices u and v.

Observe that if G contains more than one isolated vertex or any isolated edges, then no edge coloring of G is VDP. The minimum number of colors required to find a VDP-coloring of a graph G without isolated edges and with at most one isolated vertex is called the *vertex-distinguishing proper edge-coloring number* (abbreviated *VDP-coloring number*) and denoted by $\widetilde{\chi}'(G)$.

The VDP-coloring has been considered in many papers. It was introduced and studied by Burris and Schelp in [6, 7] and, independently, as *observability* of a graph, by Černý *et al.* [8], Horňák and Soták [10, 11]. In [7, 10], the VDP-coloring is also computed for some families of graphs, such as complete graphs K_n, bipartite complete graphs $K_{m,n}$, paths P_n and cycles C_n. For example, they proved that

$$\widetilde{\chi}'(K_n) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n+1, & \text{if } n \text{ is even.} \end{cases}$$

and

$$\widetilde{\chi}'(K_{m,n}) = \begin{cases} n+1, & \text{if } n > m \geq 2; \\ n+2, & \text{if } n = m \geq 2. \end{cases}$$

The following result has been conjectured by Burris and Schelp [6, 7], and later proved by Bazgan *et al.* [3].

Theorem 1 [3]. A graph G on n vertices without isolated edges and with at most one isolated vertex has $\widetilde{\chi}'(G) \leq n+1$.

Obviously, the estimation of $\widetilde{\chi}'(G)$ in Theorem 1 cannot be improved in general as $\widetilde{\chi}'(K_n) = n+1$ when n is even. However, for some families of graphs, the VDP-coloring number is rather closer to the maximum degree than to the order of the graph. The following theorem in [4] is an example of such a situation.

Theorem 2 [4]. Let G be a graph of order $n \geq 3$ without isolated edges and with at most one isolated vertex. If $\delta(G) > \frac{n}{3}$, then $\widetilde{\chi}'(G) \leq \Delta(G)+5$.

Some other results about $\widetilde{\chi}'(G)$ can be found in [1, 2, 9].

By Vizing's theorem, *any simple graph G has a proper coloring with $\Delta(G)$ or $\Delta(G)+1$ colors*, we know $\widetilde{\chi}'(G) \geq \Delta(G)$. In this paper, we will give an upper bound of the VDP-coloring numbers of the graphs G satisfying the Ore condition (that is, $\sigma_2(G) = \min\{d(x)+d(y) | \forall\ xy \notin E(G)\} \geq n$, where n is the order of G), in terms of the maximum degree of G.

2 Main Results

First, we would like to give some additional notations and useful lemmas.

Given a proper edge coloring f of G, we denote by $B_f(v) = \{u|\ u \in V(G),\ F(u) = F(v)\}$. Observe that $v \in B_f(v)$. A *semi-VDP-coloring* is a proper edge coloring with $|B_f(v)| \leq 2$ for any vertex v of G.

Let $P_1, \ldots, P_k$ be a set of vertex disjoint paths. The set $\mathcal{P} = \{P_1, \ldots, P_k\}$ is called *a long path system* if $|V(P_i)| \geq 3$ for $i = 1, \ldots, k$. If the vertices of a graph G are covered by a long path system then $\mathcal{P}$ is called a long path covering of G.

The next two lemmas were proved by Bazgan *et al.* in [3, 4]. They are useful in the proof of Theorem 1 and Theorem 2 and also important to prove our main result.

Lemma 3 [3, 4]. Let G be a graph such that the inequality $d(k-d) \geq n_d(G) - 2$ holds for any integer d, $\delta(G) \leq d \leq \Delta(G)$, where $k \geq \Delta(G)+1$

is an integer. Then there exists a semi-VDP-coloring of G with k colors.

The following lemma allows us to transform a semi-VDP-coloring of a subgraph of G to a VDP-coloring of G that uses three extra colors.

Lemma 4 [3, 4]. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a long path covering of G. If there exists a semi-VDP-coloring of $G' = G - E(\mathcal{P})$ with k colors, then there exists a VDP-coloring of G with $k+3$ colors.

Now we will present our main result of this paper.

Theorem 5. Let G be a connected graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then $\tilde{\chi}'(G) \leq \Delta(G) + 4$.

Proof. Observe that if $3 \leq n \leq 6$, then by Theorem 1, we have $\tilde{\chi}'(G) \leq n+1 = \frac{n}{2} + \frac{n}{2} + 1 \leq \Delta(G) + 4$ since $\Delta(G) \geq \frac{n}{2}$ and $n \leq 6$. So the theorem holds for $n \leq 6$ and we may assume $n \geq 7$ in the following.

Since $\sigma_2(G) \geq n$, it is well-known that G contains a Hamiltonian path. Clearly, a Hamiltonian path P_n of G is itself a long path covering of G. Let $G' = G - E(P_n)$. We will show that G' has a semi-VDP-coloring that uses $k = \Delta(G) + 1$ colors. In order to use Lemma 3, we have to verify the following inequality

$$d'(k - d') \geq n_{d'}(G') - 2 \qquad (*)$$

for any d', $\delta' \leq d' \leq \Delta'$, where $\delta' = \delta(G')$ and $\Delta' = \Delta(G')$. We consider three cases.

