Multidecomposition of $K_n - F$ into Graph-Pairs of Order 5 where F is a Hamilton Cycle or an (almost) 1-Factor

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Keywords: Multidecomposition, Graph-pair.

Abstract

A *graph-pair of order t* is two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. Given a graph-pair (G, H), we say (G, H) divides some graph K if the edges of K can be partitioned into copies of G and H with at least one copy of G and at least one copy of H. We will refer to this partition as a (G, H)- *multidecomposition* of K.

In this paper, we consider the existence of multidecompositions of $K_n - F$ into graph-pairs of order 5 where F is a Hamiltonian cycle or (almost) 1-factor.

1 Introduction

The authors in [2, 3] defined the graph-pair (G, H) as a pair of non-isomorphic graphs on the same number of vertices, say m, such that $G \cup H \cong K_m$. The authors defined the (G, H)- multidecomposition of K_m as follows: given a graph-pair (G, H), we say the graph-pair (G, H) divides K_m if the edges of K_m can be partitioned into copies of G and H with at least one copy of G and at least one copy of H. They referred to this partition as a (G, H)- *multidecomposition*.

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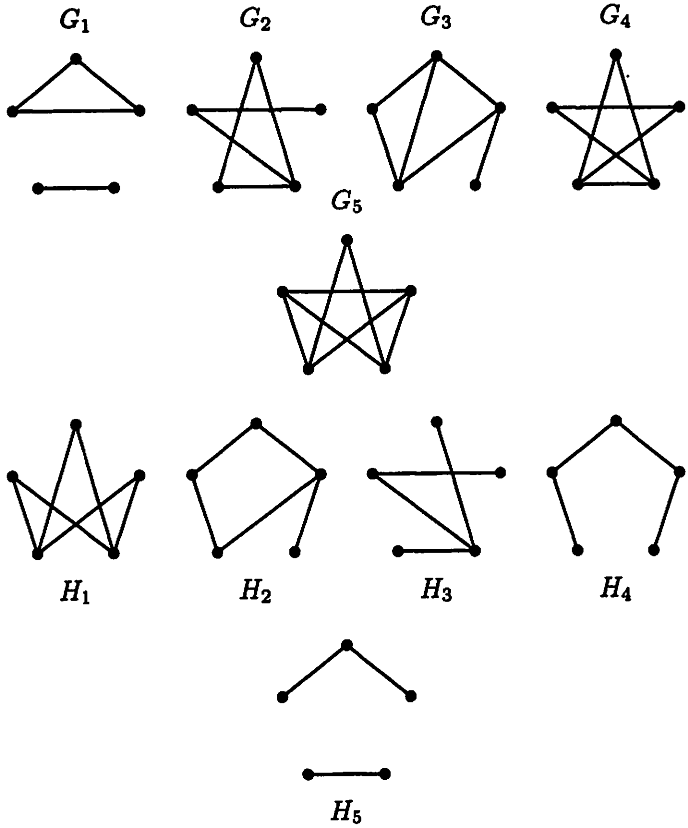


The five graph-pairs of order 5

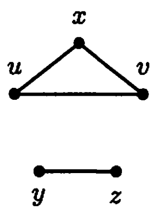
The authors in [2, 3, 4] settled the problem for the (G, H)- multidecomposition of λK_m for all the graph-pairs of order 4, and 5. We will consider simple graphs of the form $K_n - F$, where $K_n - F$ denotes the graph K_n from which a Hamiltonian cycle or (almost) 1-factor has been removed. An almost 1-factor on n vertices, where n is odd, is a 1-factor on $n-1$ vertices. In [1], the first author together with Clark and Leach settled the problem for the graph-pair of order 4. In this paper we will consider multidecompositions into the graph-pairs of order 5. For other graph-theoretic terminology used but not defined herein, see [5].

Let $[a, b] = \{x : a \leq x \leq b, x \in \mathbb{Z}\}$. We will also use $[a, f]$ to denote the set $\{a, b, c, d, e, f\}$. Let $[a, b; c, d]$ denote vertices of the bipartite graph consisting of the vertices grouped as $[a, b]$ and $[c, d]$. Let F be a Hamiltonian cycle, or (almost) 1-factor of order n. We will assume that vertices of F are labeled by $[0,1,...,n-1]$. For notional simplicity, we will use HC to denote a Hamilton cycle, and F_n to denote an (almost) 1-factor on n vertices.

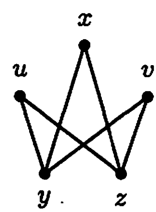
2 The First Graph Pair

We denote G_1 by $[(u,x,v)(y,z)]$ and H_1 by $[x,y,z,u,v]$.

$G_1 \cong$



$H_1 \cong$



We need the following lemma for the general theorems:

Lemma 2.1 *There exists an H_1 decomposition of $K_{3,4}$ and $K_{n,6}$ where $n \geq 2$*

Proof: Let [0,2;3,6] be a labeling of $K_{3,4}$. The following is an H_1 decomposition of $K_{3,4}$:
$H_1 \cong$ [0,3,4,1,2], [0,5,6,1,2]

For any $n \geq 2$, then n can be written as $n = 2a + 3b$ where a and b are not both zero. Since $H_1 \equiv K_{2,3}$, we can decompose $K_{2a+3b,6}$ into $2a + 3b$ copies of H_1. ∎

We are ready for the following:

Theorem 2.1 *There exists a (G_1, H_1)- multidecomposition of $K_n - HC$ if and only if $n \equiv 0$ or 3 mod 4 and $n \geq 11$.*

Proof: Since $|E(K_n - HC)|$ has $\frac{n(n-3)}{2}$ edges and since G_1 and H_1 each has an even number of edges; $n \equiv 0$ or 3 mod 4.

