Enumeration of Unicursal Planar Near-Triangulation

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Abstract

A map is called Unicursal if it has exactly two vertices of odd valency. A near-triangulation is a map with all but one of its face triangles. We use the enufunction approach to enumerate rooted Unicursal planar near-triangulation with the valency of the root-face and the number of non-rooted faces as parameters.

Keywords: Unicursal, planar near-triangulation, enumerating function, functional equation.
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1 Introduction

Since the enumeration of rooted planar maps was initiated by W. T. Tutte[1] in the early of 1960's for attacking the Four Color Problem, the theory has been developed and further generalized by R. C.Mullin [2] , W. T. Tutte[3] himself, Liskovets V.A. and Walsh T.R.S.[4], Y. P. Liu [5~10] etc. Although the theory has been developed greatly for nearly fifty years [10] , the enumerative problem of many types of maps is unable to be solved and many enumerative results still need to be improved and generalized. Liu has developed his methods to count triangulations on the disk and some more general cases of triangulations respectively with simplification[10]. But there is no research work on enumeration of Unicursal planar near-triangulation. A map is called Unicursal if it

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has exactly two vertices of odd valency. A near-triangulation is a map with all but one of its face triangles. In this paper, we will try to enumerate Unicursal planar near-triangulation and give its functional equation. Maps in this paper are only considered to be planar and rooted. If the surface is the plane, or the sphere, the map is called a planar map. A map M is said to be rooted if an edge with a direction along the edge, and a side of the edge is distinguished. We denote the root-edge of M by $R(M)$ and its tail vertex is chosen to be the root-vertex of this map, the face on the right-hand side of the root-edge is called the root-face. Without loss of generality, the root-face may be chosen as the infinite face. Terminologies not explained here refer to [10].

2 Generating Functions

The way of decomposition is very closely related to the choice of parameters which the enumeration is according to. In this paper, we will decompose Unicursal planar near-triangulation maps and provide a form of functional equation of the enumerating function of rooted Unicursal planar near-triangulation maps with the valency of the root-face and the number of non-rooted faces as parameters .

Now, we introduce the enufunction for enumerating rooted Eulerian planar near-triangulation maps in $\mathcal{U}$ and rooted Unicursal planar near-triangulation maps in $\widetilde{\mathcal{U}}$, respectively, as follows

$$f = f_{\mathcal{U}}(x,y) = \sum_{U\in\mathcal{U}} x^{m(U)}y^{n(U)} \tag{1}$$

where $m(U)$ is the valency of root-face of $U \in \mathcal{U}$ and $n(U)$ is the number of non-rooted faces.

$$\widetilde{f} = f_{\widetilde{\mathcal{U}}}(x,y) = \sum_{\widetilde{U}\in\widetilde{\mathcal{U}}} x^{m(\widetilde{U})}y^{n(\widetilde{U})} \tag{2}$$

where $m(\widetilde{U})$ is the valency of root-face of $\widetilde{U} \in \widetilde{\mathcal{U}}$ and $n(\widetilde{U})$ is the number of non-rooted faces.

On the enumerating problem of rooted Eulerian planar near-triangulation maps with the face partition, Liu [10] investigated it and obtained a result as follows, which is is represented as formula (5.3.15) in [10]:

$$x^3yf^3 + 2y^2f^2 + ((\frac{y}{x})^3 - y^2 - 1)f + 1 - y^2 - (\frac{y}{x})^3(f^{\Delta} + 1) = 0. \tag{3}$$

where f^{Δ} is the coefficient of x^3 in f.

For $\widetilde{U} = (\mathcal{X}, \mathcal{J}) \in \widetilde{\mathcal{U}}$, its root edge is denoted by $a = Kr(\widetilde{U})$. Then $\widetilde{\mathcal{U}}$ is divided into three classes as

$$\widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}_1 + \widetilde{\mathcal{U}}_2 + \widetilde{\mathcal{U}}_3 \tag{4}$$

where $\widetilde{\mathcal{U}}_1$ contains all the maps of root-edge separable in $\widetilde{\mathcal{U}}$; $\widetilde{\mathcal{U}}_2$ contains all the maps of root-edge a loop in $\widetilde{\mathcal{U}}$; Of course, $\widetilde{\mathcal{U}}_3$ is with all the maps of root-edge nonseparable in $\widetilde{\mathcal{U}}$.