Case 1. $d' = \Delta'$. If $\Delta' = \Delta - 1$, then $k - d' = 2$ and $(*)$ is implied by the inequality $2(\Delta - 1) \geq n - 2$ since $\Delta \geq \frac{n}{2}$. If $\Delta' = \Delta - 2$, then $k - d' = 3$. To get $(*)$, it suffices to verify that $3(\Delta - 2) \geq n - 2$. It holds for $n \geq 7$ as $\Delta \geq \lceil \frac{n}{2} \rceil$.

Case 2. $\delta' + 1 \leq d' \leq \Delta' - 1$. Then $k - d' = \Delta + 1 - d' \geq \Delta' + 2 - d' \geq 3$, and $n_{d'} - 2 \leq n - 4$. So $d'(k - d') \geq 3d' \geq 3(\delta' + 1) \geq 3(\delta - 1)$.

If $\delta \geq \frac{n-2}{2}$, then $d'(k - d') \geq 3(\delta - 1) \geq \frac{3}{2}n - 6 \geq n - 4 \geq n_{d'} - 2$ for $n \geq 4$.

In the other case, $\delta \leq \frac{n-3}{2}$. Since $\sigma_2(G) \geq n$, for a vertex $v \in V(G)$ such that $d(v) = \delta$ and for any vertex u not adjacent to v, $d(u) \geq \frac{n+3}{2}$. So the number of the vertices of degree less than $\frac{n+3}{2}$ in G is at most $\delta + 1$, and the number of the vertices of degree less than $\frac{n+3}{2} - 2 = \frac{n-1}{2}$ in G' is at most $\delta+1$, that is, $n_{d'} \le \delta+1$ for $\delta'+1 \le d' \le \frac{n-1}{2}$.

If $\frac{n}{2} \le d' \le \Delta'-1$, then $d'(k-d') \ge 3d' \ge \frac{3n}{2} > n-4 \ge n_{d'}-2$ for $n \ge 4$.

If $\delta'+1 \le d' \le \frac{n-1}{2}$, then $d'(k-d') \ge 3d' \ge 3(\delta-1) \ge 3(n_{d'}-2) > n_{d'}-2$.

Case 3. $d' = \delta'$. We can assume that G' is not regular in this case, otherwise, it comes to Case 1 and we are done. Then $k-d' \ge 3$, $n_{d'} \le n-1$, and $d' = \delta' \ge \delta-2$.

If $\delta \ge \frac{n}{2}$, then $d'(k-d') \ge 3d' \ge 3(\delta-2) \ge \frac{3n}{2}-6 \ge n-3 \ge n_{d'}-2$ for $n \ge 6$.

In the other case, $\delta < \frac{n}{2}$. Since $\sigma_2(G) \ge n$, $\Delta(G) \ge \frac{n+1}{2}$, and for all the vertices of G whose degrees are less than $\frac{n}{2}$, the subgraph induced by them is a clique in G, *i.e.*, a complete graph with at most $\delta+1$ vertices. Since G is connected, in the clique there is at least one vertex with degree greater than δ in G. So the number of the vertices in G with degree δ is at most δ.

Assume that $\delta' = \delta-2$. Then $n_{d'} \le \delta$, because the vertices of degree δ' can only be the vertices in G with degree δ. Thus $(*)$ holds since $d'(k-d') \ge 3d' = 3(\delta-2) > \delta-2 \ge n_{d'}-2$.

Assume that $\delta' = \delta-1$. If $\frac{n-2}{2} \le \delta < \frac{n}{2}$, then $d'(k-d') \ge 3d' = 3\delta' = 3(\delta-1) \ge \frac{3n}{2}-6 \ge n-3 \ge n_{d'}-2$ for $n \ge 6$. If $\delta \le \frac{n-3}{2}$, by a similar analysis as in Case 2, we have $n_{d'} \le \delta+1$ as $d' = \delta' = \delta-1 \le \frac{n-5}{2} < \frac{n-1}{2}$. Thus $(*)$ holds since $d'(k-d') \ge 3d' = 3\delta' = 3(\delta-1) \ge (\delta+1)-2 \ge n_{d'}-2$.

Now the inequality $(*)$ holds for $n \ge 7$, so G' has a semi-VDP-coloring using $k = \Delta(G)+1$ colors by Lemma 3. By Lemma 4 and the remark at the beginning of the proof, we have that $\tilde{\chi}'(G) \le \Delta(G)+4$ for $n \ge 3$. □

Since $\delta(G) \ge \frac{n}{2}$ implies that $\sigma_2(G) \ge n$, we can immediately get the following corollary by Theorem 5.

Corollary 6. Let G be a connected graph of order $n \ge 3$. If $\delta(G) \ge \frac{n}{2}$, then $\tilde{\chi}'(G) \le \Delta(G)+4$.

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