Let [0,6] be a labeling of the vertices of $K_7 - HC$. Since $K_7 - HC$ has 14 edges, a (G_1, H_1)- multidecomposition of $K_7 - HC$ must consist of 1 copy of H_1 and 2 copies of G_1. Because we only have 7 vertices and we cannot place any edges between adjacent vertices, when we place a copy of H_1 we must have the 3 vertices of degree 2 adjacent to each other and the 2 vertices of degree 3 adjacent to each other. Let us assume that we place a copy of H_1 at [5,0,1,3,4]. Then vertices 0 and 1 will each have degree 3. Since there is only one edge left incident to each of them, this means that {0,2} and {1,6} will need to be the loose edges on the copies of G_1. Any C_3 formed from the remaining edges must then include {1,5}, which has already been used.

Let [0,7] be a labeling of the vertices of $K_8 - HC$. Since $K_8 - HC$ has 20 edges, a (G_1, H_1)- multidecomposition of $K_8 - HC$ would need to consist of 2 copies of H_1 and 2 copies of G_1. After placing the first copy of H_1 at [3,0,1,5,6], our choices for placing the second copy of H_1 are [0,2,3,6,7], [0,2,4,6,7], [2,6,7,3,4]. Placing the H_1 copy on [0,2,4,6,7] or [2,6,7,3,4] leaves at most 1 C_3 in the remaining graph, and placing the second copy of H_1 on the vertices [0,2,3,6,7] leaves 2 C_3's, but requires that one of the loose edges of one copy of G_1 be incident to vertex 4, which is part of both C_3's.

Hence there is no (G_1, H_1)- multidecomposition of $K_7 - HC$ or $K_8 - HC$.

Let [0,10] be a labeling of the vertices of $K_{11} - HC$. The following is a (G_1, H_1)- multidecomposition of $K_{11} - HC$:

$$\begin{aligned} G_1 &\cong [(0,2,9)(3,7)], [(0,3,8)(1,10)], [(1,3,9)(2,7)], \\ &\quad [(1,5,8)(2,10)], [(2,4,8)(1,6)], [(3,5,10)(4,6)], \\ &\quad [(4,7,9)(3,6)], [(6,8,10)(5,7)] \\ H_1 &\cong [0,4,7,1,10], [0,5,6,2,9] \end{aligned}$$

Let [0,11] be a labeling of the vertices of $K_{12} - HC$. The following is a (G_1, H_1)- multidecomposition of $K_{12} - HC$:

$$\begin{aligned} G_1 &\cong [(2,4,8)(3,9)], [(2,5,9)(8,11)], [(3,5,7)(6,11)], \\ &\quad [(3,6,8)(4,7)], [(4,6,9)(5,8)], [(7,9,11)(8,10)] \\ H_1 &\cong [2,0,10,6,7], [3,0,1,4,5], [3,10,11,4,5], \\ &\quad [8,0,1,9,10], [6,1,2,7,11] \end{aligned}$$

We will prove the general case using a recursive construction. If $n \geq 11$, then $n = 3k + 4 + t$, $t \in \{6, 7, 8\}$, $k \in \mathbb{Z}$. Let $[0, n-5]$ be a labeling of the vertices of $K_{n-4} - HC$ that has a (G_1, H_1)- multidecomposition. Next we show how to find a (G_1, H_1)- multidecomposition of $K_n - HC$ on vertex set $[0, n-5] \cup [a, d]$ with the Hamiltonian Cycle being $[0, a, 1, b, 2, c, 3, d, 4, ..., n-5]$. Note that the (G_1, H_1)- multidecomposition of $K_{n-4} - HC$ does not include the edges {0,1}, {1,2}, {2,3}, and {3,4} as these are part of the Hamiltonian cycle on $n-4$ vertices. If $n \equiv 1 \bmod 3$ (so $t = 6$), then we use the following multidecomposition on the remaining edges involving vertices [0,5] and [a,d]:

$$\begin{aligned} G_1 &\cong [(1,c,d)(5,a)], [(2,a,3)(b,c)], [(3,b,4)(2,d)], \\ &\quad [(a,4,c)(0,1)], [(a,b,d)(1,2)] \\ H_1 &\cong [b,0,5,c,d] \end{aligned}$$

If $n \equiv 2 \bmod 3$ (so $t = 7$), then we use the following multidecomposition on the remaining edges involving vertices [0,6] and $[a, d]$:

$$\begin{aligned} G_1 &\cong [(0,1,c)(a,d)], [(0,b,d)(a,c)], [(1,2,d)(b,c)], \\ &\quad [(2,3,a)(6,d)], [(5,a,b)(3,4)], [(5,c,d)(3,b)] \\ H_1 &\cong [a,4,6,b,c] \end{aligned}$$

If $n \equiv 0 \bmod 3$ (so $t = 8$), then we use the following multidecomposition on the remaining edges involving vertices [0,7] and $[a, d]$:

$$\begin{aligned} G_1 &\cong [(0,1,c)(a,d)], [(0,b,d)(a,c)], [(1,2,d)(7,a)], [(2,3,a)(6,d)], \\ &\quad [(5,a,b)(7,d)], [(5,c,d)(3,b)], [(7,b,c)(3,4)] \\ H_1 &\cong [a,4,6,b,c] \end{aligned}$$

The remaining edges are between $[t, n-5]$ and $[a, d]$ which can then be partitioned using Lemma 2.1 into copies of H_1.

