A new upper bound of the basis number of the lexicographic product of graphs

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Abstract

An upper bound of the basis number of the lexicographic product of two graphs from the basis number of the factors is presented. Furthermore, the basis numbers of the lexicographic product of some classes of graphs is determined.

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1 Introduction.

The perception of cyclic structures is a crucial step in the analysis of graphs. Cycle bases of a cycle spaces have a variety of applications in sciences and engineering, for example, in structural flexibility analysis, electrical networks, and in chemical structure storage and retrieval systems (see [7], [8] and [16]). The basis number of a graph is one of the numbers which give rise to a better understanding and interpretations of a geometric properties of a graph (see [17]).

In general, required cycle bases is not very well behaved under graph operations. That is, the basis number $b(G)$ of a graph G is not monotonic (see [18]). Hence, there does not seem to be a general way of extending required cycle bases of a certain collection of partial graphs of G to a required cycle basis. Global upper bounds $b(G) \leq 2\gamma(G) + 2$ where $\gamma(G)$ is the genus of G is proven in [18].

Getting new graphs from known graphs through different kinds of graph products and operations on graphs originated as early as the beginning of graph theory as an independent subject. Actually graph products are the best natural way to enlarge the space of graphs. In the literature there are a lot of graph products. We mention out of these products; the cartesian product, the direct product, the strong product, the semi-strong product, the lexicographic product, the semi-composition product and the special product. Many researchers employed their efforts to study the properties of graphs obtained by the graph products and related some of these properties to those of the graphs incorporated in the products. The enthusiasm of studying graph products led Klavzar and Wilfried to write a whole book that focuses on materials regarding four of the above mentioned graph products (see [9]).

In this paper, we give a new upper bound of the basis number of the lexicographic product of graphs. And as a results, we determine the basis number of the lexicographic product of classes of graphs.

2 Definitions and preliminaries.

The graphs considered in this paper are finite, undirected, simple and connected. Most of the notations that follow can be found in [5]. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$.

2.1 Cycle bases.

Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which correspond to the cycles in G generate a subspace called the *cycle space* of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that for a connected graph G $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1$ (see [6]).

Given any spanning tree T of G, every graph $T + e$, $e \in E(G) - E(T)$, contains exactly one cycle C_e and the collection of cycles $\{C_e | e \in E(G) - E(T)\}$ forms a basis of $\mathcal{C}(G)$, called the *fundamental basis associated with* T. Note that each edge $e \in E(G) - E(T)$ occurs in exactly one cycle of this basis, but each edge of T itself may occur in many cycles of the basis. This obrervation led Schmeichel to define what is called the basis number.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a *d-fold* if each edge of G occurs in at most d of the cycles in $\mathcal{B}$. *The basis number*, $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The *required basis* of $\mathcal{C}(G)$ is a basis with $b(G)$-fold. Let G and H be two graphs, $\varphi : G \longrightarrow H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of $\mathcal{C}(G)$. Then $\mathcal{B}' = \{\varphi(c) | c \in \mathcal{B}\}$ is called the *corresponding (required)* basis of $\mathcal{B}$ in H. The following result will be used frequently in the sequel.

Theorem 2.1.1.(MacLane). *The Graph G is planar if and only if $b(G) \leq 2$.*

Although MacLane is the first who gave an important result regarding the basis number when he proved the above result, Schmeichel is the first who gave a formal definition of the basis number when he proved that $b(K_n) \leq 3$.

2.2 Products.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. (1) The cartesian product $G \square H$ has the vertex set $V(G \square H) = V(G) \times V(H)$ and the edge set $E(G \square H) = \{(u_1, v_1)(u_2, v_2) | u_1 u_2 \in E(G)$ and $v_1 = v_2$, or $v_1 v_2 \in E(H)$ and $u_1 = u_2\}$. (2) The Lexicographic product $G_1[G_2]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2 v_2 \in E(H)$ or $u_1 v_1 \in E(G)\}$ (See [9]).

Many authors studied the basis number of the lexicographic product. Schemichel [18] and Ali [1] proved the following results.

Theorem 2.2.1. *(Schmeichel) For any null graph N_m of order m and path, P_2, of order 2, we have that $b(P_2[N_m]) \leq 4$.*

Theorem 2.2.2. *(Ali) For any null graph N_m of order $m \geq 5$ and a complete graph K_n of order $n \geq 5$, we have that $b(\ K_n[N_m]) \leq 3 + 2b(K_n)$.*

There after, Ali and Marougi [3] gave an upper bound of the basis number of the lexicographic product of a cycle (path) with a semi-Hamiltonian graph. In fact, they gave the following result:

Theorem 2.2.3. *(Ali and Marougi)* Let C and P be a cycle and a path and H *be a semi-Hamiltonian graph. Then* $b(C[H]), b(P[H]) \leq \max\{4, 2 + b(H)\}$.

Jaradat [12] gave the following upper bound of the basis number of the lexicographic product of graph where the upper bound is achieved:

Theorem 2.2.4. *(Jaradat) For each two connected graphs G and H, $b(G[H]) \leq \max\{4, 2\Delta(G) + b(H), 2 + b(G)\}$.*

Other products were studied by the author (see [10], [13] and [14]) and Ali and Marougi [2]. Note that the upper bound in Theorem 2.3.3 is restricted to some classes of graphs of the factors and the upper bound in Theorem 2.3.4 is large if the degree in the first factor of the product is large. Therefore, it was needed to give a new upper bound deals with any graphs and gives a reasonable bound.

In this work we give a new reasonable upper bound of the basis number of the lexicographic product of any two graphs. Also, as a results, we determine the exact basis numbers of the lexicographic product of some classes of graphs.

In the rest of this paper $f_B(e)$ stand for the number of cycles in B containing e where $B \subseteq \mathcal{C}(G)$. $\mathcal{B}_G$ stand for the required basis of G. Moreover, if A is a set of cycles, then $E(A) = \cup_{c \in A} E(c)$.

3 Lexicographic product of trees.

In this section, we give a new upper bound of the basis number of the lexicographic product of trees. In fact, we give an exact basis number of the lexicographic product of some classes of trees. The graph $G[H]$ consists of $|V(G)|$ copies of the graph H and $|E(G)|$ copies of the graph $P_2[N_{|V(H)|}]$ where $N_{|V(H)|}$ is the null graph with vertex set $V(H)$. Let $\{e_1, e_2, \ldots, e_{|E(G)|}\}$ be the edge set of the graph G. We say that $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{|E(G)|}\}$ is a *foundation* of $G[H]$ if $\mathcal{B}_i$ is a basis of $\mathcal{C}\left(e_i[N_{|V(H)|}]\right)$ for each $i = 1, 2, \ldots, |E(G)|$. A tree T consisting of n equal order paths $\{P^{(1)}, P^{(2)}, \ldots, P^{(m)}\}$ is called an *n-special star* if there is a vertex, say v, such that v is an end vertex for each path in $\{P^{(1)}, P^{(2)}, \ldots, P^{(m)}\}$ and $V(P^{(i)}) \cap V(P^{(j)}) = \{v\}$ for each $i \neq j$ (see [10]).

Lemma 3.1. *Let $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{|E(G)|}\}$ be a foundation of $G[H]$. Then $\cup_{i=1}^{|E(G)|} \mathcal{B}_i$ is linearly independent subset of $\mathcal{C}(G[H])$.*

Proof. The result follows from being that $e_1[N_{|V(H)|}]$, $e_2[N_{|V(H)|}]$, $\ldots$, $e_{|E(G)|}[N_{|V(H)|}]$ are mutually edge disjoint subgraphs of $G[H]$ and $\mathcal{B}_i$ is a basis of $\mathcal{C}\left(e_i[N_{|V(H)|}]\right)$ for each $i = 1, 2, \ldots, |E(G)|$.

The following result follows immediately from the definition of the foundation.

Lemma 3.2. *Let G and H be two graphs such that G is decomposable into edge disjoint subgraphs $G_1, G_2, \ldots, G_h$. Then $\mathcal{B}$ is a foundation of $G[H]$ if and only if $\mathcal{B} = \cup_{i=1}^{h} \mathcal{B}^{(i)}$ where $\mathcal{B}^{(i)}$ is a foundation of $G_i[H]$ for each $i = 1, 2, \ldots, h$.*

The question regarding the basis number of lexicographic product of graphs can not be resolved directly, simply because graphs has no uniform form. Therefore, we recall the following result of Jaradat [15] which decomposes trees into well known subgraphs.