3 Some Lemmas

In this section, some useful lemmas will be obtained firstly.

Lemma 1 *Let* $\widetilde{\mathcal{U}}_{\langle 1\rangle} = \{\widetilde{U} - a|\forall \widetilde{U} \in \widetilde{\mathcal{U}}_1\}$, *then*

$$\widetilde{\mathcal{U}}_{\langle 1\rangle} = \mathcal{U} \times \mathcal{U} \tag{5}$$

where $\times$ *is the Cartesian production between two sets.*

Proof Because of the root-edge separable in $\widetilde{\mathcal{U}}_1$, each element of $\widetilde{\mathcal{U}}_{\langle 1\rangle}$ is a premap which consists of two submaps, easy to check that both in $\mathcal{U}$ and hence $\widetilde{\mathcal{U}}_{\langle 1\rangle} \subseteq \mathcal{U} \times \mathcal{U}$.

Conversely, for any two maps in $\mathcal{U}$, by adding a new edge as the root-edge separable, what obtained can be seen as uniquely a map in $\widetilde{\mathcal{U}}_{\langle 1\rangle}$ and hence $\widetilde{\mathcal{U}}_{\langle 1\rangle} \supseteq \mathcal{U} \times \mathcal{U}$. ♮

From Lemma1, the enufunction of $\mathcal{U}_1$ is

$$\widetilde{f}_1 = \sum_{\widetilde{U} \in \widetilde{\mathcal{U}}_1} x^{m(\widetilde{U})} y^{n(\widetilde{U})} = x^2 f^2 \tag{6}$$

Lemma 2 *Let* $\widetilde{\mathcal{U}}_{\langle 2\rangle} = \{\widetilde{U} - a|\forall \widetilde{U} \in \widetilde{\mathcal{U}}_2\}$, *then*

$$\widetilde{\mathcal{U}}_{\langle 2\rangle} = \widetilde{\mathcal{U}}(2) \odot \mathcal{U} \tag{7}$$

where $\widetilde{\mathcal{U}}(2) = \{\widetilde{U}|\forall \widetilde{U} \in \widetilde{\mathcal{U}}, m(\widetilde{U}) = 2\}$ *in which* $m(\widetilde{U})$ *is the valency of the root-face in* $\widetilde{U}$.

Proof For a map $\widetilde{U} \in \widetilde{\mathcal{U}}_{\langle 2\rangle}$, because there is a map $\widetilde{U}'$, such that $\widetilde{U} = \widetilde{U}' - a, a = e_r(\widetilde{U}')$, the root-edge of $\widetilde{U}'$, by considering the root-edge a as a loop we see that the inner and outer domains determine respectively two kinds of maps, one is in $\widetilde{\mathcal{U}}(2)$ and the other in $\mathcal{U}$. ♮

From Lemma2, the enufunction of $\mathcal{U}_2$ is

$$\widetilde{f}_2 = \sum_{\widetilde{U} \in \widetilde{\mathcal{U}}_2} x^{m(\widetilde{U})} y^{n(\widetilde{U})} = \frac{y}{x} \widetilde{f}^{(2)} f = \frac{y^2}{x^2} \widetilde{f}^{(3)} f \tag{8}$$

where $\widetilde{f}^{(i)} = \widetilde{f}_{\widetilde{\mathcal{U}}(i)}, i = 2, 3$.

