Upper Chromatic Number of Steiner Triple Systems [1]

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Abstract: The upper chromatic number $\overline{\chi}(\mathcal{H})$ of a C-hypergraph $\mathcal{H} = (X, \mathcal{C})$ is the maximum number of colors that can be assigned to the vertices of $\mathcal{H}$ in such a way that each $C \in \mathcal{C}$ contains at least a monochromatic pair of vertices. This paper gives an upper bound for the upper chromatic number of Stiener triple systems of order n and proved that it is best possible for any $n (\equiv 1$ or $3 (\bmod 6))$.

Keywords: C-hypergraph; upper chromatic number; Steiner triple systems

1. Introduction

A mixed hypergraph is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is a finite set of vertices, $|X| = n \geq 1$, and $\mathcal{C}$ and $\mathcal{D}$ are two arbitrary families of X. The elements of $\mathcal{D}$ are called D-edges, while those of $\mathcal{C}$ are called C-edges. The size of every D-edges and every C-edges is at least 2. The difference between C-edges and D-edges is in the requirements for a coloring.

Definition 1.1 (V. Voloshin [9]). A *strict k-coloring* of a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a coloring of vertices of X in such a way that the following conditions hold:

(1) any C-edge $C \in \mathcal{C}$ has at least two vertices of the same color;
(2) any D-edge $D \in \mathcal{D}$ has at least two vertices colored differently;
(3) all vertices are colored;
(4) the number of used colors is exactly k.

If $\mathcal{H}$ has a strict k-coloring, then we say that $\mathcal{H}$ is k-colorable.

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Definition 1.2 (V. Voloshin [9]). The maximum (minimum) k for which there exists a srtict k-coloring of a mixed hypergraph $\mathcal{H}$ is called the *upper (lower) chromatic number* of $\mathcal{H}$ and is denoted by $\overline{\chi}(\mathcal{H})$ ($\chi(\mathcal{H})$).

The classical weak vertex coloring of a hypergraph can be seen as a strict coloring of a mixed hypergraph in which there are only D-edges. In the language of mixed hypergraphs, classical hypergraphs are called D-hypergraphs, while mixed hypergraphs with only C-edges are called C-hypergraphs, mixed hypergraphs with $\mathcal{C} = \mathcal{D}$ are called *bi-hypergraphs* and the subsets of X are called *bi-edges*.

In a D-hypergraph, the lower chromatic number coincides with the (weak) chromatic number and the upper chromatic number trivially equals n. In a C-hypergraph, the lower chromatic number trivially equals 1 but the upper chromatic number represents a value that is hard to deternime. And in any coloring of a bi-hypergraph, each bi-edge is neither monochromatic nor polychromatic.

In this paper, we concentrate on the coloring theory of Steiner triple systems. A *Steiner system* $S(t, k, n)$ is a pair $(X, \mathcal{S})$ where X is an n-element set of vertices, and $\mathcal{S}$ is a family of k-element subsets of X (called *blocks*) such that any t distinct vertices of X appear together in precisely one block. A Steiner system with parameters $t = 2$ and $k = 3$ is called a *Steiner triple system* (denoted by STS(n)). It is well known that the condition $n \equiv 1$ or 3 (mod 6) is necessary and sufficient for the existence of an STS(n) . Systems of the type $S(3, 4, n)$ represent *Steiner quadruple systems,* or SQS(n), where the condition for existence is $n \equiv 2$ or 4 (mod 6).

When coloring a STS(n) in which each block is considered just as a C-edge, we have a C-hypergraph and we denote it by CSTS(n), in the case of SQS we denote it by CSQS(n). If, on the other hand, each block is assumed to be both a C-edge and a D-edge, we have a bi-hypergraph, or a BSTS(n) for triple systems and BSQS(n) for quadruple systems. We study CSTS only in this paper. The advanced results about BSTS, CSQS and BSQS were listed in [7].

It is easy to notice that the upper chromatic number of any CSTS(n) is no less than 3 if $n \geq 7$. This paper discusses the upper bound for the upper chromatic number of CSTS(n)s. Lorenzo Milazzo and Zsolt Tuza gave an upper bound $\lceil \log_2(n+1) \rceil$ and proved that it is best possible when $n = 2^k - 1$ (see [5]). This paper improves the upper bound as $\lfloor \log_2(n+1) \rfloor$ and proves that this new bound is best possible for any $n (\equiv 1$ or 3 (mod 6)).