Proposition 3.3 *(Jaradat) Let T be a tree of order ≥ 2. Then T can be decomposed into edge disjoint subgraphs $S^{(1)}, S^{(2)}, \ldots, S^{(r)}$ for some integer r, such that, the following holds:*
(i) For each $i \geq 1, S^{(i)}$ is either a star or a path of order 2 such that $S^{(1)}$ is a path incident with an end vertex, say $v_1^{(1)}$.
(ii) For each $v \in V(T)$, if $d_T(v) \geq 2$, then $|\{i : v \in V(S^{(i)})| = 2$, and if $d_T(v) = 1$, then $|\{i : v \in V(S^{(i)})| = 1$.
(iii) $V(S^{(i)}) \cap (\cup_{j=1}^{i-1} V(S^{(j)})\ ) = v_1^{(i)}$ where $d_{S^{(i)}}(v_1^{(i)}) = Max_{v \in V(S^{(i)})}\, d_T(v)$, and *$d_{\cup_{j=1}^{i-1} V(S^{(j)})}(v_1^{(i)}) = 1$ for each $i = 2, 3, \ldots, r$, and $v_1^{(i)} \neq v_1^{(j)}$ for each $i \neq j$.*

Let $T_1 = \cup_{t=1}^{h} S_*^{(t)}, T_2 = \cup_{i=1}^{k} S^{(i)}$ be decompositions of T_1 and T_2 as in the above proposition, respectively, with $V(S_*^{(t)}) = \left\{u_1^{(t)}, u_2^{(t)}, \ldots, u_{m_t}^{(t)}\right\}$ and $V(S^{(i)}) = \left\{v_1^{(i)}, v_2^{(i)}, \ldots, v_{n_i}^{(i)}\right\}$ for each $t = 1, 2, \ldots, h$ and $i = 1, 2, \ldots, k$. Thus, we may assume that $\left\{u_1^{(1)}, u_2^{(1)}, \ldots, u_{m_1}^{(1)}, u_2^{(2)}, \ldots, u_{m_2}^{(2)}, u_2^{(3)}, \ldots, u_{m_3}^{(3)}, \ldots, u_2^{(h)}, \ldots, u_{m_h}^{(h)}\right\}$ and $\left\{v_1^{(1)}, v_2^{(1)}, \ldots, v_{n_1}^{(1)}, v_2^{(2)}, \ldots, v_{n_2}^{(2)}, v_2^{(3)}, \ldots, v_{n_3}^{(3)}, \ldots, v_2^{(k)}, \ldots, v_{n_k}^{(k)}\right\}$ are ordering of the vertices of T_1 and T_2, respectively. Moreover, we may assume that $u_{m_h}^{(h)}$ and $v_{n_k}^{(k)}$ are end vertices in T_1 and T_2, respectively. Note that for $i \geq 2, v_1^{(i)} = v_s^{(l)}$ for some $1 \leq l \leq i-1; 1 \leq s \leq n_l$, similarly, $t \geq 2, v_1^{(t)} = v_r^{(f)}$ for some $1 \leq f \leq t-1; 1 \leq r \leq m_f$. Throughout this work, for simplicity, we plot the vertices of $T_1[T_2]$ in the coordinates plane and arrange them in the following manner: $(u_l^{(t)}, v_j^{(i)})$ is directly to the lift of $(u_{l+1}^{(t)}, v_j^{(i)})$ and below of $(u_l^{(t)}, v_{j+1}^{(i)})$; the vertex $(u_l^{(t)}, v_{n_i}^{(i)})$ is directly to the lift of $(u_{l+1}^{(t)}, v_{n_i}^{(i)})$ and below of $(u_l^{(t)}, v_2^{(i+1)})$; also, the vertex $(u_{m_t}^{(t)}, v_j^{(i)})$ is directly to the lift of $(u_2^{(t+1)}, v_j^{(i)})$ and below of $(u_2^{(t)}, v_{j+1}^{(i)})$; the vertex $(u_{m_t}^{(t)}, v_{n_i}^{(i)})$ is directly to the lift of $(u_2^{(t+1)}, v_{n_i}^{(i)})$ and below of $(u_{m_t}^{(t)}, v_2^{(i+1)})$. Finally, from the proof of Proposition 3.3 (see [15]) one can chose $S_*^{(t)}$ and $S^{(i)}$ and label their vertices in such a way that $u_{m_t}^{(t)} = u_1^{(t+1)}$ if T_1 has no 3-special star of order 7 as a subgraph and $v_{n_i}^{(i)} = v_1^{(i+1)}$ if T_2 has no 3-special star of order 7 as a subgraph.

For each $t = 1, 2, \ldots, h$; $i = 2, 3, \ldots, k$; $r = 2, 3, \ldots, m_t$ and $j = 3, \ldots, n_i$, set

$$C_{t,r}^{(i,j)} = (u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_j^{(i)})(u_r^{(t)}, v_1^{(i)})(u_r^{(t)}, v_{j-1}^{(i)})(u_1^{(t)}, v_1^{(1)}),$$

$$C_{t,r}^{(i,2)} = (u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_2^{(i)})(u_r^{(t)}, v_1^{(i)})(u_1^{(t)}, v_1^{(1)}),$$

and

$$C_{t,r}^{(1,2)} = (u_1^{(t)}, v_{n_k}^{(k)})(u_r^{(t)}, v_2^{(i)})(u_r^{(t)}, v_1^{(i)})(u_1^{(t)}, v_{n_k}^{(k)})$$

Moreover, for each $t = 1, 2, \ldots, h$; $i = 2, 3, \ldots, k$ and $j = 3, \ldots, n_i$, set

$$C_{t,1}^{(i,j)} = (u_2^{(t)}, v_1^{(1)})(u_1^{(t)}, v_j^{(i)})(u_1^{(t)}, v_1^{(i)})(u_1^{(t)}, v_{j-1}^{(i)})(u_2^{(t)}, v_1^{(1)}),$$

and for each $t = 1, 2, \ldots, h$; $i = 1, 2, 3, \ldots, k$ set

$$C_{t,1}^{(i,2)} = (u_2^{(t)}, v_1^{(1)})(u_1^{(t)}, v_2^{(i)})(u_2^{(t)}, v_1^{(i)})(u_2^{(t)}, v_1^{(1)}).$$

Let $\mathcal{B}_{t,r}^{(i)} = \left\{C_{t,r}^{(i,j)} | 2 \le j \le n_i\right\}$ and $\mathcal{B}_{t,r} = \cup_{i=1}^{k} \mathcal{B}_{t,r}^{(i)}$.

Lemma 3.4. *$\mathcal{B}_t = \cup_{r=1}^{m_t} \mathcal{B}_{t,r}$ is linearly independent set.*

Proof. First, we proceed using mathematical induction on n_i to show that $\mathcal{B}_{t,r}^{(i)}$ is linearly independent for each $i = 1, 2, \ldots, k$. If $n_i = 2$, then $\mathcal{B}_{t,r}^{(i)}$ consists only of one cycle $C_{t,r}^{(i,2)}$. So $\mathcal{B}_{t,r}^{(i)}$ is linearly independent. Assume $n_i > 2$ and it is true for less than n_i. Note that $\mathcal{B}_{t,r}^{(i)} = \left\{C_{t,r}^{(i,j)} | 2 \le j \le n_i - 1\right\} \cup \left\{C_{t,r}^{(i,n_i)}\right\}$. Since $C_{t,r}^{(i,n_i)}$ contains $(u_r^{(t)}, v_{n_i}^{(i)})(u_r^{(t)}, v_1^{(i)})$ which is not in $C_{t,r}^{(i,j)}$ for each $j = 1, 2, \ldots, n_i - 1$, and by the inductive step $\mathcal{B}_{t,r}^{(i)}$ is linearly independent. We now show that $\mathcal{B}_{t,r} = \cup_{i=1}^{k} \mathcal{B}_{t,r}^{(i)}$ is linearly independent. It is an easy task to see that for $i \ne l$, say $i < l$, we have that

$$E(C_{t,r}^{(i,j)}) \cap E(C_{t,r}^{(l,s)}) = \begin{cases} (u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_j^{(i)}), & \text{if } s = 2, r \geq 2 \\ & \text{and } v_j^{(i)} = v_1^{(l)}, \\ (u_2^{(t)}, v_1^{(1)})(u_1^{(t)}, v_j^{(i)}), & \text{if } s = 2, r = 1 \\ & \text{and } v_j^{(i)} = v_1^{(l)}, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{3.1}$$