For $\widetilde{U}_3 \in \widetilde{\mathcal{U}}_3$, let

$$\hat{U} = \widetilde{U}_3 - \{\mathcal{J}r, \mathcal{J}\alpha\beta(\mathcal{J}r), (\mathcal{J}\alpha\beta)^2(\mathcal{J}r)\}$$

be the map obtained by deleting the three edges incident with

$$\mathcal{J}r, \mathcal{J}\alpha\beta(\mathcal{J}r), (\mathcal{J}\alpha\beta)^2(\mathcal{J}r)$$

ie., those on the boundary of the nonrooted face which is incident with the root-edge, from $\widetilde{U}_2$. Then $\widetilde{\mathcal{U}}_3$ can be divided into three classes: $\widetilde{\mathcal{U}}_{3i}, i = 1,2,3$ such that

$$\widetilde{\mathcal{U}}_{3i} = \{\widetilde{U} | \forall \widetilde{U} \in \widetilde{\mathcal{U}}_3, \hat{U} \textit{ has } i \textit{ joints}\}$$

for a joint, ie., a connected component, $i = 1,2$, and 3. Thus,

$$\widetilde{\mathcal{U}}_3 = \sum_{i=1}^{3} \widetilde{\mathcal{U}}_{3i}.$$

Further, write

$$\hat{\mathcal{U}}_{3i} = \{\hat{U} | \forall \widetilde{U}_{3i} \in \widetilde{\mathcal{U}}_{3i}\}$$

for $i = 1,2,3$.

Lemma 3 *For* $\widetilde{\mathcal{U}}_{31}$, *we have*

$$\hat{\mathcal{U}}_{31} = \widetilde{\mathcal{U}} - \sum_{j=1}^{3} \widetilde{\mathcal{U}}(j) \tag{9}$$

where $\widetilde{\mathcal{U}}(j) = \{\widetilde{U} | \widetilde{U} \in \widetilde{\mathcal{U}}, m(\widetilde{U}) = j\}, j = 1,2,3$ *in which* $m(\widetilde{U})$ *is the valency of root-face in* $\widetilde{U}$.

Proof We see that the set on the right hand is the set of all rooted Unicursal planar near-triangulations with the outer face valency not being $1,2$ and 3, and hence we can get above result . ♮

We evaluate the contributions $\tilde{f}_{3i}$ of $\widetilde{\mathcal{U}}_{3i}$ to $\tilde{f}$ for $i = 1,2,3$ respectively.

For $\widetilde{\mathcal{U}}_{31}$, from (9) in lemma 3, we soon obtain that

$$\tilde{f}_{31} = x^{-3}y^3(\tilde{f} - \sum_{j=1}^{3} \tilde{f}^{(j)}) = x^{-3}y^3[\tilde{f} - \tilde{f}^{(3)}(\frac{y^2}{x^2} + \frac{y}{x} + 1)] \tag{10}$$

$\tilde{f}^{(j)} = \tilde{f}_{\mathcal{U}(j)}, j = 1,2,3$, the enufunction of rooted Unicursal planar near-triangulations with the number of nonrooted face as the parameter.

Let $\widetilde{U} \in \widetilde{\mathcal{U}}_{32}$ with root r. Write o, u and v as the vertices incident with $r, \beta r$ and $\beta\mathcal{J}r$ respectively. Of course, o is the root-vertex, or say the first vertex, and u is the nonrooted end of the root-edge, or say the second vertex, and v is called the third vertex of $\widetilde{U}$.

If the $i-$ vertex is connected to any of the other two in $\hat{U}$, then $\hat{U}$ is said to be $i-$disjoint. $\widetilde{\mathcal{U}}_{32}$ can be divided into 3 classes named by $\widetilde{\mathcal{U}}_{32}^{(j)}, j = 1,2,3$.