2. Upper bound for upper chromatic number of CSTSs

In this section we will discuss the upper bound for upper chromatic numbers of CSTSs. Before our discussion we give two symbols which we will use in our discussion.

Let Y be a set of order n and K_n be the complete graph on Y. If n is even, then K_n has 1-factorizations, a 1-factorization of K_n is denoted by $OF(K_n)$; if n is odd, then K_n has no 1-factorization. In this case, let GK_n be the complete graph on $Y \cup \{y\}$ (where y is a vertex not in Y). If $\{F_1, F_2, \cdots, F_n\}$ is an $OF(GK_n)$, then omit y from every F_i, we can obtain a near-1-factorization of K_n, denoted by near-$OF(K_n)$. Clearly, every near-1-factor has an isolated vertex.

We also need the concept of isomorphic in our discussion. Two Steiner triple systems $(X_1, \mathcal{S}_1)$ and $(X_2, \mathcal{S}_2)$ are *isomorphic* if there exists a bijection between X_1 and X_2 that maps each block of $\mathcal{S}_1$ onto a block of $\mathcal{S}_2$ and vice versa.

The already known results about the upper bound for upper chromatic number of CSTSs are as follows.

Theorem 2.1(Milazzo-Tuza [5]). *Let $\mathcal{H}$ be a CSTS(n) with $n \leq 2^k - 1$. Then $\overline{\chi} \leq k$.*

Theorem 2.2(Milazzo-Tuza [5]). *If there exists a CSTS(n) with upper chromatic number k, then there exists a $CSTS(2n+1)$ with upper chromatic number $k+1$.*

Theorem 2.3(Milazzo-Tuza [6]). *Suppose that a colorable CSTS(n) of order $n < 2^k \times 10 - 1$ has a strict coloring with upper chromatic number $\overline{\chi}$ colors where the first three color classes are of cardinalities $n_1 = 1, n_2 = n_3 = 4$. Then $\overline{\chi} < k + 3$.*

The proofs of theorem 2.2 and theorem 2.3 are based on the following construction.

Let $\mathcal{H}_1 = (X_1, \mathcal{S}_1)$ be a CSTS(n). Take a set X_2 of vertices with $|X_2| = n+1$ and $X_1 \cap X_2 = \emptyset$. Recalling that $n+1$ is even, let $K_{n+1} = (X_2, \mathcal{F})$, $\mathcal{F} = \{F_1, F_2, ..., F_n\}$ be an $OF(K_{n+1})$. Define the collection $\mathcal{S}$ of triples on $X = X_1 \cup X_2$ in this way:

(1) every triple belonging to $\mathcal{S}_1$ belongs to $\mathcal{S}$, too;

(2) if $x_i \in X_1 (i = 1, 2, ..., n)$ and $y_1, y_2 \in X_2$, then $\{x_i, y_1, y_2\} \in \mathcal{S}$ if and only if $(y_1, y_2) \in F_i$.

It is easy to see that $\mathcal{H} = (X, \mathcal{S})$ is a CSTS($2n+1$) and if $\mathcal{H}_1$ is k-colorable, then $\mathcal{H}$ is $(k+1)$-colorable.

We call the above method of construction *Double-plus-one construction* for CSTSs, i.e., construct a CSTS($2n+1$) from a CSTS(n).

Theorem 2.4(Milazzo-Tuza [5]). *Let $\mathcal{H}$ be a* CSTS(n) *such that* $n \le 2^k - 1$ *and* $\overline{\chi}(\mathcal{H}) = k$. *Then* $n = 2^k - 1$.

Theorem 2.1 and theorem 2.4 show that an upper bound for the upper chromatic number of CSTS(n)s is $\lfloor \log_2(n+1) \rfloor$ (the upper bound given in [5] is $\lceil \log_2(n+1) \rceil$). The above two theorems together with Double-plus-one construction for CSTSs show that the upper bound is best possible when $n = 2^k - 1$. We will prove that it is also best possible for any other n, that is to say, for any n such that $2^k - 1 < n < 2^{k+1} - 1$ and $n \equiv 1$ or 3(mod 6), there exists a CSTS(n) with upper chromatic number k. We prove this by induction.

We first discuss the structure of positive integers $n \equiv 1$ or 3(mod 6)) and have the following propositions.