Note that the construction does not include edges $\{0,a\}, \{a,1\}, \{1,b\}, \{b,2\}, \{2,c\}, \{c,3\}, \{3,d\}$, and $\{d,4\}$ as these are used to change the Hamiltonian cycle on $n-4$ vertices to the Hamiltonian cycle on n vertices: $[0,a,1,b,2,c,3,d,4,\ldots,n-5]$. ∎

Theorem 2.2 *There exists a (G_1, H_1)- multidecomposition of $K_n - F_n$ if and only if $n \geq 7$.*

Proof: Let [0,6] be a labeling of the vertices of $K_7 - F_7$. The following is a (G_1, H_1)- multidecomposition of $K_7 - F_7$:

$$\begin{aligned} G_1 &\cong [(1,3,4)(2,5)], [(2,4,6)(0,3)], [(3,5,6)(0,4)] \\ H_1 &\cong [2,0,1,5,6] \end{aligned}$$

Let [0,7] be a labeling of the vertices of $K_8 - F_8$. The following is a (G_1, H_1)-multidecomposition of $K_8 - F_8$:

$$\begin{aligned} G_1 &\cong [(0,2,4)(1,3)], [(0,3,5)(1,4)], [(1,2,5)(3,4)] \\ H_1 &\cong [0,6,7,1,2], [3,6,7,4,5] \end{aligned}$$

Let [0,8] be a labeling of the vertices of $K_9 - F_9$. The following is a (G_1, H_1)-multidecomposition of $K_9 - F_9$:

$$\begin{aligned} G_1 &\cong [(0,2,4)(5,6)], [(0,3,5)(1,4)], [(1,2,5)(4,7)], \\ &\quad [(4,6,8)(1,3)], [(5,7,8)(3,4)] \\ H_1 &\cong [6,0,3,7,8], [6,1,2,7,8] \end{aligned}$$

Let [0,9] be a labeling of the vertices of $K_{10} - F_{10}$. The following is a (G_1, H_1)- multidecomposition of $K_{10} - F_{10}$:

$$\begin{aligned} G_1 &\cong [(0,2,4)(1,6)], [(0,3,5)(2,6)], [(1,2,5)(0,6)], [(1,3,4)(2,7)] \\ H_1 &\cong [2,8,9,3,4], [3,6,7,4,5], [5,8,9,6,7], [7,0,1,8,9] \end{aligned}$$

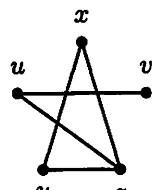
Let [0,10] be a labeling of the vertices of $K_{11} - F_{11}$. The following is a (G_1, H_1)- multidecomposition of $K_{11} - F_{11}$:

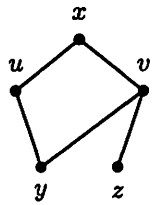
$$\begin{aligned} G_1 &\cong [(0,2,4)(5,6)], [(0,3,5)(1,4)], [(1,2,5)(4,7)], \\ &\quad [(4,6,10)(1,3)], [(5,7,10)(3,4)] \\ H_1 &\cong [0,8,9,1,2], [3,8,9,4,5], [6,0,3,7,10], \\ &\quad [6,1,2,7,10], [6,8,9,7,10] \end{aligned}$$

For $n \geq 12$, we will find the (G_1, H_1)- multidecomposition recursively. We place the following G_1 decomposition of $K_6 - F_6$ on vertices [0,5]; $G_1 \cong$ [(0,2,4)(1,3)], [(0,3,5)(1,4)], [(1,2,5)(3,4)], and we place a $K_{n-6} - F_{n-6}$ design on vertices [6,n-1]. Finally, apply lemma 2.1 to partition the edges in the bipartite graph on the vertices [0,5;6,$n-1$] into copies of H_1. ∎

3 The Second Graph Pair

We denote both G_2 and H_2 by $[x, y, z, u, v]$.

$G_2 \cong$ 

$H_2 \cong$ 

We need the following lemma for the general theorems:

Lemma 3.1 *There is an H_2 decomposition of $K_{2,5}$ and $K_{n,10}$ for $n \geq 4$.*

Proof: Let [0,1;2,6] be a labeling of $K_{2,5}$. The following is an H_2 decomposition of $K_{2,5}$:
$H_2 \cong [2,3,6,0,1], [5,4,6,1,0]$

Note that any $n \geq 4$, can be expressed as $2a + 5b$ for some $a, b \in \mathbb{Z}$ and both $K_{2,10}$ and $K_{5,10}$ can be decomposed into copies of $K_{2,5}$. ∎

We are ready for the following:

Theorem 3.1 *There exists a (G_2, H_2)- multidecomposition of $K_n - HC$ if and only if $n \equiv 0$ or $3 \bmod 5$ and $n \geq 8$.*

Proof: Since G_2 and H_2 each has five edges and $K_n - HC$ has $\frac{n(n-3)}{2}$ edges, then $n \equiv 0$ or 3 mod 5.

Let [0,7] be a labeling of the vertices of $K_8 - HC$. The following is a (G_2, H_2)- multidecomposition of $K_8 - HC$:

$$\begin{aligned} G_2 &\cong [0,2,4,1,3], [3,5,0,6,2] \\ H_2 &\cong [3,4,5,6,7], [5,7,6,2,1] \end{aligned}$$

Let [0,9] be a labeling of the vertices of $K_{10} - HC$. The following is a (G_2, H_2)- multidecomposition of $K_{10} - HC$:

$$\begin{aligned} G_2 &\cong [0,2,5,1,6] \\ H_2 &\cong [2,3,8,7,6], [3,4,7,0,1], [4,1,5,8,9], \\ &\quad [7,6,2,9,4], [8,7,6,5,0], [8,9,5,2,3] \end{aligned}$$

If $n \geq 11$, then $n = 2k + 5 + t, t \in \{6, 7\}$, $k \in \mathbb{Z}$. We are going to use a recursive construction, so let $[0, n-6]$ be a labeling of the vertices of $K_{n-5} - HC$ that has a (G_2, H_2)- multidecomposition. Next we show how to find a (G_2, H_2)- multidecomposition of $K_n - HC'$ with the new Hamiltonian cycle HC' being $[0, a, 1, b, 2, c, 3, d, 4, e, 5, ..., n-6]$. Note that the (G_2, H_2)- multidecomposition of $K_{n-5} - HC$ does not include the edges {0,1}, {1,2}, {2,3}, {3,4}, and {4,5} as these are part of the Hamiltonian cycle on $n - 5$ vertices.

