Multicompetition numbers of some multigraphs

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Abstract

In 1990, Anderson et al. [1] generalized the competition graph of a digraph to the competition multigraph of a digraph and defined the multicompetition number of a multigraph. The competition multigraph $CM(D)$ of a digraph $D = (V, A)$ is the multigraph $M = (V, E')$ where two vertices of V are joined by k parallel edges if and only if they have exactly k common preys in D. The multicompetition number $k^*(M)$ of the multigraph M is the minimum number p such that $M \cup I_p$ is the competition multigraph of an acyclic digraph, where I_k is a set of k isolated vertices. In this paper, we study the multicompetition numbers for some multigraphs and generalize some results

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provided by Kim and Roberts [9], and by Zhao and He [18] on general competition graphs, respectively.

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1 Introduction

In 1968, the notion of competition graph was introduced by Cohen [4] in connection with a problem in ecology. Let $D = (V, A)$ be a digraph, in which V is the vertex set and A is the set of directed arcs. The *competition graph* $C(D)$ of D is an undirected graph G with the same vertex set as D and with an edge $uv \in E(G)$ if and only if there exists some vertex x such that $(u, x), (v, x) \in A(D)$. We say that a graph G is a competition graph if there exists a digraph D such that $C(D) = G$. If (x, a) and (y, a) are arcs of digraph D, we say that a is a *common prey* of x and y.

Roberts [14] observed that, for any graph, the graph with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The minimum number of such isolated vertices was called the *competition number* of the graph G and was denoted by $k(G)$. We use I_r to denote the graph only consisting of r isolated vertices, and $G \cup I_r$ the graph consisting of the disjoint union of G and I_r.

Since the notion of competition graph was introduced, there has been a very large literature on it. For surveys of the literature of competition graphs, see [6, 7, 11, 14, 17]. Several variants of competition graphs also have been studied, see [1, 2, 3, 8, 12, 15, 16]. Much of the study focused on competition numbers, since the characterization of competition graphs of acyclic digraphs equals to compute the competition number of an arbitrary graph. But it seems to be difficult to compute the competition numbers of graphs in general. Opsut [13] showed that the computation of the competition number of a graph is an NP-hard problem. The competition numbers are known only for some special graph classes, see [7, 9, 10].

A *clique* of a graph G is a complete subgraph of G. An *edge clique cover* of a graph G is a family of cliques of G such that each edge of G is contained in some clique in the family. The minimum size of an edge clique cover of G is called the *edge clique cover number* of the graph G, and is denoted by $\theta_e(G)$. For the competition number and the edge clique cover number of a graph G, Opsut [13] provided the following result.

Theorem 1 (Opsut [13]) *For any graph G, $\theta_e(G) \leq k(G) + |V(G)| - 2$.*

In 1990, Anderson et al. [1] generalized the competition graph of a digraph to the competition multigraph of a digraph and defined the multicompetition number of a multigraph. The *competition multigraph* $CM(D)$ of a digraph $D = (V, A)$ is the multigraph $M = (V, E')$ where two vertices of V are joined by k parallel edges if and only if they have exactly k common preys in D. We call a multigraph M a competition multigraph if there exists a digraph D such that $CM(D) = M$. The *multicompetition number* $k^*(M)$ of the multigraph M is the minimum number p such that $M \cup I_p$ is the competition multigraph of an acyclic digraph.

For a multigraph $M = (V, E')$, the *underlying graph* $M \downarrow$ of M is obtained by replacing each multiple edge by a simple edge, and the *multiplicity* of each edge $uv \in E'$, denoted by $\mu_M(u, v)$ (or $\mu(u, v)$), is the number of edges joining vertices u and v in M. Let $|E'(M)|$ denote the edge number of M, and it is easy to see that $|E'(M)| = \sum_{xy \in E(M\downarrow)} \mu(x, y)$. For any subgraph G_1 of $M \downarrow$, the *corresponding multigraph* M_1 of G_1 is a submultigraph of M such that $G_1 = M_1 \downarrow$ and $\mu_{M_1}(x, y) = \mu_M(x, y)$ for any edge $xy \in E(G_1)$. An *edge clique partition* of a multigraph M is a family $\{Q_1, \ldots, Q_n\}$ of not necessarily distinct cliques of $M \downarrow$ such that $\mu(v_i, v_j) = |\{k : v_iv_j \in E(Q_k)\}|$ for each $v_iv_j \in E(M \downarrow)$. For a multigraph M, let $\theta_e^*(M)$ be the *edge clique partition number* of M, i.e., the smallest number of cliques which form an edge clique partition of M.

Anderson et al. [1] showed the relation between competition numbers and multicompetition numbers as follows.

Theorem 2 (Anderson et al. [1]) *For a multigraph M, $k(M \downarrow) \leq k^*(M)$.*

For any multigraph M, they provided the following bounds for the multicompetition number.

Theorem 3 (Anderson et al. [1]) *If M is a multigraph, then $\theta_e^*(M) - |V(M)| + 2 \leq k^*(M) \leq \theta_e^*(M)$.*

If a multigraph M is triangle-free and connected, they also got the following explicit formula for the multicompetition number.