Proposition 2.1. *If a positive integer n satisfies that $n \equiv 1$ or 3(mod 6), then it also satisfies $n \equiv 1$ or 3 or 7 or 9(mod 12), and vice versa.*

The result of proposition 2.1 is easy to verify.

Proposition 2.2. *If a positive integer n satisfies that $n \equiv 1$ or 3(mod 6), then there exists a positive integer l such that $l \equiv 1$ or 3(mod 6) and $n = 2l + 1$ or $n = 2l + 7$.*

Proof. Assume that $n = 6k + 1$. Then

$$\begin{aligned}
&\text{if } k = 4m, \text{ then } n = 6 \cdot 4m + 1 = 2(12(m-1) + 9) + 7;\\
&\text{if } k = 4m+1, \text{ then } n = 6(4m+1) + 1 = 2(12m + 3) + 1;\\
&\text{if } k = 4m+2, \text{ then } n = 6(4m+2) + 1 = 2(12m + 3) + 7;\\
&\text{if } k = 4m+3, \text{ then } n = 6(4m+3) + 1 = 2(12m + 9) + 1.
\end{aligned}$$

Assume that $n = 6k + 3$. Then

$$\begin{aligned}
&\text{if } k = 4m, \text{ then } n = 6 \cdot 4m + 3 = 2(12m + 1) + 1;\\
&\text{if } k = 4m+1, \text{ then } n = 6(4m+1) + 3 = 2(12m + 1) + 7;\\
&\text{if } k = 4m+2, \text{ then } n = 6(4m+2) + 3 = 2(12m + 7) + 1;\\
&\text{if } k = 4m+3, \text{ then } n = 6(4m+3) + 3 = 2(12m + 7) + 7.
\end{aligned}$$

This completes the proof. □

According to the structure of n, since we already have *Double-plus-one construction* for CSTSs, we need to find a method to construct a CSTS($2n + 7$) from a CSTS(n), the following two results are about this method.

Lemma 2.1. *If $n \equiv 1$ or $9 (mod\ 12)$ and there exists a k-colorable* CSTS(n), *then there exists a $(k+1)$-colorable* CSTS$(2n+7)$.

Proof. Let $\mathcal{H} = (X, \mathcal{S})$ be a CSTS(n) with $X = \{x_1, x_2, \cdots, x_n\}$. Put $n + 7 = 2m$. Then m is even since $n \equiv 1$ or $9(\text{mod } 12)$. Let $Y = \{y_1, y_2, \cdots, y_m\}$ and $Z = \{z_1, z_2, \cdots, z_m\}$ such that $Y \cap Z = \emptyset$, $T = Y \cup Z$ and $T \cap X = \emptyset$. Let $K_m = (Y, \mathcal{F})$, $\mathcal{F} = \{F_1, F_2, \cdots, F_{m-1}\}$ be an $OF(K_m)$ containing two 1-factors (let these be , w.l.o.g., F_{m-2} and F_{m-1}) whose union is a Hamiltonian circuit (let, again w.l.o.g., this Hamiltonian circuit be $F_{m-2} \cup F_{m-1} = (y_1, y_2, \cdots, y_m, y_1)$). Such a 1-factorization is well known to exist (cf., e.g., [3,4]).

Let

$$\begin{aligned}
\mathcal{C} &= \{\{z_i, y_{i+3}, y_{i+4}\}, \{z_i, z_{i+1}, y_{i+2}\} : i = 1, 2, \cdots, m\},\\
\mathcal{D} &= \{\{x_i, y_p, y_q\}, \{x_i, z_p, z_q\} : (y_p, y_q) \in F_i, i = 1, 2, \cdots, m-3\},\\
\mathcal{E} &= \{\{x_{m-2+k}, y_j, z_{j+k}\} : j = 1, 2, \cdots, m;\ k = 0, 1, \cdots, m-5\}
\end{aligned}$$

(the subscripts of y's and z's are reduced modulo m to the range $\{1, 2, \cdots, m\}$ whenever necessary).

Put $X^* = X \cup T$ and $\mathcal{S}^* = \mathcal{S} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$. It is easily verified that $\mathcal{H}^* = (X^*, \mathcal{S}^*)$ is a CSTS$(2n+7)$ (cf. [3,4]).