If n is odd (so $t = 6$), use the following multidecomposition on the edges involving vertices [0,5] and $[a, e]$:

$$\begin{aligned} G_2 &\cong [0,d,e,3,4], [1,d,c,e,2], [a,b,e,1,2] \\ H_2 &\cong [b,a,4,d,c], [b,a,5,3,4], [c,b,1,5,0], [d,a,3,5,2] \end{aligned}$$

If n is even (so $t = 7$), use the following multidecomposition on the edges involving vertices [0,6] and $[a, e]$:

$$\begin{aligned} G_2 &\cong [1,d,c,4,5], [6,d,e,0,1], [a,3,4,b,c], [a,b,6,c,e], [e,1,2,3,b] \\ H_2 &\cong [2,5,c,d,a], [b,a,3,d,e], [c,b,d,5,0] \end{aligned}$$

The remaining edges are between $[t, n-6]$ and $[a, e]$, and can be partitioned into copies of H_2 using Lemma 3.1.

Note that the construction does not include edges $\{0, a\}, \{a, 1\}, \{1, b\}, \{b, 2\}, \{2, c\}, \{c, 3\}, \{3, d\}, \{d, 4\}, \{4, e\}$, and $\{e, 5\}$ as these are used to change the Hamiltonian cycle on $n-5$ vertices to the Hamiltonian cycle on n vertices: $[0, a, 1, b, 2, c, 3, d, 4, e, 5, ..., n-6]$. ∎

Theorem 3.2 *There exists a (G_2, H_2)- multidecomposition of $K_n - F_n$ if and only if $n \equiv 0$, 1, or 2 mod 10 and $n \geq 10$.*

Proof: Since $|E(K_n - F_n)|$ is either $\frac{n(n-2)}{2}$ if n is even or $\frac{(n-1)^2}{2}$ if n is odd, and since G_2 and H_2 have 5 edges; $n \equiv 0$, 1, or 2 mod 10.

Let [0,9] be a labeling of $K_{10} - F_{10}$. The following is a (G_2, H_2)- multidecomposition of $K_{10} - F_{10}$

$$\begin{aligned} G_2 &\cong [0,3,5,7,8],[0,4,2,9,7],[1,2,5,6,3],[1,3,4,9,6],[2,6,8,5,9] \\ H_2 &\cong [0,1,3,8,9],[0,1,4,7,6],[3,4,2,8,7] \end{aligned}$$

Let [0,10] be a labeling of $K_{11} - F_{11}$. The following is a (G_2, H_2)- multidecomposition of $K_{11} - F_{11}$

$$\begin{aligned} G_2 &\cong [0,3,7,2,6],[0,8,5,1,3],[1,4,8,6,9], \\ &\quad [2,5,9,4,10],[3,6,10,8,7],[7,9,10,5,3] \\ H_2 &\cong [0,1,3,10,9],[0,1,10,6,2],[2,3,0,8,4],[4,5,1,6,7] \end{aligned}$$

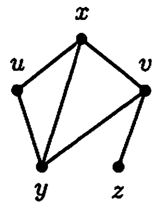
Let [0,11] be a labeling of $K_{12} - F_{12}$. The following is a (G_2, H_2)- multidecomposition of $K_{12} - F_{12}$

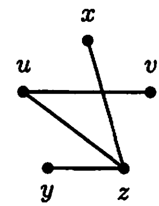
$$\begin{aligned} G_2 &\cong [0,3,5,7,8],[0,4,2,9,7],[1,2,5,6,3],[1,3,4,9,6],[2,6,8,5,9] \\ H_2 &\cong [0,1,2,10,11],[0,1,3,8,9],[0,1,4,7,6],[3,4,2,8,7] \\ &\quad [3,4,2,11,10],[5,6,7,10,11],[8,9,7,11,10] \end{aligned}$$

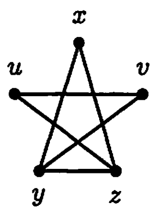
Let $[0,n-1]$ be a labeling of $K_n - F_n$. Again we are using a recursive construction to find a (G_2, H_2)- multidecomposition of $K_n - F_n$. Place a $K_{10} - F_{10}$ (G_2, H_2)- multidecomposition on [0,9], a (G_2, H_2)- multidecomposition on $K_{n-10} - F_{n-10}$ on the remaining vertices labeled $[10, n-1]$, and use Lemma 3.1 to finish the edges in the bipartite graph with vertices labeled $[0,9;10,n-1]$. ∎

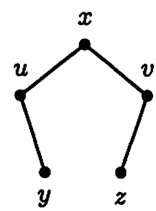
4 The Third and Fourth Graph Pairs

We denote G_3, H_3, G_4, and H_4 by $[x, y, z, u, v]$.

$G_3 \cong$ 

$H_3 \cong$ 

$G_4 \cong$ 

$H_4 \cong$ 

We need the following lemma for the general case:

Lemma 4.1 *There exists an H_i decomposition of $K_{n,4}$, where $i \in \{3,4\}$ and $n \geq 2$.*

Proof: For any $n \geq 2$, n can be written as $n = 2a + 3b$ where a and b are not both zero. Thus it is sufficient to show that there exists an H_3 decomposition of $K_{2,4}$ and $K_{3,4}$ and an H_4 decomposition of $K_{2,4}$ and $K_{3,4}$.