And so,

$$E(\cup_{i=1}^{(l-1)} \mathcal{B}_{t,r}^{(i)}) \cap E(\mathcal{B}_{t,r}^{(l)}) = \begin{cases} (u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_j^{(i)}), & \text{if } r \geq 2 \\ & \text{and for some} \\ & 1 \leq i \leq l-1, \\ & 1 \leq j \leq n_i, \\ (u_2^{(t)}, v_1^{(1)})(u_1^{(t)}, v_j^{(i)}), & \text{if } r = 1 \\ & \text{and for some} \\ & 1 \leq i \leq l-1, \\ & 1 \leq j \leq n_i, \\ \varnothing, & \text{otherwise,} \end{cases} \tag{3.2}$$

because $E(\cup_{i=1}^{(l-1)} S^{(i)}) \cap E(S^{(l)}) = v_1^{(l)} = v_j^{(i)}$ for some $1 \leq i \leq l-1, 1 \leq j \leq n_i$. We now show that $\mathcal{B}_{t,r}$ is linearly independent for each t, r. Assume that $\sum_{s=1}^{\alpha} \sum_{f=1}^{\gamma_s} C_{t,r}^{(i_s,j_f)} = 0 \pmod 2$. Then $\sum_{s=1}^{\alpha-1} \sum_{f=1}^{\gamma_s} C_{t,r}^{(i_s,j_f)} = \sum_{f=1}^{\gamma_\alpha} C_{t,r}^{(i_\alpha,j_f)} \pmod 2$. Thus,

$$E(\oplus_{s=1}^{\alpha-1} \oplus_{f=1}^{\gamma_s} C_{t,r}^{(i_s,j_f)}) = E(\oplus_{f=1}^{\gamma_\alpha} C_{t,r}^{(i_\alpha,j_f)}). \tag{3.3}$$

Since $\mathcal{B}_{t,r}^{(i)}$ is linearly independent for each $i = 1, 2, \ldots, k$ and $C_{t,r}^{(i_\alpha,j_f)} \in \mathcal{B}_{t,r}^{(i_\alpha)}$, the ring sum $\oplus_{f=1}^{\gamma_\alpha} C_{t,r}^{(i_\alpha,j_f)}$ is a cycle or a pairwise edge-disjoint union of cycles. Hence, $\oplus_{f=1}^{\gamma_\alpha} C_{t,r}^{(i_\alpha,j_f)}$ consists of at least 3 edges and so by equation 3.3 the ring sum $\oplus_{s=1}^{\alpha-1} \oplus_{f=1}^{\gamma_s} C_{t,r}^{(i_s,j_f)}$ consists of the same edges, on the other hand and by using equations (3.1) and (3.2) $\oplus_{f=1}^{\gamma_\alpha} C_{t,r}^{(i_\alpha,j_f)}$ and $E(\oplus_{s=1}^{\alpha-1} \oplus_{f=1}^{\gamma_s} C_{t,r}^{(i_s,j_f)})$ consists of at most one edge, a contradiction. Thus, $\mathcal{B}_{t,r}$ is linearly independent set for each $t = 1, 2, \ldots, h; r = 1, 2, \ldots, m_t$. To this end, we show that $\cup_{r=1}^{m_t} \mathcal{B}_{t,r}$ is linearly independent. Note that if $r_1 \neq r_2$, then

$$E(\mathcal{B}_{t,r_1}) \cap E(\mathcal{B}_{t,r_2}) = \begin{cases} (u_1^{(t)}, v_{n_k}^{(k)})(u_2^{(t)}, v_1^{(1)}), & \text{if one of } r_1 \text{ and } r_2 \text{ is} \\ & 1 \text{ and the other is } 2, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{3.4}$$

By equation (3.4), $E(\mathcal{B}_{t,r_1}) \cap E(\mathcal{B}_{t,r_2}) = \phi$ for each $r_1 \neq r_2$ and $r_1, r_2 \geq 2$ and so $\cup_{r=2}^{k} \mathcal{B}_{t,r}$ is linearly independent. Moreover, any linear combination of cycles of $\mathcal{B}_{t,1}$consists of at least three edges. Thus, as in the above argument and by equation 3.4 any linear combination of cycles of $\mathcal{B}_{t,1}$ can not be written as a linear combination of cycles of $\cup_{i=2}^{k} \mathcal{B}_{t,r}$. Therefore, $\mathcal{B}_t = \cup_{r=1}^{k} \mathcal{B}_{t,r}$ is linearly independent. The proof is complete.

Remark 2.1. *One can notices that (1) if* $e = (u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_j^{(i)})$ *or* $(u_r^{(t)}, v_1^{(1)})(u_1^{(t)}, v_j^{(i)})$*, then* $f_{\mathcal{B}}(e) \leq \begin{cases} 2, & \text{if } j = n_i \text{ and } v_{n_i}^{(i)} = v_1^{(i+1)}, \\ 3, & \text{otherwise.} \end{cases}$ *(2) if* $e = (u_r^{(t)}, v_1^{(1)})(u_r^{(t)}, v_j^{(i)})$, $v_{n_i}^{(i)} = v_1^{(i+1)}$*, then* $f_{\mathcal{B}}(e) \leq 2$.

For $r = 2, 3, \ldots, m_t - 1$, let $P_3^{(r)} = u_r^{(t)} u_1^{(t)} u_{r+1}^{(t)}$. Note that $V(P_3^{(r_2)}) \cap V(P_3^{(r_1)}) = \{u_1^{(t)}\}$ for any $r_1 \neq r_2$ and $u_1^{(t)} u_{r+1}^{(t)} \in E(P_3^{(r)})$ but $u_1^{(t)} u_{r+1}^{(t)} \notin \cup_{j=2}^{r-1} E(P_3^{(j)})$. Moreover, $E(S_*^{(t)}) = \cup_{j=2}^{m_t-1} E(P_3^{(j)})$. For each $t = 1, 2, \ldots, h$; $i = 1, 2, \ldots, k$, and $r = 3, 4, \ldots, m_t$, we set

$$a_{t,r,j}^{(1,i)} = (u_r^{(t)}, v_1^{(1)})(u_1^{(t)}, v_j^{(i)})(u_{r-1}^{(t)}, v_1^{(1)})(u_1^{(t)}, v_{j-1}^{(i)})(u_r^{(t)}, v_1^{(1)})$$

for $j = 3, 4, \ldots, n_i$ and

$$a_{t,r,j}^{(n,i)} = (u_r^{(t)}, v_{n_k}^{(k)})(u_1^{(t)}, v_j^{(i)})(u_{r-1}^{(t)}, v_{n_k}^{(k)})(u_1^{(t)}, v_{j+1}^{(i)})(u_r^{(t)}, v_{n_k}^{(k)})$$

for $j = 2, \ldots, n_i - 1$. Moreover, for $t = 1, 2, \ldots, h; i = 2, 3, \ldots, k$ and $r = 3, 4, \ldots, m_t$, we set

$$\begin{aligned}
a_{t,r,2}^{(1,i)} &= (u_r^{(t)}, v_1^{(1)})(u_1^{(t)}, v_2^{(i)})(u_{r-1}^{(t)}, v_1^{(1)})(u_1^{(t)}, v_{n_{(i-1)}}^{(i-1)})(u_r^{(t)}, v_1^{(1)}) \\
a_{t,r,1}^{(n,i)} &= (u_r^{(t)}, v_{n_k}^{(k)})(u_1^{(t)}, v_{n_{(i-1)}}^{(i-1)})(u_{r-1}^{(t)}, v_{n_k}^{(k)})(u_1^{(t)}, v_2^{(i)})(u_r^{(t)}, v_{n_k}^{(k)}) \\
a_{t,r,2}^{(1,1)} &= (u_r^{(t)}, v_1^{(1)})(u_1^{(t)}, v_2^{(1)})(u_{r-1}^{(t)}, v_1^{(1)})(u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_1^{(1)}) \\
a_{t,r,1}^{(n,1)} &= (u_r^{(t)}, v_{n_k}^{(k)})(u_1^{(t)}, v_1^{(1)})(u_{r-1}^{(t)}, v_{n_k}^{(k)})(u_1^{(t)}, v_2^{(i)})(u_r^{(t)}, v_{n_k}^{(k)}).
\end{aligned}$$