Let c be a strict k-coloring of $\mathcal{H}$ with color classes $C_1, C_2, \cdots, C_k$. Then $C_1, C_2, \cdots, C_k, T$ is a strict $(k+1)$-coloring of $\mathcal{H}^*$. Therefore $\mathcal{H}^*$ is $(k+1)$-colorable. □

Lemma 2.2. *If $n \equiv 3$ or $7 (mod\ 12)$ and there exists a* CSTS(n) *with upper chromatic number k, then there exists a $(k+1)$-colorable* CSTS$(2n+7)$.

Proof. Let $\mathcal{H} = (X, \mathcal{S})$ be a CSTS(n) with $X = \{x_1, x_2, \cdots, x_n\}$. Put $n + 7 = 2m$ (then $m \equiv 1(\text{mod } 2)$). Let $Y = \{y_1, y_2, \cdots, y_m\}$ and $Z = \{z_1, z_2, \cdots, z_m\}$ such that $Y \cap Z = \emptyset$, $T = Y \cup Z$ and $T \cap X = \emptyset$. Let $K_m = (Y, \mathcal{F})$ be a near-$OF(K_m)$, $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ containing two near-1-factors (let these be, w.l.o.g., F_{m-1}, F_m) whose union is a Hamiltonian path (let, again w.l.o.g., this Hamiltonian path be $F_{m-1} \cup F_m = (y_1, y_2, \cdots, y_m)$). Such a near-$OF(K_m)$ is known to exist ([3]). Furthermore we may assume that the edge (y_1, y_m) belongs to the factor F_{m-2}.

Now let

$$\begin{aligned}
\mathcal{C} &= \{\{z_i, y_{i+3}, y_{i+4}\}, \{z_i, z_{i+1}, y_{i+2}\} : i = 1, 2, \cdots, m\},\\
\mathcal{D} &= \{\{x_i, y_p, y_q\}, \{x_i, z_p, z_q\} : (y_p, y_q) \in F_i,\ i = 1, 2, \cdots, m-3\},\\
\mathcal{D}' &= \{\{x_i, y_{j(i)}, z_{j(i)}\} : i = 1, 2, \cdots, m-2,\\
&\qquad y_{j(i)} \text{ is the isolated vertex of } F_i\},\\
\mathcal{D}'' &= \{\{x_{m-2}, y_p, y_q\}, \{x_{m-2}, z_p, z_q\} : (y_p, y_q) \in F_{m-2} \setminus (y_1, y_m)\}\\
&\quad \cup \{\{x_{m-2}, y_1, z_1\}, \{x_{m-2}, y_m, z_m\}\},\\
\mathcal{E} &= \{\{x_{m-2+k}, y_j, z_{j+k}\} : j = 1, 2, \cdots, m;\ k = 1, 2, \cdots, m-5\}
\end{aligned}$$

(the subscripts of y's and z's are reduced modulo m to the range $\{1, 2, \cdots, m\}$ whenever necessary).

Put $X^* = X \cup T$ and $\mathcal{S}^* = \mathcal{S} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{D}' \cup \mathcal{D}'' \cup \mathcal{E}$. It is again easily verified that $\mathcal{H}^* = (X^*, \mathcal{S}^*)$ is a CSTS$(2n + 7)$ (see [3]). Moreover, if c is a strict k-coloring of $\mathcal{H}$ with color classes $C_1, C_2, \cdots, C_k$, then $C_1, C_2, \cdots, C_k, T$ is a strict $(k + 1)$-coloring of $\mathcal{H}^*$. Therefore $\mathcal{H}^*$ is $(k + 1)$-colorable. □

We call the constructions in Lemma 2.1 and Lemma 2.2 *Double-plus-seven construction* for CSTSs.

Notice that for any $n \equiv 1$ or $3 (\mathrm{mod}\ 6)$ and $2^k - 1 \leq n < 2^{k+1} - 1$, we have $2^{k+1} - 1 \leq 2n + 1 < 2^{k+2} - 1$. Similarly, for any $n \equiv 1$ or $3 (\mathrm{mod}\ 6)$ and $2^k - 1 \leq n < 2^{k+1} - 3$, we have that $2^{k+1} - 1 \leq 2n + 7 < 2^{k+2} - 1$. Therefore, we get the following result.