Let [0,1;2,5] be a labeling of the vertices of $K_{2,4}$. The following is an H_3 decomposition of $K_{2,4}$:
$H_3 \cong$ [2,3,0,4,1],[2,3,1,5,0]

The following is an H_4 decomposition of $K_{2,4}$:
$H_4 \cong$ [2,5,3,0,1], [4,3,5,0,1]

Let [0,2;3,6] be a labeling of the vertices of $K_{3,4}$. The following is an H_3 decomposition of $K_{3,4}$:
$H_3 \cong$ [0,1,3,2,4], [0,2,5,1,4], [1,2,6,0,4]

The following is an H_4 decomposition of $K_{3,4}$:
$H_4 \cong$ [4,5,3,2,0], [5,6,3,0,1], [6,4,3,1,2] ∎

We are ready for the following:

Theorem 4.1 *For $i \in \{3,4\}$, there exists a (G_i, H_i)- multidecomposition of $K_n - HC$ if and only if $n \equiv 0$ or $3 \bmod 4$ and $n \geq 7$.*

Proof: Since $K_n - HC$ has $\frac{n(n-3)}{2}$ edges, and since each of G_3, G_4, H_3, and H_4 has an even number of edges, $n \equiv 0$ or 3 mod 4.

Let [0,6] be a labeling of the vertices of $K_7 - HC$. The following is a (G_3, H_3)- multidecomposition of $K_7 - HC$:

$$\begin{aligned} G_3 &\cong [5,3,6,0,1] \\ H_3 &\cong [2,1,4,6,3],[6,5,2,0,4] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_7 - HC$:

$$\begin{aligned} G_4 &\cong [4,0,2,5,3] \\ H_4 &\cong [1,0,3,5,6],[4,2,3,6,1] \end{aligned}$$

Let [0,7] be a labeling of the vertices of $K_8 - HC$. The following is a (G_3, H_3)- multidecomposition of $K_8 - HC$:

$$\begin{aligned} G_3 &\cong [0,2,1,6,4],[1,3,2,7,5] \\ H_3 &\cong [1,4,6,3,0],[2,4,7,5,0] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_8 - HC$:

$$\begin{aligned} G_4 &\cong [0,2,4,1,6],[2,5,7,3,0] \\ H_4 &\cong [3,0,1,6,5],[7,3,6,1,4] \end{aligned}$$

Let $i \in \{3,4\}$ and let $[0, n-5]$ be a labeling of the vertices of $K_{n-4} - HC$ that has a (G_i, H_i)- multidecomposition. Next we show how to find a (G_i, H_i)- multidecomposition of $K_n - HC$ with the Hamiltonian Cycle being $[0, a, 1, b, 2, c, 3, d, 4, ..., n-5]$. Note that the (G_i, H_i)- multidecomposition of $K_{n-4} - HC$ does not include the edges {0,1}, {1,2}, {2,3}, and {3,4} as these are part of the Hamiltonian cycle on $n-4$ vertices. The proof will be using a recursive construction.

The following is a (G_3, H_3)- multidecomposition on the edges involving vertices $[0,4]$ and $[a,d]$:

$$\begin{aligned} G_3 &\cong [0,1,4,d,c] \\ H_3 &\cong [4,a,3,b,0],[4,c,b,a,d],[b,c,d,2,3],[c,4,a,2,1] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition on the edges involving vertices $[0,4]$ and $[a,d]$:

$$\begin{aligned} G_4 &\cong [d,1,2,a,c] \\ H_4 &\cong [4,2,b,3,c],[a,b,c,3,d],[b,1,a,0,4],[d,a,c,b,0] \end{aligned}$$

The remaining edges are between $[5, n-4]$ and $[a, d]$ which can then be partitioned into copies of H_3 and H_4 respectively using Lemma 4.1. ∎

Note that the construction does not include edges $\{0,a\},\{a,1\},\{1,b\},\{b,2\},$ $\{2,c\},\{c,3\},\{3,d\}$, and $\{d,4\}$ as these are used to change the Hamiltonian cycle on $n-4$ vertices to the Hamiltonian cycle on n vertices: $[0, a, 1, b, 2, c, 3, d, 4, ..., n-5]$.

Theorem 4.2 *If $i \in \{3,4\}$, there exists a (G_i, H_i)- multidecomposition of $K_n - F_n$ for any $n \geq 7$.*

Proof: Let [0,6] be a labeling of the vertices of $K_7 - F_7$. The following is a (G_3, H_3)- multidecomposition of $K_7 - F_7$:

$$\begin{aligned} G_3 &\cong [0,2,6,4,5] \\ H_3 &\cong [0,4,3,1,6],[0,4,6,3,5],[4,5,1,2,6] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_7 - F_7$:

$$\begin{aligned} G_4 &\cong [0,2,4,1,5] \\ H_4 &\cong [3,0,1,5,6],[4,1,5,3,6],[6,1,3,2,0] \end{aligned}$$

Let [0,7] be a labeling of the vertices of $K_8 - F_8$. The following is a (G_3, H_3)- multidecomposition of $K_8 - F_8$:

$$\begin{aligned} G_3 &\cong [0,2,6,4,5],[1,3,0,6,7] \\ H_3 &\cong [0,4,6,2,1],[0,5,3,4,1],[2,4,7,5,1] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_8 - F_8$:

$$\begin{aligned} G_4 &\cong [0,2,4,1,6],[1,3,5,2,7] \\ H_4 &\cong [0,3,5,6,7],[0,4,6,3,5],[7,2,6,1,4] \end{aligned}$$