Let

$$a_{t,r}^{(1,i)} = \cup_{j=2}^{n_i} \left\{ a_{t,r,j}^{(1,i)} \right\} \text{and } a_{t,r}^{(n,i)} = \cup_{j=1}^{n_i-1} \left\{ a_{t,r,j}^{(n,i)} \right\}$$

Lemma 3.5. $\cup_{r=3}^{m_t} \cup_{i=1}^{k} a_{t,r}^{(1,i)}$ *and* $\cup_{r=3}^{m_t} \cup_{i=1}^{k} a_{t,r}^{(n,i)}$ *are linearly independent sets of cycles.*

Proof. First, we use mathematical induction on n_i to prove that $a_{t,r}^{(1,i)}$ is linearly independent for each $i = 1, 2, \ldots, k$. If $n_i = 2$, then $a_{t,r}^{(1,i)}$ is linearly independent because it consists only of one cycle $a^{(1,i)}_{t,r,2}$. Assume that $n_i \geq 3$ and it is true for smaller values of n_i. It is an easy matter to see that $a^{(1,i)}_{t,r,n_i}$ contains the edge $(u^{(t)}_r, v^{(1)}_1)(u^{(t)}_1, v^{(i)}_{n_i})$ which is not in any cycle of $\cup_{j=2}^{n_i-1}\left\{a^{(1,i)}_{t,r,j}\right\}$. Therefore, $a^{(1,i)}_{t,r}$ is linearly independent. We now proceed using mathematical induction on k to show that $\cup_{i=1}^{k} a^{(1,i)}_{t,r}$ is linearly independent. Note that for $i \neq l$, say $i < l$ we have that

$$E(a^{(1,i)}_{t,r,j}) \cap E(a^{(1,l)}_{t,r,s}) = \begin{cases} \left\{(u^{(t)}_r, v^{(1)}_1)(u^{(t)}_1, v^{(i)}_{n_i}), \right. & \text{if } l = i+1 \\ \left. (u^{(t)}_{r-1}, v^{(1)}_1)(u^{(t)}_1, v^{(i)}_{n_i})\right\}, & \text{and } s = 2, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{3.5}$$

And so,

$$E(a^{(1,i)}_{t,r}) \cap E(a^{(1,l)}_{t,r}) = \begin{cases} \left\{(u^{(t)}_r, v^{(1)}_1)(u^{(t)}_1, v^{(i)}_{n_i}), \right. & \text{if } l = i+1, \\ \left. (u^{(t)}_{r-1}, v^{(1)}_1)(u^{(t)}_1, v^{(i)}_{n_i})\right\}, & \\ \varnothing, & \text{otherwise.} \end{cases} \tag{3.6}$$

Thus, for each $g = 2, 3, \ldots, k$

$$E(\cup_{i=1}^{g-1} a^{(1,i)}_{t,r}) \cap E(a^{(1,g)}_{t,r}) = \{(u^{(t)}_r, v^{(1)}_1)(u^{(t)}_1, v^{(g-1)}_{n_{g-1}}), (u^{(t)}_{r-1}, v^{(1)}_1)\,(u^{(t)}_1, v^{(g-1)}_{n_{g-1}})\}. \tag{3.7}$$

The first step of the induction whenever $k = 1$ follows from the above argument and noting that $\cup_{i=1}^{k} a^{(1,i)}_{t,r} = a^{(1,1)}_{t,r}$. Assume that $k \geq 2$ and it is true for less than k. Now suppose that $\sum_{i=1}^{k} C_i = 0 \pmod 2$ where C_i is a linear combination of cycles of $a^{(1,i)}_{t,r}$, then $\sum_{i=1}^{k-1} C_i = C_k \pmod 2$. And so $E(C_1 \oplus C_2 \oplus \cdots \oplus C_{k-1}) = E(C_k)$ where $\oplus$ is the ring sum. By inductive step and being $a^{(1,k)}_{t,r}$ is linearly independent, we conclude that$C_1 \oplus C_2 \oplus \cdots \oplus C_{k-1}$ and C_k are two identical cycles or two identical edge disjoint union of cycles which contradict equation 3.7. Hence, $\cup_{i=1}^{k} a^{(1,i)}_{t,r}$ is linearly independent. By noting that $\cup_{i=1}^{k} a^{(1,i)}_{t,r} \subset \mathcal{C}(P_3^{(r-1)}[N_T])$ for each $r = 3, 4, \ldots, m_t$ and any linear combinations of cycles of $\cup_{i=1}^{k} a^{(1,i)}_{t,r}$ must contain an edge of the form $(u^{(t)}_r, v^{(1)}_1)(u^{(t)}_1, v^{(i)}_j)$ which is not in $\cup_{s=2}^{r-2}\mathcal{C}(P_3^{(s)}[N_T])$ (because $u^{(t)}_1 u^{(t)}_r$ is an edge of $P_3^{(r-1)}$ but it is not an edge of $\cup_{s=2}^{r-2} P_3^{(s)}$), we have that $\cup_{r=3}^{m_t} \cup_{i=1}^{k} a^{(1,i)}_{t,r}$ is linearly independent. In a similarly argument one can show that $\cup_{r=3}^{m_t} \cup_{i=1}^{k} a^{(n,i)}_{t,r}$ is linearly independent. The proof is complete.

Lemma 3.6. *Let A, B and D be any linearly independent sets of cycles such that $E(A) \cap E(B) = \varnothing$ and each cycle of D contains at least one* *edge of $E(A)$ and at least one edge of $E(B)$. Then $A \cup B \cup D$ is linearly independent set of cycles.*

Proof. Since any linear combination of cycles of D contains at least one edge in common with the edge set of A and since $E(A) \cap E(B) = \varnothing$, $D \cup B$ is a linearly independent set. Now, let C be any linear combination of of cycles of $D \cup B$. Since C is a cycle or an edge disjoint union of cycles and since $E(D) \cap E(B) \neq \varnothing$ and $E(A) \cap E(B) = \varnothing$, as a result $E(C) \cap E(D \cup B)$ is either an empty set or an edge set of a forest. Thus, C can not be written as a linear combination of cycles of A. Therefore, $A \cup B \cup D$ is linearly independent set. The proof is complete.

The following lemma follows easily from the definition of $a_{t,r}^{(n,i)}$ and $a_{t,r}^{(1,i)}$.

Lemma 3.7. $\cup_{r=3 \text{ and } r \text{ is odd}}^{m_t} \left(\cup_{i=1}^{k} E(a_{t,r}^{(n,i)}) \right)$ and $\cup_{r=4 \text{ and } r \text{ is even}}^{m_t} \left(\cup_{i=1}^{k} E(a_{t,r}^{(1,i)}) \right)$ are disjoint sets.

Lemma 3.8. *Let $\mathcal{B}_f$ be a union of cycles of a foundation of $T_1[T_2]$. Then*

$$\mathcal{B}(T_1[T_2]) = \mathcal{B}_f \cup \left(\cup_{t=1}^{h} \mathcal{B}_t \right) \cup$$

$$\left[\cup_{t=1}^{h} \left(\left[\cup_{r=3 \text{ and } r \text{ is odd}}^{m_t} \left(\cup_{i=1}^{k} a_{t,r}^{(n,i)} \right) \right] \cup \left[\cup_{r=4 \text{ and } r \text{ is even}}^{m_t} \left(\cup_{i=1}^{k} a_{t,r}^{(1,i)} \right) \right] \right) \right]$$

is a basis of $\mathcal{C}(T_1[T_2])$ where $\mathcal{B}_t$ is as in Lemma 3.4 for each $t = 1, 2, \ldots, h$.