Theorem 2.5. *For any n such that $n \equiv 1$ or $3 (\mathrm{mod}\ 6)$ and $2^k - 1 \leq n < 2^{k+1} - 3$, if there exists a* CSTS(n) *with upper chromatic number k, then there exists a* CSTS$(2n + 7)$ *with upper chromatic number $k + 1$.*

But if $n = 2^k - 3$, then $2^{k-1} - 1 < n < 2^k - 1$ and $2n + 7 = 2^{k+1} + 1 > 2^{k+1} - 1$. We can not get a CSTS$(2^{k+1} + 1)$ using Double-plus-one construction. If we construct a CSTS$(2^{k+1} + 1)$ from a CSTS$(2^k - 3)$ using *Double-plus-seven construction* , we only know that this CSTS$(2^{k+1} + 1)$ is k-colorable. But we hope to guarantee that it is $(k + 1)$-colorable. So we need to find other method to construct a CSTS$(2^{k+1} + 1)$ with upper chromatic number $k + 1$. Furthermore, we only need to consider the case of k being even, since we have the following result.

Lemma 2.3. *If $n = 2^{2k_1+1} - 3$, then $n \equiv 5 (\mathrm{mod}\ 6)$.*

Proof. We prove this result by induction on k_1. It is easy to notice that the conclusion is true for $k_1 = 1$. Consider that

$$\begin{aligned} n &= 2^{2(k_1+1)+1} - 3 = 2^{2k_1+1+2} - 3 \\ &= 4 \cdot 2^{2k_1+1} - 3 = 3 \cdot 2^{2k_1+1} + 2^{2k_1+1} - 3 \\ &= 6 \cdot 2^{2k_1} + (2^{2k_1+1} - 3), \end{aligned}$$

hence, if $2^{2k_1+1} - 3 \equiv 5 (\mathrm{mod}\ 6)$, we also have that $2^{2(k_1+1)+1} - 3 \equiv 5 (\mathrm{mod}\ 6)$. This completes the proof. □

If $n = 2^{2k_1} - 3$, then $2n + 7 = 2^{2k_1+1} + 1$. In the following we try to find other method to construct a CSTS$(2^{2k_1+1} + 1)$ with upper chromatic number $2k_1 + 1$.

Let us discuss another kind of constructions of $n = 2^{2k_1+1}+1$. If $n = 2^{2k_1+1}+1$, then $n = 2(2^{2k_1}-1)+3 = 2m+3$(where $m = 2^{2k_1}-1$). So we try to construct a CSTS$(2m+3)$ from a CSTS(m) when $m = 2^{2k_1}-1$. We do this by two steps. The first step is to construct a CSTS$(4v+3)$ from a CSTS(v) when $v = 2^{2p}-1$ (thus $4v+3 = 2^{2p+2}-1$). We get the following result.

Lemma 2.4. *For any positive integer p, we can get a* CSTS$(2^{2p}-1)$ $\mathcal{H}^* = (X^*, \mathcal{S}^*)$ *satisfying the following conditions:*

(1) *the upper chromatic number of* $\mathcal{H}^*$ *is* $2p$*;*

(2) X^* *can be parted into* $\frac{2^{2p}-1}{3}$ *parts such that every part contains three vertices and these three vertices form a triple in* $\mathcal{H}^*$.

Proof. We begin with a CSTS(15). Let

$X_3 = \{x_1, x_2, x_3\}$, $X_4^1 = \{a_1, a_2, a_3, a_4\}$, $X_4^2 = \{b_1, b_2, b_3, b_4\}$ and $X_4^3 = \{c_1, c_2, c_3, c_4\}$ such that $X_3 \cap X_4^i = \emptyset$, $X_4^i \cap X_4^j = \emptyset$, $i, j = 1, 2, 3$, $i \neq j$ and $X_7^1 = X_3 \cup X_4^1$, $X_7^2 = X_3 \cup X_4^2$, $X_7^3 = X_3 \cup X_4^3$. Let $\mathcal{H}_7^1 = (X_7^1, \mathcal{S}_7^1)$, $\mathcal{H}_7^2 = (X_7^2, \mathcal{S}_7^2)$ and $\mathcal{H}_7^3 = (X_7^3, \mathcal{S}_7^3)$ be three CSTS(7)s such that they are isomorphic with each other and $\{x_1, x_2, x_3\} \in \mathcal{S}_7^1 \cap \mathcal{S}_7^2 \cap \mathcal{S}_7^3$.

