Super Edge-connectivity of Transformation Graphs G^{-++}

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Abstract: In this paper, we prove that for any graph G, $\lambda(G^{-++}) = \delta(G^{-++})$ and all but for a few exceptions, G^{-++} is super edge-connectivity where G^{-++} is transformation graph of a graph G introduced in [1].

1 INTRODUCTION

Consider the following network reliability defined on simple graphs. A network is an undirected simple graph $G = (V, E)$ with a probability of failure p associated with each edge. We assume that the edge-failure probabilities are equal and independent; it is also assumed that the vertices do not fail. The *global reliability* $R(G;p)$ of a connected graph G is defined to be the probability that G remains connected. An *edge-cut set* of a graph G is defined to be the set of edges whose removal disconnects G. The total number of edge-cut sets of size i in G is denoted by $C_i(G)$ and the *edge-connectivity* of G is denoted by $\lambda(G)$. The global reliability $R(G;p)$ of G can be expressed as

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$$R(G;p) = 1 - \sum_{i=1}^{|E|} C_i(G)p^i(1-p)^{|E|-i}$$

In general, the calculation of *R(G;p)* is NP-hard [2]. To minimize C_λ, in [3], Bauer defined a connected graph G to be *super edge-connected*, or *super-*λ, if every edge-cut set of size λ isolates one vertex with the minimum vertex degree of G.

In the design of communication networks, constructing networks with higher connectivity is one of methods to improve the reliability of networks. There are many different construction of graphs such as total graphs, line graphs, jump graphs, middle graphs etc. In these graphs, the total graph of G, usually denoted by T(G), has $V(G) \cup E(G)$ as its vertex set, and two vertices of T(G) are adjacent if and only if they are adjacent or incident in G. Wu baoyin and Meng jixiang defined the following transformation graphs which inspired by the definition of total graph of a graph[1].

Definition 1.1 Let $G = (V(G), E(G))$ be a graph, and x, y, z be three variables taking values + or -. The transformation graph G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set, and for $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds:

(1)$\alpha, \beta \in V(G)$, α and β are adjacent in G if x=+; and are not adjacent in G if x=-.

(2)$\alpha, \beta \in E(G)$, α and β are adjacent in G if y=+; and are not adjacent in G if y=-.

(3)$\alpha \in V(G), \beta \in E(G)$, α and β are incident in G if z=+; and are not incident in G if z=-.

Thus, as defined above, there are eight kinds of transformation graphs, among which G^{+++} is usually known as the total graph of G. In this paper ,we study the connectivity of G^{-++}. In [1], the authors have proved that:

theorem 1.2: G^{-++} is connected for any graph G.

2 SUPER EDGE-CONNECTIVITY OF G^{-++}

Throughout the paper we only consider finite and simple graphs. Undefined symbols and concepts can be found in [5].

Let $V(G^{-++}) = V(G) \cup E(G)$, for any $x \in V(G)$, $d_{G^{-++}}(x) = (|V(G)| - 1 - d_G(x)) + d_G(x) = |V(G)| - 1$, for any $e = xy \in E(G)$, $d_{G^{-++}}(e) = (d_G(x) + d_G(y) - 2)) + 2 = (d_G(x) + d_G(y))$. Thus, $\delta(G^{-++}) = \min\{|V(G)| - 1, \delta(e)\}$ where $\delta(e) = \min\{d_G(x) + d_G(y) : e = xy \in E(G)\}$. we will prove that:

Theorem 2.1: For a given graph G, G^{-++} is super-λ if and only if $G \ncong K_{1,n}$ or $K_{1,n} \cup K_1$.

Proof: If $G \cong K_{1,n}$, then $\delta(G^{-++}) = n$. Let $d_G(x) = n$, thus $S = \{\hat{e} = ye \in E(G^{-++}) : y \in V(G)\backslash\{x\}, e = yx \in E(G)\}$ is minimum edge-cut set, $|S| = \lambda(G^{-++}) = n = \delta(G^{-++})$. Clearly, G^{-++} is not super-λ. If $G \cong K_{1,n} \cup K_1$, then $\delta(G^{-++}) = n + 1$. Let $d_G(x) = n$, $S = \{\hat{e} = ye \in E(G^{-++}) : y \in V(G)\backslash\{x\}, e = yx \in E(G)\} \cup \{\hat{e} = xz \in E(G^{-++}) : z \in V(K_1)\}$ is minimum edge-cut set, $|S| = \lambda(G^{-++}) = n + 1 = \delta(G^{-++})$. Clearly, G^{-++} is not super-λ. Thus, the necessity is proved. Now we prove the sufficiency.

Suppose $S \subseteq E(G^{-++})$ be a minimum edge-cut set of G^{-++}, $G^{-++} - S$ has exactly two components, say G_1 and G_2. If one component has one vertex, then the result holds. For any graph G, $\lambda(G) \leq \delta(G)$. So we only prove that if each component has at least two vertices, $|S| > \delta(G^{-++})$. If G is empty, then G^{-++} is complete, the result is obvious. We assume $E(G) \neq \emptyset$.

Case 1: one component contains V(G).

Without loss of generality, suppose $V(G) \subseteq V(G_1)$. Let $E_i = E(G) \cap V(G_i)$, i=1,2.