Proof. By Lemma 3.2, we may assume that $\mathcal{B}_f = \cup_{t=1}^{h} \mathcal{B}_f^{(t)}$ where $\mathcal{B}_f^{(t)}$ is a foundation of $S_*^{(t)}[T_2]$ for each $t = 1, 2, \ldots, h$. Note that

$$\begin{aligned} \dim \mathcal{C}(T_1[T_2]) &= |E(T_1)||V(T_2)|^2 + |E(T_2)||V(T_1)| - \\ &\quad |V(T_1)||V(T_2)| + 1 \\ &= |E(T_1)||V(T_2)|^2 - |V(T_1)| + 1. \end{aligned} \tag{3.8}$$

Now, for $t = 1, 2, \ldots, h$

$$\begin{aligned} \left| \mathcal{B}_f^{(t)} \right| &= \sum_{i=2}^{m_t} \dim u_1^{(t)} u_i^{(t)} [N_{|V(T_2)|}] \\ &= \sum_{i=2}^{m_t} \left(|V(T_2)|^2 - 2|V(T_2)| + 1 \right) \\ &= \sum_{i=2}^{m_t} (|V(T_2)| - 1)^2 \\ &= (m_t - 1)(|V(T_2)| - 1)^2, \end{aligned} \tag{3.9}$$

and

$$|\mathcal{B}_t| = \sum_{r=1}^{m_t}\sum_{i=1}^{k}(n_i - 1) \tag{3.10}$$
$$= \sum_{r=1}^{m_t}\left(\sum_{i=1}^{k} n_i - k\right).$$

But,

$$\sum_{i=1}^{k} n_i = |V(T_2)| + k - 1. \tag{3.11}$$

Therefore,

$$|\mathcal{B}_t| = \sum_{r=1}^{m_t}(|V(T_2)| - 1) \tag{3.12}$$
$$= m_t\,(|V(T_2)| - 1).$$

Now

$$\cup_{i=1}^{k} a_{t,r}^{(1,i)} = \cup_{i=1}^{k} a_{t,r}^{(n,i)} \tag{3.13}$$
$$= \sum_{i=1}^{k}(n_i - 1)$$
$$= |V(T_2)| - 1.$$

Thus,

$$\left|\cup_{r=3r=3 \textit{ and } r \textit{ is odd}}^{m_j}\left(\cup_{i=1}^{k} a_{t,r}^{(n,i)}\right)\right| + \left|\cup_{r=4 \textit{ and } r \textit{ is even}}^{m_j}\left(\cup_{i=1}^{k} a_{t,r}^{(1,i)}\right)\right| \tag{3.14}$$
$$= \sum_{r=3}^{m_t}(|V(T_2)| - 1)$$
$$= (m_t - 2)\,(|V(T_2)| - 1).$$

From (3.9), (3.10) and (3.14), we have

$$\begin{aligned}
&\left|\mathcal{B}^{(t)}\right| + |\mathcal{B}_t| + \left|\cup_{r=3 \text{ and } r \text{ is odd}}^{m_t}\left(\cup_{i=1}^{k} a_{t,r}^{(n,i)}\right)\right| + \\
&\left|\cup_{r=4 \text{ and } r \text{ is even}}^{m_t}\left(\cup_{i=1}^{k} a_{t,r}^{(1,i)}\right)\right| \\
= \; & (m_t - 1)(|V(T_2)| - 1)^2 + m_t\left(|V(T_2)| - 1\right) + (m_t - 2)\left(|V(T_2)| - 1\right) \\
= \; & (m_t - 1)\left[(|V(T_2)| - 1)^2 + 2\left(|V(T_2)| - 1\right)\right] \\
= \; & (m_t - 1)\left[|V(T_2)|^2 - 1\right].
\end{aligned}$$

Therefore, By (3.8)

$$\begin{aligned}
|\mathcal{B}(T_1[T_2])| &= \sum_{j=1}^{h}\left((m_t - 1)\left[|V(T_2)|^2 - 1\right]\right) \\
&= \left[|V(T_2)|^2 - 1\right]\sum_{t=1}^{h}(m_t - 1) \\
&= \left[|V(T_2)|^2 - 1\right]\left[|V(T_1)| + h - 1 - h\right] \\
&= \left[|V(T_2)|^2 - 1\right]|E(T_1)| \\
&= |V(T_2)|^2|E(T_1)| - |V(T_1)| + 1 \\
&= \dim \mathcal{C}(T_1[T_2]).
\end{aligned}$$

Thus to prove that $\mathcal{B}(T_1[T_2])$ is a basis of $\mathcal{C}(T_1[T_2])$, it suffices to show that the cycles of $\mathcal{B}(T_1[T_2])$ are independent. For simplicity we assume that

$$\mathcal{F}_t = \left[\cup_{r=3 \text{ and } r \text{ is odd}}^{m_t}\left(\cup_{i=1}^{k} a_{t,r}^{(n,i)}\right)\right] \cup \left[\cup_{r=4 \text{ and } r \text{ is even}}^{m_t}\left(\cup_{i=1}^{k} a_{t,r}^{(1,i)}\right)\right]$$

We now prove that

$$\mathcal{B}(T_1[T_2])_t = \mathcal{B}^{(t)} \cup \mathcal{B}_t \cup \mathcal{F}_t$$

is linearly independent set of $\mathcal{C}(S_*^{(t)}[T_2])$ for each $t = 1, 2, \ldots h$. By Lemma 3.1 and Lemma 3.4 each of $\mathcal{B}^{(t)}$ and $\mathcal{B}_t$ is linearly independent. By Lemma 3.5 and Lemma 3.7 we have that $\mathcal{F}_t$ is a linearly independent set. Any non-trivial linear combinations of cycles of $\mathcal{F}_t$ is a cycle or an edge disjoint union of cycles each of which consists of 4 edges two of them from $u_1^{(t)}u_{s_1}^{(t)}[N_{|V(H)}]$ and the other two from $u_1^{(t)}u_{s_2}^{(t)}[N_{|V(H)}]$ for some $s_1 \neq s_2$. Therefore, by Lemma 3.6, $\mathcal{B}^{(t)} \cup \mathcal{F}_t$ is linearly independent. Any linear combination of cycles of $\mathcal{B}_t$ must contains at least one edge of $u_l^{(t)}\Box T_2$ for some l which is not in any cycle of $\mathcal{B}^{(t)} \cup \mathcal{F}_t$. Therefore, $\mathcal{B}(T_1[T_2])_t$ is linearly independent for each $t = 1, 2, \ldots, h$. To this end, we use mathematical induction on h to prove $\mathcal{B}(T_1[T_2])$ is linearly independent. The result is done for $h = 1$.

Assume that $h \geq 2$ and the theorem is true for smaller than h. We now prove that the cycles of $\mathcal{B}(T_1[T_2])_h$ are linearly independent of cycles of $\cup_{j=1}^{h-1}\mathcal{B}(T_1[T_2])_j$. Since

$$V(\cup_{t=1}^{j-1} S_*^{(t)}) \cap V(S_*^{(j)}) = \{u_1^{(j)}\},$$

we have

$$E(\cup_{t=1}^{j-1}\left(S_*^{(t)}[T_2]\right)) \cap E(S_*^{(j)}[T_2]) = E(u_1^{(j)}\Box T_2). \quad (3.15)$$

Suppose that $\sum_{i=1}^{\alpha} C_{j_i} = C_h \pmod 2$ where C_{j_i} and C_h are nontrivial linear combinations of cycles of $\mathcal{B}(T_1[T_2])_{j_i}$ and $\mathcal{B}(T_1[T_2])_h$, respectively, then $E(C_{j_1} \oplus C_{j_2} \oplus \cdots \oplus C_{j_\alpha}) = E(C_h)$ where the ring sum $C_{j_1} \oplus C_{j_2} \oplus \cdots \oplus C_{j_\alpha}$ and C_h are cycles or edge disjoint union of cycles. On the other hand, by equation 3.15, $C_{j_1} \oplus C_{j_2} \oplus \cdots \oplus C_{j_\alpha} \subseteq u_1^{(h)}\Box T_2$ which contradicts the fact that $u_1^{(h)}\Box T_2$ is a tree. Hence, $\mathcal{B}(T_1[T_2])$ is a basis of $\mathcal{C}(T_1[T_2])$. The proof is complete.