We need to construct the triples among X_4^1, X_4^2, X_4^3. Let

$$\begin{aligned} \mathcal{S}'_{15} = \; & \{\{a_1,b_1,c_1\},\{a_1,b_2,c_4\},\{a_1,b_3,c_2\},\{a_1,b_4,c_3\} \\ & \{a_2,b_1,c_3\},\{a_2,b_2,c_2\},\{a_2,b_3,c_4\},\{a_2,b_4,c_1\} \\ & \{a_3,b_1,c_4\},\{a_3,b_2,c_1\},\{a_3,b_3,c_3\},\{a_3,b_4,c_2\} \\ & \{a_4,b_1,c_2\},\{a_4,b_2,c_3\},\{a_4,b_3,c_1\},\{a_4,b_4,c_4\}\}. \end{aligned}$$

Let $X_{15} = X_7^1 \cup X_7^2 \cup X_7^3$ and $\mathcal{S}_{15} = \mathcal{S}_7^1 \cup \mathcal{S}_7^2 \cup \mathcal{S}_7^3 \cup \mathcal{S}'_{15}$. Then $\mathcal{H}_{15} = (X_{15}, \mathcal{S}_{15})$ is a CSTS(15). Since $C_1 = \{x_1\}, C_2 = \{x_2, x_3\}, C_3 = \{a_1, a_2, a_3, a_4\}, C_4 = \{b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\}$ can give a strict 4-coloring of $\mathcal{H}_{15}$ and $15 = 2^4 - 1$, we have that $\overline{\chi}(\mathcal{H}_{15}) = 4$.

Notice that $\{x_1, x_2, x_3\}, \{a_1, b_1, c_1\}, \{a_2, b_2, c_2\}, \{a_3, b_3, c_3\}, \{a_4, b_4, c_4\}$ are all in $\mathcal{S}_{15}$.

Similarly, suppose that $\mathcal{H} = (X, \mathcal{S})$ be a CSTS(v) such that $v = 2^{2p}-1 = 3q$ (it is ease to verify by induction that $v = 2^{2p}-1 \equiv 0 \pmod 3$), $\overline{\chi}(\mathcal{H}) = 2p$ and the vertex set X can be separated into q parts such that every part contains three vertices and these three vertices form a triple in $\mathcal{H}$. Then let Y, Z and U be three disjoint vertex sets of order $v+1$ such that $X \cap Y = X \cap Z = X \cap U = \emptyset$. Assume that $X = \{x_1, x_2, \cdots, x_v\}$, $Y = \{y_1, y_2, \cdots, y_{v+1}\}$, $Z = \{z_1, z_2, \cdots, z_{v+1}\}$, $U = \{u_1, u_2, \cdots, u_{v+1}\}$ and $\mathcal{H}_Y = (X \cup Y, \mathcal{S}_Y)$, $\mathcal{H}_Z = (X \cup Z, \mathcal{S}_Z)$, $\mathcal{H}_U = (X \cup U, \mathcal{S}_U)$ are three CSTS$(2v+1)$s constructed from $\mathcal{H}$ using Double-plus-one construction.

Now let

$$\begin{aligned} \mathcal{S}' = \; & \bigcup_{i=1}^{v+1}\{\{y_i, z_i, u_i\}\} \\ & \bigcup_{i=2}^{v}\{\{y_i, z_j, u_{i+j}\} : j < i\} \\ & \bigcup_{j=2}^{v-1}\{\{y_i, z_j, u_{i+j+1}\} : i < j\} \\ & \bigcup\{\{y_{v+1}, z_j, u_{j+1}\} : j = 1, 2, \cdots, v-1\} \\ & \bigcup\{\{y_i, z_v, u_{2i}\}, \{y_i, z_{v+1}, u_{2i+1}\} : i = 1, 2, \cdots, \tfrac{v+1}{2}\} \\ & \bigcup\{\{y_i, z_v, u_{2i+1}\}, \{y_i, z_{v+1}, u_{2i}\} : \tfrac{v+1}{2} + 1 \leq i \leq v+1\}. \end{aligned}$$

(the subscripts of z's and u's are reduced modulo $v+1$ to range $\{1, 2, \cdots, v+1\}$).

Let $X^* = X \cup Y \cup Z \cup U$ and $\mathcal{S}^* = \mathcal{S}_Y \cup \mathcal{S}_Z \cup \mathcal{S}_U \cup \mathcal{S}'$. Then $\mathcal{H}^* = (X^*, \mathcal{S}^*)$ is a CSTS($2^{2p+2} - 1$). From the construction of $\mathcal{H}^*$, we can see that the vertex set X^* of $\mathcal{H}^*$ can be separated into $q + 3q + 1 = 4q + 1$ parts such that every part contains three vertices and these three vertices form a triple in $\mathcal{H}^*$.