Let [0,8] be a labeling of the vertices of $K_9 - F_9$. The following is a (G_3, H_3)- multidecomposition of $K_9 - F_9$:

$$\begin{aligned} G_3 &\cong [0,2,6,4,5],[1,2,8,6,7] \\ H_3 &\cong [0,5,7,4,3],[1,2,8,0,6],[3,5,1,4,6],[4,5,8,3,0],[5,7,3,6,8] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_9 - F_9$:

$$\begin{aligned} G_4 &\cong [0,2,4,1,6],[0,5,6,3,8],[0,7,8,2,1],[1,3,5,2,7] \\ H_4 &\cong [4,0,8,3,6],[4,1,5,8,7] \end{aligned}$$

Let [0,9] be a labeling of the vertices of $K_{10} - F_{10}$. The following is a (G_3, H_3)- multidecomposition of $K_{10} - F_{10}$:

$$\begin{aligned} G_3 &\cong [0,2,6,4,5],[0,3,9,7,6],[1,3,9,4,5],[1,6,9,8,2] \\ H_3 &\cong [2,3,8,0,9],[4,3,9,7,2],[5,8,7,1,9],[6,7,4,8,5] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_{10} - F_{10}$:

$$\begin{aligned} G_4 &\cong [0,2,4,1,6],[1,3,5,2,7] \\ H_4 &\cong [0,1,2,8,9],[0,3,5,6,7],[0,4,6,3,5],[3,2,1,8,9], \\ &\quad [4,5,6,8,9],[7,2,6,1,4],[7,6,5,8,9] \end{aligned}$$

Let [0,10] be a labeling of the vertices of $K_{11} - F_{11}$. The following is a (G_3, H_3)- multidecomposition of $K_{11} - F_{11}$:

$$\begin{aligned} G_3 &\cong [0,2,6,4,5],[1,2,8,6,7],[9,10,8,1,0] \\ H_3 &\cong [0,5,7,4,3],[1,2,8,6,0],[2,3,9,5,10],[2,3,10,4,9], \\ &\quad [3,5,1,4,6],[4,5,8,3,0],[5,7,3,6,9],[6,8,10,7,9] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_{11} - F_{11}$:

$$\begin{aligned} G_4 &\cong [0,2,4,1,6],[0,5,6,3,8],[0,7,8,2,1], \\ &\quad [0,9,10,8,6],[1,3,5,2,7] \\ H_4 &\cong [2,1,4,10,9],[3,1,7,9,10],[4,0,10,3,6], \\ &\quad [4,1,5,8,7],[5,4,7,10,9] \end{aligned}$$

Let [0,11] be a labeling of the vertices of $K_{12} - F_{12}$. The following is a (G_3, H_3)- multidecomposition of $K_{12} - F_{12}$:

$$\begin{aligned} G_3 &\cong [0,2,6,4,5],[0,3,9,7,6],[1,3,9,4,5],[1,6,9,8,2] \\ H_3 &\cong [0,1,10,2,11],[0,1,11,3,10],[2,5,8,0,9], \\ &\quad [4,5,10,6,11],[4,5,11,7,2],[4,7,9,3,8], \\ &\quad [5,8,7,1,9],[6,7,4,8,11],[7,8,10,9,11] \end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_{12} - F_{12}$:

$$\begin{aligned} G_4 &\cong [0,2,4,1,6],[0,5,9,6,10],[1,3,5,2,7], \\ &\quad [1,7,9,3,0],[5,6,8,4,3],[7,8,10,9,11] \\ H_4 &\cong [4,3,11,10,6],[8,6,2,0,1],[10,9,11,2,0], \\ &\quad [11,4,8,7,2],[11,7,8,5,3],[11,9,10,4,1] \end{aligned}$$

Let [0,12] be a labeling of the vertices of $K_{13} - F_{13}$. The following is a (G_3, H_3)- multidecomposition of $K_{13} - F_{13}$:

$$\begin{aligned}
G_3 &\cong [0,2,6,4,5],[1,2,8,6,7] \\
H_3 &\cong [0,5,7,4,3],[2,10,8,0,6],[3,5,1,4,6],[4,5,8,3,0], \\
&\quad [5,7,3,6,8],[6,8,11,9,10],[9,10,0,12,1],[9,10,1,11,0], \\
&\quad [9,10,2,12,3],[9,10,3,11,2],[9,10,4,12,5],[9,10,5,11,4], \\
&\quad [10,11,7,12,9],[10,12,6,9,7],[10,11,12,8,1]
\end{aligned}$$

The following is a (G_4, H_4)- multidecomposition of $K_{13} - F_{13}$:

$$\begin{aligned}
G_4 &\cong [0,2,4,1,6],[0,5,6,3,8],[0,7,8,2,1], \\
&\quad [0,9,10,8,6],[1,3,5,2,7],[4,7,9,12,10] \\
H_4 &\cong [2,1,3,12,9],[4,0,10,3,6],[4,3,5,11,12], \\
&\quad [6,5,7,11,12],[9,8,12,1,11],[10,8,12,4,3], \\
&\quad [10,9,11,5,2],[11,10,12,1,0],[11,5,12,7,8]
\end{aligned}$$

Let [0,13] be a labeling of the vertices of $K_{14} - F_{14}$. The following is a (G_3, H_3)- multidecomposition of $K_{14} - F_{14}$:

$$\begin{aligned}
G_3 &\cong [0,2,6,4,5],[1,3,0,6,7] \\
H_3 &\cong [0,4,6,2,1],[0,5,3,4,1],[2,4,7,5,1],[4,5,13,9,10], \\
&\quad [4,11,9,7,8],[4,12,8,13,10],[7,8,10,4,12],[8,9,0,10,1], \\
&\quad [8,10,2,9,1],[8,10,6,9,5],[8,11,5,10,12],[8,13,11,12,9], \\
&\quad [9,10,3,8,1],[11,12,0,13,1],[11,13,2,12,1],[11,13,6,12,5], \\
&\quad [12,13,3,11,1],[12,13,7,11,4]
\end{aligned}$$