Let $\{x_1, x_2, \ldots, x_{|V(T_2)|}\}$ and $\{y_1, y_2, \ldots, y_{|V(T_2)|}\}$ be the two vertex sets of the complete bipartite graph $K_{|V(T_2)|,|V(T_2)|}$. Then

$$\mathcal{S} = \{x_i y_j x_{i+1} y_{j+1} x_i | 1 \leq i, j \leq |V(T_2)| - 1\} \quad (3.16)$$

is the Shmeichel 4-fold basis of $\mathcal{C}(K_{|V(T_2)|,|V(T_2)|})$ (see [18] Theorem 2.4). Let $e = u_1^{(t)}u_r^{(t)}$ and let $N_{|V(T_2)|}$ be the null graph with vertex set $V(T_2)$. By replacing x_1 by $(u_1^{(t)}, v_1^{(1)})$, x_2 by $(u_1^{(t)}, v_2^{(1)}), \ldots, x_{n_1}$ by $(u_1^{(t)}, v_{n_1}^{(1)})$, x_{n_1+1} by $(u_1^{(t)}, v_2^{(2)})$, x_{n_1+2} by $(u_1^{(t)}, v_3^{(2)}), \ldots$, $x_{n_1+n_2-1}$ by $(u_1^{(t)}, v_{n_2}^{(2)})$, $x_{n_1+n_2}$ by $(u_1^{(t)}, v_2^{(3)}), \ldots$, and $x_{|V(T_2)|}$ by $(u_1^{(t)}, v_{n_k}^{(k)})$ and y_1 by $(u_r^{(t)}, v_1^{(1)})$, y_2 by $(u_r^{(t)}, v_2^{(1)}), \ldots, y_{n_1}$ by $(u_r^{(t)}, v_{n_1}^{(1)})$, y_{n_1+1} by $(u_r^{(t)}, v_2^{(2)})$, y_{n_1+2} by $(u_r^{(t)}, v_3^{(2)})$, $\ldots$, $y_{n_1+n_2-1}$ by $(u_r^{(t)}, v_{n_2}^{(2)})$, $y_{n_1+n_2}$ by $(u_r^{(t)}, v_2^{(3)}), \ldots$, and $y_{|V(T_2)|}$ by $(u_r^{(t)}, v_{n_k}^{(k)})$ in equation 3.16, we obtain that $\mathcal{S}$ is a 4-fold basis of $\mathcal{C}(e[N_{|V(T_2)|}])$. From now on $\mathcal{B}_S^{(e)}$ stand for the Shmeichel 4-fold basis of $\mathcal{C}(e[N_{|V(T_2)|}])$ as in equation 3.16 after the above replacement.

Remark 3.2. *Notice that for each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, if $e = (u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_j^{(i)})$ or $(u_r^{(t)}, v_1^{(1)})(u_1^{(t)}, v_j^{(i)})$ or $(u_1^{(t)}, v_{n_k}^{(k)})(u_r^{(t)}, v_j^{(i)})$ or $(u_r^{(t)}, v_{n_k}^{(k)})(u_1^{(t)}, v_j^{(i)})$ then $f_{\mathcal{B}_e}(e) \leq 2$. Moreover, if $e = (u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_{n_k}^{(k)})$ or $(u_r^{(t)}, v_1^{(1)})(u_1^{(t)}, v_{n_k}^{(k)})$ or $(u_1^{(t)}, v_1^{(1)})(u_r^{(t)}, v_1^{(1)})$ or $(u_1^{(t)}, v_{n_k}^{(k)})(u_r^{(t)}, v_{n_k}^{(k)})$,, then $f_{\mathcal{B}_e}(e) = 1$. Finally, if e is not of the above forms, then $f_{\mathcal{B}_e}(e) \leq 4$.*

Lemma 3.9. *$\{\mathcal{B}_S^{(e)} : e \in E(T_1)\}$ is a foundation for $T_1[T_2]$.*

The following Lemmas are needed in the proof of next result.

Lemma 3.10. *Let m, n be two positive integers with $m \geq 2$ and*

$$4(m-1)(n^2-1) \leq 3\left(n^2(m-1)+m(n-1)\right).$$

Then $n \leq 5$.

Proof. Since $4(m-1)(n^2-1) \leq 3\left(n^2(m-1)+m(n-1)\right)$, we have that $4n^2(m-1)-4m+4 \leq 3n^2(m-1)+3mn-3m$. And so $n^2(m-1)+4 \leq m(3n+1)$. Thus, $n^2/(3n+1)+4/\left[(3n+1)(m-1)\right] \leq m/(m-1) < 2$, and so $n^2/(3n+1) \leq 2$ which implies that $n \leq 6$. For $n=6$, the direct substitution in the inequality yells that $17m \leq 22$ which is a contradiction. Therefore, $n \leq 5$.

Lemma 3.11. *Let m, n be two positive integers and*

$$(m-1)(n^2-1) \leq s+\left\lfloor\left(3n^2(m-1)+3m(n-1)-3s\right)/4\right\rfloor$$

where $s \leq 3m(n-1)$. Then 1) if $m=2$, then $n \leq 11$, 2) if $m=3$, then $n \leq 9$ 3) if $n \geq 4$, then $m \leq 8$.

Proof. Suppose that

$$(m-1)(n^2-1) \leq s+\left\lfloor\left(3n^2(m-1)+3m(n-1)-3s\right)/4\right\rfloor.$$

Since

$$\left\lfloor\left(3n^2(m-1)+3m(n-1)-3s\right)/4\right\rfloor \leq \left(3n^2(m-1)+3m(n-1)-3s\right)/4,$$

we have $4(m-1)(n^2-1) \leq 4s+\left(3n^2(m-1)+3m(n-1)-3s\right)$ which implies that $4(m-1)(n^2-1) \leq 3n^2(m-1)+3m(n-1)+s$. Thus, $4(m-1)(n^2-1) \leq 3n^2(m-1)+3m(n-1)+3m(n-1)$ and so $(m-1)n^2-4(m-1) \leq 6m(n-1)$. Which gives, $\left(n^2-4\right) \leq 6m(n-1)/(m-1)$. Thus, if $m=2$, then $n^2+8 \leq 12n$ and so $n \leq 11$. if $m=3$, then $n^2+5 \leq 9n$ and so $n \leq 9$. Finally, if $m \geq 4$, then $n^2+4 \leq 8n$ and so $n \leq 8$. The proof is complete.

Theorem 3.12. *For any T_1 and T_2 with $|V(T_1)| \geq 2$ and $|V(T_2)| \geq 2$, we have $b(T_1[T_2]) \leq 5$. Moreover, $b(T_1[T_2]) = 4$ whenever T_2 contains no subgraph isomorphic to 3-special star of order 7 and ($|V(T_1)| = 2$ and $|V(T_2)| \geq 12$) or ($|V(T_1)| = 3$ and $|V(T_2)| \geq 10$) or ($|V(T_1)| \geq 4$ and $|V(T_2)| \geq 9$).*

Proof. Let $\mathcal{B}_S = \cup_{e \in E(T_2)} \mathcal{B}_S^{(e)}$ and

$$\mathcal{B}(T_1[T_2]) = \mathcal{B}_S \cup \left(\cup_{t=1}^{h} \mathcal{B}_t\right) \cup$$

$\cup_{t=1}^{h}\left(\left[\cup_{r=3 \text{ and } r \text{ is odd}}^{m_t}\left(\cup_{i=1}^{k}a_{t,r}^{(n,i)}\right)\right]\cup\left[\cup_{r=4 \text{ and } r \text{ is even}}^{m_t}\left(\cup_{i=1}^{k}a_{t,r}^{(1,i)}\right)\right]\right)$.