Lemma 4 *For $\widetilde{\mathcal{U}}_{32}$, we have*

$$\hat{\mathcal{U}}_{32}^{(1)} = \widetilde{\mathcal{U}} \times \mathcal{U} \tag{11}$$

$$\hat{\mathcal{U}}_{32}^{(2)} = \widetilde{\mathcal{U}} \times \mathcal{U} \tag{12}$$

$$\hat{\mathcal{U}}_{32}^{(3)} = \widetilde{\mathcal{U}} \tag{13}$$

Proof Because $\hat{U}$ has two components here such that one component of $\hat{U}$ is incident with only the first vertex, or the second vertex and the other is incident with the other two among the three vertices for $\widetilde{\mathcal{U}}_{32}$. The former is allowed to be any map in $\widetilde{\mathcal{U}}$ and the latter is a map in $\mathcal{U}$. Therefore , the former two formulae are true. (14) can be found by similar discussion. ♮

According to lemma 4 , we see that

$$f_{\widetilde{\mathcal{U}}_{32}}^{(1)} = y^2 \tilde{f} f \tag{14}$$

$$f_{\widetilde{\mathcal{U}}_{32}}^{(2)} = y^2 \tilde{f} f \tag{15}$$

$$f_{\widetilde{\mathcal{U}}_{32}}^{(3)} = y^2 \tilde{f} \tag{16}$$

Lemma 5 *For $\widetilde{\mathcal{U}}_{33}$, we have*

$$\hat{\mathcal{U}}_{33} = \widetilde{\mathcal{U}} \times \mathcal{U} \times \mathcal{U} \tag{17}$$

where$\times$ is the Cartesian production

Proof Because all the three of the first, the second and the third vertices are cut-vertices, each component of $\hat{U}, \widetilde{U} \in \widetilde{\mathcal{U}}_{33}$, is allowed to be map in $\widetilde{\mathcal{U}}$ and $\mathcal{U}$, respectively. The lemma follows. ♮

For $\widetilde{\mathcal{U}}_{33}$, from (18)in lemma 5 , we soon obtain that

$$\tilde{f}_{33} = x^3 y \tilde{f} f^2 \tag{18}$$

4 Main results

Theorem 1 *The equation about $\tilde{f} = \tilde{f}(x, y)$*

$$\tilde{f} = \frac{x^{-3}y^3\tilde{f}^{(3)}(x^{-2}y^2 + x^{-1}y + 1) - x^{-2}y^2\tilde{f}^{(3)}f - x^2 f^2}{x^{-3}y^3 + 2y^2 f + x^3 y f^2 + y^2 - 1} \tag{19}$$

is well defined in the ring $\mathcal{L}\{\mathcal{R}; x, y\}$, $\tilde{f}^{(3)}$ is the coefficient of x^3 in $\tilde{f}$. The solution of the equation is $\tilde{f} = f_{\widetilde{\mathcal{U}}}$.

Proof Because of (4) ,

$$\tilde{f}_{\tilde{\mathcal{U}}} = \tilde{f}_{\tilde{\mathcal{U}}_1} + \tilde{f}_{\tilde{\mathcal{U}}_2} + \tilde{f}_{\tilde{\mathcal{U}}_3}.$$

In virtue of (6),(8),(10),(11),(14)-(16),(18), we soon obtain that

$$\begin{aligned}\tilde{f} &= x^2 f^2 + \frac{y^2}{x^2}\tilde{f}^{(3)} f + x^{-3}y^3[\tilde{f} - \tilde{f}^{(3)}(\frac{y^2}{x^2} + \frac{y}{x} + 1)] \\ &+ 2y^2\tilde{f}f + y^2\tilde{f} + x^3y\tilde{f}f^2\end{aligned}$$

By grouping and rearranging its term, it is seen that $\tilde{f}$ is a solution of the equation. This is the second statement.

The first statement can be done from the well definedness of the equation system about coefficients obtained by equating terms on two sides of the equation when $\tilde{f}$ is in form as a power series of x and y. ♮

From the equation (19), we can see that enufunction of rooted Unicursal planar near-triangulation maps $\tilde{f}$ has a close relationship with rooted Eulerian planar near-triangulation maps f. The cubic equation (3) has no easy way to solve up until now. This means that further work can also be done on how to find an explicit solution of equation (19).

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