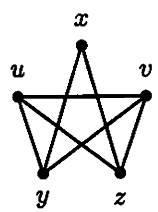
The following is a (G_4, H_4)- multidecomposition of $K_{14} - F_{14}$:

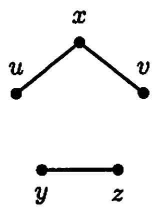
$$\begin{aligned}
G_4 &\cong [0,2,4,1,6],[1,3,5,2,7] \\
H_4 &\cong [0,1,3,12,13],[0,3,5,6,7],[0,4,6,3,5],[1,0,2,8,10], \\
&\quad [2,0,1,11,13],[2,3,0,8,9],[3,0,1,10,9],[3,1,2,11,12], \\
&\quad [5,4,6,8,10],[5,4,6,11,13],[6,4,7,9,8],[6,4,7,12,11], \\
&\quad [7,2,6,1,4],[7,4,5,10,9],[7,4,5,13,12], \\
&\quad [8,10,12,13,11],[9,10,11,12,13],[10,11,12,9,8]
\end{aligned}$$

For any $n \geq 15$, place a $K_8 - F_8$-design on the vertices [0,7] and place a $K_{n-8} - F_{n-8}$-design on the vertices $[8, n-1]$. The remaining edges in the bipartite graph on the vertices $[0,7; 8, n-1]$ can be partitioned using Lemma 4.1 into copies of H_3 or H_4. ∎

5 The Fifth Graph Pair

We denote G_5 by $[x, y, z, u, v]$ and H_5 by $[(u, x, v)(y, z)]$

$G_5 \cong$ 

$H_5 \cong$ 

We need the following lemma:

Lemma 5.1 *There exists an H_5 decomposition of $K_{n,3}$ for $n \geq 2$.*

Proof: For any $n \geq 4$, n can be written as $n = 2a + 3b$ where a and b are not both zero. Hence, it is sufficient to show that there is an H_5 decomposition of $K_{2,3}$ and an H_5 decomposition of $K_{3,3}$. Let [0,1;2,4] be a labeling of $K_{2,3}$. The following is an H_5- decomposition of $K_{2,3}$:
$H_5 \cong$ [(2,0,3)(1,4)], [(2,1,3)(0,4)]
Let [0,2;3,5] be a labeling of $K_{3,3}$. The following is an H_5- decomposition of $K_{3,3}$:
$H_5 \cong$ [(0,3,1)(2,4)], [(0,4,1)(2,5)], [(0,5,1)(2,3)] ∎

We are ready for the following:

Theorem 5.1 *There exists a (G_5, H_5)- multidecomposition of $K_n - HC$ if and only if $n = 8$ or $n \geq 10$.*

Proof: An edge count implies that there are no (G_5, H_5)- multidecomposition of $K_6 - HC$ and $K_7 - HC$.

Since $K_9 - HC$ has 27 edges, a multidecomposition must consist of 3 copies of G_5 and 2 copies of H_5. A vertex degree count argument shows that this is not possible.

Let [0,7] be a labeling of $K_8 - HC$. The following is a (G_5, H_5)- multidecomposition of $K_8 - HC$:

$$\begin{aligned} G_5 &\cong [0,3,5,1,7], [7,2,4,0,6] \\ H_5 &\cong [(1,4,2)(3,5)], [(1,6,3)(2,5)] \end{aligned}$$

Let [0,9] be a labeling of $K_{10} - HC$. The following is a (G_5, H_5)- multidecomposition of $K_{10} - HC$:

$$\begin{aligned} G_5 &\cong [0,3,7,1,9],[0,4,6,2,8] \\ H_5 &\cong [(2,9,5)(4,7)],[(4,1,5)(6,9)],[(5,0,2)(3,6)],[(5,2,7)(0,8)], \\ &\quad [(7,3,8)(4,6)],[(7,5,8)(4,9)],[(8,1,6)(3,5)] \end{aligned}$$

Let [0,11] be a labeling of $K_{12} - HC$. The following is a (G_5, H_5)- multidecomposition of $K_{12} - HC$:

$$\begin{aligned} G_5 &\cong [0,5,8,2,11],[1,6,9,3,0],[2,7,10,4,1] \\ H_5 &\cong [(1,5,10)(7,9)],[(2,0,4)(5,7)],[(2,9,4)(6,8)], \\ &\quad [(3,8,4)(6,10)],[(4,2,6)(1,3)],[(5,3,7)(6,11)], \\ &\quad [(6,4,11)(7,10)],[(7,0,10)(5,9)],[(7,11,9)(8,10)], \\ &\quad [(8,1,11)(6,9)],[(10,3,11)(5,8)] \end{aligned}$$

The remaining cases will be shown using a recursive construction. Let $[0,n-4]$ be a labeling of the vertices of $K_{n-3}-HC$ that has a (G_5, H_5)- multidecomposition. Next we show how to find a (G_5, H_5)- multidecomposition of $K_n - HC$ with the Hamiltonian Cycle being $[0,a,1,b,2,c,3,...,n-4]$. Note that the (G_5, H_5)- multidecomposition of $K_{n-3}-HC$ does not include the edges {0,1}, {1,2}, and {2,3} as these are part of the Hamiltonian cycle on $n-3$ vertices.

We use the following multidecomposition on the edges involving vertices [0,3] and $[a,c]$:

$H_5 \cong [(1,0,b)(a,c)],[(2,1,c)(a,b)],[(3,2,a)(b,c)],[(a,3,b)(0,c)]$

The remaining edges are between $[4,n-3]$ and $[a,c]$ which can then be partitioned into copies of H_5 using Lemma 5.1. ∎

Note that the construction does not include edges $\{0,a\},\{a,1\},\{1,b\},\{b,2\},\{2,c\}$, and $\{c,3\}$ as these are used to change the Hamiltonian cycle on $n-4$ vertices to the Hamiltonian cycle on n vertices: $[0,a,1,b,2,c,3,...,n-4]$.