By Lemma 3.9 and Theorem 2.6 $\mathcal{B}(T_1[T_2])$ is a basis of $\mathcal{C}(T_1[T_2])$. Note that if T_2 contains no subgraph isomorphic to a 3-special stars of order 7, then $v_{n_i}^{(i)} = v_1^{(i+1)}$. Therefor, by Remarks 3.1 and 3.2 one can show that the fold of any edge does not exceed 5 and if T_2 contains no subgraph isomorphic to a 3-special stars of order 7, then the fold of any edge does not exceed 4. To complete the proof of the theorem we show that $\mathcal{C}(T_1[T_2])$ has no 3-fold basis under the constraints which stated in the theorem on $|V(T_1)|$ and $|V(T_2)|$. Assume that $\mathcal{B}^*$ is a three fold basis of $\mathcal{C}(T_1[T_2])$ under the stated constraints. Then we consider the following three cases:

Case a. $\mathcal{B}^*$ consists only of 3-cycles. Then $|\mathcal{B}^*| \leq 3|V(T_1)|(\,|V(T_2)|-1)$ because every cycle must contains at least one edge of $\cup_{u\in V(T_1)}(u\square T_2)$ and the fold of every edge is at most 3. This is equivalent to the inequality that $|V(T_2)|^2|V(T_1)| - |V(T_2)|^2 - |V(T_1)| + 1 \leq 3|V(T_1)|(\,|V(T_2)|-1)$ and so $|V(T_2)|^2(|V(T_1)|-1) \leq |V(T_1)|(3|V(T_2)|-2)+1$, which implies that $|V(T_2)|^2 \leq (|V(T_1)|(3|V(T_2)|-2))/(|V(T_1)|-1) + 1/(|V(T_1)|-1) \leq 2(3|V(T_2)|-2)+1$. Thus, $|V(T_2)|^2 \leq 6|V(T_2)|-3$. Hence, $|V(T_2)| \leq 5$, a contradiction.

Case b. $\mathcal{B}^*$ consists only of cycles of length greater than or equal to 4. Then $4\,|\mathcal{B}^*| \leq 3\,|E(T_1[T_2])|$ because every edge is of fold 3 and the length of every cycle of $\mathcal{B}^*$ is greater than or equal to 4. That is $4(|V(T_2)|^2|V(T_1)| - |V(T_2)|^2 - |V(T_1)|+1) \leq 3(||E(T_1)||V(T_2)|^2 + |E(T_2)||V(T_1)|)$. By Lemma 3.10, it follows that $|V(T_2)| \leq 5$, a contradiction.

Case c. $\mathcal{B}^*$ consists of s 3-cycles and t cycles of length greater than or equal to 4. By Case a, $s \leq 3|V(T_1)|(\,|V(T_2)|-1)$. Since the fold of every edge of $T_1[T_2]$ is three and at most $3s$ edge are joint to make the s 3-cycles, we have that $t \leq \lfloor(3[|V(T_2)|^2(|V(T_1)|-1) + |V(T_1)|(|V(T_2)| \quad -1)] - 3s)/4\rfloor$. Hence, $|\mathcal{B}^*| = s+t \leq s+\lfloor(3[|V(T_2)|^2(|V(T_1)|-1) + |V(T_1)|(|V(T_2)|-1)] - 3s)/4\rfloor$. By Lemma 3.11, $|V(T_2)| \leq 11$, a contradiction. The proof is complete.

Remark 3.3. *One can easily see from the proof of Theorem 3.12 that if* $e \in E(u_r^{(t)}\square T_2)$ *where* $u_r^{(t)}$ *is an end vertex of* T_1*, then* $f_{\mathcal{B}(T_1[T_2])}(e) \leq 2$.

By specializing trees in the above theorem into paths and stares we obtain the following results.

Corollary 3.13. *Let* S_n *and* S_m *be two stars of order* n, m*, respectively. Then* $b(S_n[S_m]) = 4$ *whenever* ($n = 2$ *and* $m \geq 12$) *or* ($n = 3$ *and* $m \geq 10$) *or* ($n \geq 4$ *and* $m \geq 9$).

Corollary 3.14. *Let P_n and P_m be two paths of order n, m, respectively. Then $b(P_n[P_m]) = 4$ whenever ($n = 2$ and $m \geq 12$) or ($n = 3$ and $m \geq 10$) or ($n \geq 4$ and $m \geq 9$).*

Corollary 3.15. *Let P_n and S_m be a path and a star of order n, m, respectively. Then $b(P_n[S_m]) = 4$ whenever ($n = 2$ and $m \geq 12$) or ($n = 3$ and $m \geq 10$) or ($n \geq 4$ and $m \geq 9$). Also, $b(S_m[P_n]) = 4$ whenever ($m = 2$ and $n \geq 12$) or ($m = 3$ and $n \geq 10$) or ($m \geq 4$ and $n \geq 9$).*

4 Lexicographic product of graphs.

Throughout this section, T^G stands for the complement graph of a spanning tree T in G. Also, T_G stands for a spanning tree of G such that $\Delta(T_G) = \min\{\ \Delta(T) | T$ is a spanning tree of $G\}$. Moreover, for $e = uw \in E(G)$ we set

$$\mathcal{A}_e = \mathcal{B}_{e,1} \cup \mathcal{B}_{e,2} \cup \mathcal{B}_S^{(e)}$$

where $\mathcal{B}_{e,1}$ and $\mathcal{B}_{e,2}$ are the linearly independent sets as in Lemma 3.4 which obtained from $\mathcal{B}_{t,1}$ and $\mathcal{B}_{t,2}$, respectively, by replacing $u_1^{(t)}$ by u and $u_2^{(t)}$ by w . Thus, by theorem 3.12, $\mathcal{A}_e$ is linearly independent subset of $\mathcal{C}(e[T_2])$.

Theorem 4.1. *Let G, T_1 and T_2 be a graph, a spanning tree of G and a tree, respectively. Then, $b(G[T_2]) \leq \max\ \{5, 2(2 + \Delta(T_1^G)), 2 + b(G)\}$. Moreover, If T_1 contains no subgraph isomorphic to a 3-special star of order 7, then $b(G[T_2]) \leq \max\{4, 2(2 + \Delta(T_1^G)), 2 + b(G)\}$.*

Proof. Note that $G[T_2] = T_1[T_2] \cup \left(\cup_{e \in E(T_1^G)} e[N_{|V(T_1)|}]\right)$. Let $\mathcal{B}' = \mathcal{B}(T_1[T_2])$ where $\mathcal{B}(T_1[T_2])$ is as in Theorem 3.12. Let $\mathcal{B}'' = \cup_{e \in E(T_1^G)} \mathcal{A}_e$ and $\mathcal{B}'''$ be the corresponding required basis of $\mathcal{B}_G$ in $G \Box v_{nk}^{(k)}$. Set $\mathcal{B}(G[T_2]) = \mathcal{B}' \cup \mathcal{B}'' \cup \mathcal{B}'''$. Since

$$\begin{aligned} |\mathcal{B}(G[T_2])| &= \left|\mathcal{B}'\right| + \left|\mathcal{B}''\right| + \left|\mathcal{B}'''\right| \\ &= |V(T_2)|^2|E(T_1)| - |E(T_1)| + \dim \mathcal{C}(G)(|V(T_2)|^2 - 1) \\ &\quad + \dim \mathcal{C}(G) \\ &= |V(T_2)|^2 (|E(T_1)| + \dim \mathcal{C}(G)) - (|E(T_1)| + \dim \mathcal{C}(G)) \\ &\quad + \dim \mathcal{C}(G) \\ &= |V(T_2)|^2|E(G)| - |E(G)| + \dim \mathcal{C}(G) \\ &= |V(T_2)|^2|E(G)| - |V(G)| + 1 \\ &= \dim \mathcal{C}(G[T_2]), \end{aligned}$$

it suffices to show that $\mathcal{B}(G[T_2])$ is linearly independent and satisfies the fold which is stated in the Theorem. Since each linear combination of cycles of $\mathcal{B}^{'''}$ contains at least one edge $e \in E(G\Box v_{n_k}^{(k)}) - E(T_1\Box v_{n_k}^{(k)})$ and no cycle of $\mathcal{B}^{'}$ contains such edge, as a result $\mathcal{B}^{'} \cup \mathcal{B}^{'''}$ is linearly independent. By a similar argument as in lemma 3.4, we show that $\mathcal{B}^{''}$ is a linearly independent set. Note that for any $e = uw \in E(T^G)$, we have that

$$\begin{aligned} & E(uw\,[T_2]) \cap \left(E(T_1\,[T_2] \cup E(G\Box v_{n_k}^{(k)})\right) \\ \subseteq \; & \left\{E(u\Box T_2) \cup E(w\Box T_2) \cup (u, v_{n_k}^{(k)})(w, v_{n_k}^{(k)})\right\} \end{aligned}$$

which forms edges of a tree. Thus, if $\sum_{i=1}^{t} l_i = \sum_{i=1}^{s} c_i \pmod 2$ where $c_i \in \mathcal{B}^{'} \cup \mathcal{B}^{'''}$ and $l_i \in \mathcal{B}^{''}$, then $l_1 \oplus l_2 \oplus \cdots \oplus l_t$ is a subgraph of the forest. Which contradicts the fact that $l_1 \oplus l_2 \oplus \cdots \oplus l_t$ is a cycle or an edge disjoint union of cycles. Thus, $\mathcal{B}$ is linearly independent and so $\mathcal{B}$ is a basis. Let $e \in E(G\,[T_2])$.