It is easy to find that if c is a strict $2p$-coloring of $\mathcal{H}$ with color classes $C_1, C_2, \cdots, C_{2p}$, then $C_1, C_2, \cdots, C_{2p}, Y, Z \cup U$ is a strict $(2p+2)$-coloring of $\mathcal{H}^*$, that is to say, $\mathcal{H}^*$ is $(2p+2)$-colorable. Furthermore, since $|X^*| = 2^{2p+2} - 1$, we have that $\overline{\chi}(\mathcal{H}^*) \leq 2p+2$. Therefore $\overline{\chi}(\mathcal{H}^*) = 2p+2$. □

We call the above method of constructions *Four-time-plus-three construction* for CSTSs. Based on this construction, we can obtain the following conclusion.

Lemma 2.5. *There exists a* CSTS($2^{2k+1}+1$) *with upper chromatic number* $2k+1$.

Proof. We prove this result by construction. Let $n = 2^{2k-1} - 1$ and $X = \{x_1, x_2, \cdots, x_n\}$. Suppose that $\mathcal{H} = (X, \mathcal{S})$ is a CSTS(n) constructed using *Four-time-plus-three construction* . Without loss of generality, assume that

$\mathcal{S}' = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \cdots, \{x_{n-2}, x_{n-1}, x_n\}\} \subseteq \mathcal{S}$.

Let $Y = \{y_1, y_2, \cdots, y_n\}$ and $Z = \{z_1, z_2, z_3\}$ such that $X \cap Y = \emptyset$, $X \cap Z = \emptyset$ and $Y \cap Z = \emptyset$. Define the collection $\mathcal{S}^*$ of triples on $X^* = X \cup Y \cup Z$ as follows.

(1) every triple belonging to $\mathcal{S}$ belongs to $\mathcal{S}^*$,too;

(2)$\{y_1, y_2, y_3\}, \{y_4, y_5, y_6\}, \cdots, \{y_{n-2}, y_{n-1}, y_n\} \in \mathcal{S}^*$ and $\{z_1, z_2, z_3\} \in \mathcal{S}^*$;

(3)$\{\{z_1, x_i, y_i\} : i = 1, 2, \cdots, n\} \subseteq \mathcal{S}^*$;

(4)$\{\{z_2, x_{3i+1}, y_{3i+2}\}, \{z_2, x_{3i+2}, y_{3i+3}\}, \{z_2, x_{3i+3}, y_{3i+1}\} : i = 0, 1, \cdots, m-1\} \subseteq \mathcal{S}^*$;

(5)$\{\{z_3, x_{3i+1}, y_{3i+3}\}, \{z_3, x_{3i+2}, y_{3i+1}\}, \{z_3, x_{3i+3}, y_{3i+2}\} : i = 0, 1, \cdots, m-1\} \subseteq \mathcal{S}^*$;

(6)$\{\{x_i,y_j,y_l\},\{x_j,y_i,y_l\},\{x_l,y_i,y_j\}\} \subseteq \mathcal{S}^*$ if and only if $\{x_i,x_j,x_l\} \in \mathcal{S}\backslash\mathcal{S}'$.

It is easy to see that $\mathcal{H}^* = (X^*,\mathcal{S}^*)$ is a CSTS$(2n+3)$ and $\mathcal{H}$ is its subsystem. Let c be a strict $2k$-coloring of $\mathcal{H}$ with color classes $C_1, C_2, \cdots, C_{2k}$. Then $C_1, C_2, \cdots, C_{2k}, Y \cup Z$ is a strict $(2k+1)$-coloring of $\mathcal{H}^*$. Furthermore, since $2n+3 = 2^{2k+1}+1 < 2^{2k+2}-1$, we have that $\overline{\chi}(\mathcal{H}^*) < 2k+2$. Therefore, we get that $\overline{\chi}(\mathcal{H}^*) = 2k+1$. □

We call the above method of construction *Double-plus-three construction* for CSTSs, i.e., construct a CSTS$(2n+3)$ from a CSTS(n) when $n = 2^{2k}-1 = 3m$.

From the above discussion, we can obtain our main result.

Theorem 6. *For any positive integer n such that $n \equiv 1$ or $3(mod\ 6)$, there exists a* CSTS(n) *with upper chromatic number $\lfloor log_2(n+1)\rfloor$.*

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