We need the following lemma for the general case

Lemma 5.2 *There exists an H_5 decomposition of $K_6 - F_6$*

Proof: Let [0,5] be a labeling of the vertices of $K_6 - F$. The following is an H_5 decomposition of $K_6 - F$
$H_5 \cong$ [(0,4,3)(2,5)], [(2,0,3)(1,4)], [(3,5,1)(2,4)], [(2,1,3)(0,5)]. ∎

We are ready for the following:

Theorem 5.2 *There exists a* (G_5, H_5)*- multidecomposition of* $K_n - F_n$ *for any* $n \geq 9$.

Proof:

Let [0,8] be a labeling of the vertices of $K_9 - F_9$. The following is a (G_5, H_5)- multidecomposition of $K_9 - F_9$:

$$\begin{aligned} G_5 &\cong [1,6,7,4,8],[2,7,8,3,5] \\ H_5 &\cong [(0,2,1)(3,4)],[(0,3,1)(2,4)],[(0,4,1)(2,5)], \\ &\quad [(0,5,1)(2,6)],[(0,6,3)(1,8)],[(7,0,8)(5,6)] \end{aligned}$$

Let [0,9] be a labeling of the vertices of $K_{10} - F_{10}$. The following is a (G_5, H_5)- multidecomposition of $K_{10} - F_{10}$:

$$\begin{aligned} G_5 &\cong [0,8,9,2,7],[3,0,1,2,4],[6,0,1,5,7],[8,5,6,3,9] \\ H_5 &\cong [(5,2,6)(3,7)],[(6,4,7)(3,8)],[(8,1,9)(3,4)],[(8,4,9)(5,6)] \end{aligned}$$

Let [0,10] be a labeling of the vertices of $K_{11} - F_{11}$. The following is a (G_5, H_5)- multidecomposition of $K_{11} - F_{11}$:

$$\begin{aligned} G_5 &\cong [0,2,8,1,4],[0,9,10,4,7],[1,6,7,2,8], \\ &\quad [6,3,5,9,10],[8,3,5,0,7] \\ H_5 &\cong [(4,0,6)(1,3)],[(4,3,5)(8,10)],[(9,1,10)(2,5)], \\ &\quad [(9,2,10)(4,6)],[(9,6,10)(1,5)]. \end{aligned}$$

Let [0,11] be a labeling of the vertices of $K_{12} - F_{12}$. The following is a (G_5, H_5)- multidecomposition of $K_{12} - F_{12}$:

$$\begin{aligned} G_5 &\cong [3,8,9,5,6],[5,0,1,4,6],[5,10,11,4,8], \\ &\quad [6,10,11,7,9],[7,0,1,2,8],[9,0,1,3,10] \\ H_5 &\cong [(0,11,1)(2,4)],[(2,5,3)(4,9)],[(3,7,8)(2,9)], \\ &\quad [(4,3,6)(5,7)],[(6,2,7)(3,11)],[(10,2,11)(4,7)] \end{aligned}$$

Let [0,12] be a labeling of the vertices of $K_{13} - F_{13}$. The following is a (G_5, H_5)- multidecomposition of $K_{13} - F_{13}$:

$$\begin{aligned} G_5 &\cong [0,7,8,1,2],[0,9,10,3,6],[3,11,12,6,8], \\ &\quad [4,7,8,3,5],[10,4,5,11,12],[11,7,9,10,12] \\ H_5 &\cong [(0,2,4)(1,9)],[(0,3,1)(5,6)],[(0,4,1)(2,10)],[(0,5,1)(2,6)], \\ &\quad [(0,6,1)(2,11)],[(0,11,1)(2,12)],[(0,12,1)(2,5)], \\ &\quad [(2,9,4)(1,10)],[(3,4,6)(7,8)],[(5,9,7)(8,10)] \end{aligned}$$

Let [0,13] be a labeling of the vertices of $K_{14} - F_{14}$. The following is a (G_5, H_5)- multidecomposition of $K_{14} - F_{14}$:

$$\begin{aligned}
G_5 &\cong [3,8,9,5,6],[5,0,1,4,6],[5,10,11,4,8],\\
&\quad [6,10,11,7,9],[7,0,1,2,8],[9,0,1,3,10]\\
H_5 &\cong [(0,11,1)(2,4)],[(0,12,1)(2,13)],[(0,13,1)(2,12)],\\
&\quad [(2,5,3)(4,9)],[(3,7,8)(2,9)],[(3,12,4)(5,13)],\\
&\quad [(3,13,4)(5,12)],[(4,3,6)(5,7)],[(6,2,7)(3,11)],\\
&\quad [(6,12,7)(8,13)],[(6,13,7)(8,12)],[(9,12,10)(11,13)],\\
&\quad [(10,2,11)(4,7)],[(9,13,10)(11,12)]
\end{aligned}$$

If $n \geq 15$; place an H_5 decomposition of $K_6 - F_6$ on vertices [0,5], a $K_{n-6} - F_{n-6}$ design on vertices $[6, n-1]$, and finally apply lemma 5.1 to partition the edges between the vertices $[0,5;6,n-1]$. ∎

6 Conclusions

We have established the necessary and sufficient conditions for a complete graph to have a multidecomposition into a graph-pair of order 5 with a Hamiltonian cycle or (almost) 1-factor leave. A nice extension of this would be to find the multidecomposition for specified feasible number of copies of G or H. Future topics suggested by this work would allow for a different leave, a different set of graph pairs (order 6 or higher), or the possibility of a λ-fold graph or any combination of the above.

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