(1) If $e \in E(G\Box v_{n_k}^{(k)})$, then $f_{\mathcal{B}'\cup\mathcal{B}''}(e) \leq 2$ and $f_{\mathcal{B}'''}(e) \leq b(G)$.

(2) If $e \in E(\cup_{u\in V(G)} E(u\Box T_2))$, then $f_{\mathcal{B}'}(e) \leq 4$, $f_{\mathcal{B}''}(e) \leq 2\Delta(T_1^G)$ and $f_{\mathcal{B}'''}(e) = 0$.

(3) If $e \in E(e^{'}\,[T_2]) - [E(G\Box v_{n_k}^{(k)}) \cup E(\cup_{u\in V(G)} E(u\Box T_2))]$, then

$$f_{\mathcal{B}'\cup\mathcal{B}''}(e) \leq \begin{cases} 4, & \text{if } T_1 \text{ has no subgraph isomorphic} \\ & \text{to a 3-special star of order 7,} \\ 5, & \text{otherwise.} \end{cases}$$

and $f_{\mathcal{B}'''}(e) = 0$ for $e^{'} \in E(G)$. The proof is complete.

Theorem 4.2. *Let G , H and T_1 be two graphs and a spanning tree of G, respectively. Then $b(G[H]) \leq \max\{5, 4 + 2\Delta(T_1^G) + b(H), 2 + b(G)\}$. Moreover, If T_1 contains no subgraph isomorphic to a 3-special star of order 7, then $b(G[H]) \leq \max\{4, 4 + 2\Delta(T_1^G) + b(H), 2 + b(G)\}$.*

Proof. Let $\mathcal{B}^{'} = \mathcal{B}(G[T_2])$ where $\mathcal{B}(G[T_2])$ is the basis of $\mathcal{C}(G\,[T_2])$ as in Theorem 4.1 where T_2 is a spanning tree of H. Let $\mathcal{B}^{''} = \bigcup_{v\in V(G)} \mathcal{B}_v$ where $\mathcal{B}_v$ is the corresponding required basis of $\mathcal{B}_H$ in $v\Box H$. Now, $E(v\Box H) \cap E(w\Box H) = \varnothing$ for each $u \neq w$. Thus, $\mathcal{B}^{''}$ is linearly independent. Moreover, each cycle of $\mathcal{B}^{''}$ contains an edge of $E(N_{|V(G)|}\Box T_2^H)$ which is not in any cycle of $\mathcal{B}^{'}$ where $N_{|V(G)|}$ is a null graph with vertex set $V(G)$. Thus $\mathcal{B}(G[H]) = \mathcal{B}^{'} \cup \mathcal{B}^{''}$ is linearly independent. Since

$$
\begin{aligned}
|\mathcal{B}(G[H])| &= \left|\mathcal{B}'\right| + \left|\mathcal{B}''\right| \\
&= |V(T_2)|^2\,|E(G)| - |V(G)| + 1 + |V(G)| \dim\, \mathcal{C}(H) \\
&= |V(T_2)|^2\,|E(G)| + |V(G)||E(H)| - |V(G)||V(H)| + 1 \\
&= \dim\, \mathcal{C}(G\,[H]),
\end{aligned}
$$

$\mathcal{B}(G[H])$ is a basis for $\mathcal{C}(G[H])$. It is an easy matter to see that $\mathcal{B}(G[H])$ satisfies the fold which stated in the theorem. The proof is complete.

In the rest of this work $T_{\min}$ stands for a spanning tree for G such that $\Delta(T^G_{\min}) = \min\{\ \Delta(T^G)|T$ is a spanning tree of $G\}$. The following result follows immediately from Theorem 4.2 and from the definition of $T_{\min}$.

Corollary 4.3. *Let G* and *H* be two graphs. Then $b(G[H]) \leq \max\{5, 4+2\Delta(T^G_{\min}) + b(H), 2 + b(G)\}$. *Moreover, If there is a $T_{\min}$ containing no subgraph isomorphic to a 3-special star of order 7, then* $b(G[H]) \leq \max\{4, 4 + 2\Delta(T^G_{\min}) + b(H), 2 + b(G)\}$.

The following corollary is a straightforward consequence from the proof of Theorems 4.1 and 4.2 and from Remark 3.1.

Corollary 4.4. *Let G and H be two graphs. If G has a spanning tree T such that T^G is a matching and each edge of T^G joins two end vertices of T, then $b(G[H]) \leq \max\{5, 4 + b(H), 2 + b(G)\}$. In addition, If H has a spanning tree contains no 3-special star of order 7, then $b(G[H]) \leq \max\{4, 4 + b(H), 2 + b(G)\}$.*

The following follows immediately by the same line of proof of Theorems 4.1 and 4.2 and by the aid of Corollary 4.4 and Remark 3.1.

Corollary 4.5. *If H is semi-Hamiltonian graph, then $b(G[H]) \leq \max\{4, 2 + \Delta(T^G_{\min}) + b(H), 2 + b(G)\}$. In addition, If G has a spanning tree T such that T^G is a matching and every edge of T^G joins two end vertices of T, then $b(G[H]) \leq \max\{4, 2 + b(H), 2 + b(G)\}$.*

By specializing graphs in the above results into paths, cycles, theta graphs, ladders and circular ladder and by using arguments similar to the last paragraph in Theorems 3.12 we obtain the following results.

Corollary 4.6. *(Ali and Marougi) $b(P_n\,[C_m]) = 4$ for $n, m \geq 7; b(C_n\,[P_m]) = b(C_n\,[C_m]) = 4$ for $n, m \geq 6$.*

Corollary 4.7. *(Jaradat and Alzoubi)* $b(P_n[L_m]) = b(P_n[CL_m]) = 4$ *for* $n \geq 2$ *and* $m \geq 7$*; also,* $b(C_n[L_m]) = b(C_n[CL_m]) = 4$ *for* $n \geq 4$ *and* $m \geq 5$.

Corollary 4.8. *(Alzoubi and Jaradat)* $b(\theta_n[P_m]) = b(\theta_n[C_m]) = 4$ *for* $n \geq 5$ *and* $m \geq 10$, $b(P_n[\theta_m]) = b(C_n[\theta_m]) = 4$ *for* $n \geq 4$ *and* $m \geq 8$ *and* $b(\theta_n[\theta_m]) = 4$ *for* $n \geq 5$ *and* $m \geq 13$.

Corollary 4.9. $b(S_n[\theta_m]) = b(S_n[C_m]) = 4$ *for* $n \geq 4$ *and* $m \geq 8$*;* $b(\theta_n[S_m]) = b(C_n[S_m]) = 4$ *for* $n \geq 5$ *and* $m \geq 10$*;* $b(L_n[P_m]) = b(L_n[C_m]) = b(CL_n[P_m]) = 4$ *for* $n \geq 3$ *and* $m \geq 6$*;* $b(\theta_n[L_m]) = b(\theta_n[CL_m]) = 4$ *for* $n \geq 4$ *and* $m \geq 4$ *and* $b(S_n[L_m]) = b(S_n[CL_m]) = 4$ *for* $n \geq 2$ *and* $m \geq 7$.

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