Subcase 1.1: $E_1 = \emptyset$.

Then $|S| = 2|E(G)| \geq \delta(e) - 1 + |E(G)| > \delta(G^{-++})$.

Subcase 1.2: $E_1 \neq \emptyset$.

Subcase 1.2.1: There is no edge between E_1 and E_2 in G^{-++}.

Then G is not connected and $G[E_2]$ is a component of G. Thus

$|S| = 2|E_2| = \sum\limits_{x \in V(G[E_2])} d_{G[E_2]}(x) = \sum\limits_{x \in V(G[E_2])} d_G(x)$.

Since $|E_2| \geq 2$, we have $|V(G[E_2])| \geq 3$, so $|S| > \delta(e)$.

Subcase 1.2.2: There are edges between E_1 and E_2 in G^{-++}.

Let $[E_1, E_2] = \{\hat{e} = e_i e_j \in E(G^{-++}) : e_i \in E_1, e_j \in E_2\}$

Claim 1: for any $\hat{e} = e_i e_j \in [E_1, E_2]$, all e_i and e_j are the neighbor edges of vertex x in G. Let $|E_G(x) \cap E_i| = m_i$, m=1,2 and $m_1 + m_2 = d_G(x)$. Thus, vertex x is a cut vertex in G, and we have:

$$
\begin{aligned}
|S| &= 2|E_2| + m_1 \cdot m_2 \geq 2|E_2| + d_G(x) - 1 \\
&= \sum_{y \in V(G[E_2])} d_{G[E_2]}(y) + d_G(x) - 1 \\
&= \sum_{y \in V(G[E_2] \setminus \{x\})} d_G(y) + d_{G[E_2]}(x) + d_G(x) - 1 \\
&> d_G(y) + d_G(x) \geq \delta(e).
\end{aligned}
$$

Claim 2: for any $\hat{e} = e_i e_j \in [E_1, E_2]$, e_i and e_j are the neighbor edges of at least two vertices in G. Suppose the set T is composed of these vertices.

If there exist $x, y \in T$ such that they are adjacent in G. Let $|E_G(x) \cap E_2| = m_x$, $|E_G(y) \cap E_2| = m_y$, then

$$
\begin{aligned}
|S| &\geq 2|E_2| + m_x \cdot (d_G(x) - m_x) + m_y \cdot (d_G(y) - m_y) \\
&\geq 4 + (d_G(x) - 1) + (d_G(y) - 1) > \delta(G^{-++}).
\end{aligned}
$$

Otherwise, there exist $e = xz \in E_G(x) \cap E_2$ such that $E_G(z) \subseteq E_2$. If not, then $x, z \in T$ such that they are adjacent in G, contradiction. So we have $d_G(z) = d_{G[E_2]}(z)$ and

$$|S| \geq 2|E_2| + m_x \cdot (d_G(x) - m_x) + m_y \cdot (d_G(y) - m_y)$$
$$\geq \sum_{v \in V(G[E_2])} d_{G[E_2]}(v) + (d_G(x) - 1) + (d_G(y) - 1)$$
$$\geq d_G(z) + d_{G[E_2]}(x) + d_G(x) + d_G(y) - 2$$
$$> d_G(z) + d_G(x) \geq \delta(G^{-++}).$$

Case 2: $V_i = V(G) \cap V(G_i) \neq \emptyset$ and $E_i = E(G) \bigcap V(G_i)$, i=1,2.

For any $x \in V_1, y \in V_2$. If x and y are adjacent with edge e in graph G, then they contribute one edge for S in graphs G^{-++}. If x and y are not adjacent in graph G, then they are adjacent in graphs G^{-++}, they also contribute one edge for S. Thus

$|S| \geq |V_1|(|V(G)| - |V_1|) \geq |V(G)| - 1.$

$|S| = |V(G)| - 1$ only if $|V_1| = 1$ (or $|V_2| = 1$) and $E_1 = \{e = xy \in V(G^{-++}) : x \in V_1, y \in V_2\}$ and for any $e_1 \in E_1, e_2 \in E_2$, e_1 and e_2 are not adjacent in graphs G^{-++}.

If $E_2 \neq \emptyset$. Thus G is not connected and $G[E_2] \cong K_{1,|E_1|}$ is a component of G, we have $\delta(e) \leq |E_1| + 1 \leq (|V(G)| - 3) + 1 \leq |V(G)| - 2$, so $|S| = |V(G)| - 1 > \delta(G^{-++})$.

If $E_2 = \emptyset$, then $E_1 = E(G)$, $G \cong K_{1,|E(G)|} \cup mK_1$ where $|V(G)| = |E(G)|+1+m$. Thus $\delta(G^{-++}) = \min\{|V(G)|-1, |E(G)|+1\} = \min\{|V(G)|-1, |V(G)| - m\}$. Only if m=0 or 1, that is $G \cong K_{1,n}$ or $K_{1,n} \cup K_1$, G^{-++} is not super-λ. □

Clearly, we contain the edge-connectivity of G^{-++}:

Corollary 2.2: For any graph G, $\lambda(G^{-++}) = \delta(G^{-++})$.

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