Theorem 4 (Anderson et al. [1]) *If M is a connected triangle-free multigraph, then $k^*(M) = |E'(M)| - |V(M)| + 2$.*

In this paper, we continue to study the multicompetition numbers for some multigraph classes. In Section 2, we study the multicompetition numbers for Type 1 multigraphs (the multigraphs whose underlying graphs have exactly one triangle). In Section 3, we study the multicompetition numbers for Type 2 multigraphs (the multigraphs whose underlying graphs have at least one not K_2 clique, and any two different maximal cliques of the underlying graph have at most one common vertex). In Section 4, we show that the results in this paper generalized some results provided by Kim and Roberts in [9], and by Zhao and He in [18], respectively.

2 Multigraphs of type 1

In 1997, Kim and Roberts [9] investigated the competition number for the graphs with exactly one triangle. In this section we study the multicompetition number for the multigraphs whose underlying graphs have exactly one triangle. We first introduce a lemma.

Lemma 5 (Harary et al. [5]) *Let $D = (V, A)$ be a digraph. Then D is acyclic if and only if there exists an ordering of vertices, $\sigma = [v_1, v_2, \dots, v_n]$, such that one of the following two conditions holds,*

(1) *For all $i, j \in \{1, \dots, n\}$, $(v_i, v_j) \in A$ implies that $i < j$,*

(2) *For all $i, j \in \{1, \dots, n\}$, $(v_i, v_j) \in A$ implies that $i > j$.*

In the following we use the similar but much more complex ways than those used in [9] to get some analogue results. A *multitree* is a multigraph whose underlying graph is a tree. A *multiforest* is a multigraph such that each connected component of it is a multitree.

Lemma 6 *For a multitree T and a vertex v of T, there is an acyclic digraph D so that $T \cup I_k$ is the competition multigraph of D and so that v has only outgoing arcs in D, where $k = |E'(T)| - |V(T)| + 2$, $I_k = \{u_1, u_2, \dots, u_k\}$ and $u_1, u_2, \dots, u_k$ are new vertices not in T.*

Proof. By Theorem 4, $k^*(T) = |E'(T)| - |V(T)| + 2$. Now we construct an acyclic digraph D so that $T \cup I_k$ is the competition multigraph of D, where $k = |E'(T)| - |V(T)| + 2$, $I_k = \{u_1, u_2, \dots, u_k\}$ and $u_1, u_2, \dots, u_k$ are new vertices not in T.

Let $T_1 = T$, $V(D_1) = V(T)$ and $A(D_1) = \emptyset$. Choose a vertex v_1 from T_1 such that the degree of v_1 in $T_1 \downarrow$ is 1. If v_1' is adjacent to v_1 in T_1, then let $T_2 = T_1 - v_1$, $V(D_2) = V(D_1) \cup \{u_{1,1}, u_{1,2}, \dots, u_{1,\mu(v_1,v_1')}\}$ for $\mu(v_1, v_1')$ vertices not in D_1, and $A(D_2) = \bigcup_{j=1}^{\mu(v_1,v_1')}\{(v_1, u_{1,j}), (v_1', u_{1,j})\}$. Choose a vertex v_2 from T_2 such that the degree of v_2 in $T_2 \downarrow$ is 1. If v_2' is adjacent to v_2 in T_2, let $T_3 = T_2 - v_2$, $V(D_3) = V(D_2) \cup \{u_{2,1}, u_{2,2}, \dots, u_{2,\mu(v_2,v_2')-1}\}$ for $\mu(v_2, v_2') - 1$ vertices not in $V(D_2)$, and $A(D_3) = A(D_2) \cup \{(v_2, v_1), (v_2', v_1)\} \cup \bigcup_{j=1}^{\mu(v_2,v_2')-1}\{(v_2, u_{2,j}), (v_2', u_{2,j})\}$. Suppose that we have defined T_i and D_i, $i \geq 3$, choose a vertex v_i from T_i such that the degree of v_i in $T_i \downarrow$ is 1. If v_i' is adjacent to v_i in T_i, then let $T_{i+1} = T_i - v_i$, $V(D_{i+1}) = V(D_i) \cup \{u_{i,1}, u_{i,2}, \dots, u_{i,\mu(v_i,v_i')-1}\}$, and $A(D_{i+1}) = A(D_i) \cup \{(v_i, v_{i-1}), (v_i', v_{i-1})\} \cup \bigcup_{j=1}^{\mu(v_i,v_i')-1}\{(v_i, u_{i,j}), (v_i', u_{i,j})\}$. Repeat the last step until $D_{|V(T)|}$ has been defined. Let $D = D_{|V(T)|}$.

In the procedure, we may avoid selecting v until we select all other vertices since there are at least two vertices of degree 1 in a tree with more than one vertex, thus we may guarantee that v has only outgoing arcs in D. Note that in the whole process for constructing D, we add $\sum_{xy\in E(T\downarrow)}(\mu(x,y)-1)+1=|E'(T)|-V(T)+2$ new vertices to $V(D_1)$ and get $V(D_{|V(T)|})$. It is easy to check that $CM(D)=T\cup I_k$, where $k=|E'(T)|-|V(T)|+2$ and $I_k=\{u_1,u_2,\ldots,u_k\}=\{u_{1,1},u_{1,2},\ldots,u_{1,\mu(v_1,v_1')}\}\cup\bigcup_{i=2}^{|V(T)|-1}\{u_{i,1},u_{i,2},\ldots,u_{i,\mu(v_i,v_i')-1}\}$. ∎

Theorem 7 *If a multigraph M is connected and $M\downarrow$ has exactly one triangle, then*

$$|E'(M)|-|V(M)|-2s+2\le k^*(M)\le |E'(M)|-|V(M)|-s+2,$$

where $s=\min\{\mu(u,v): uv$ is an edge in the triangle of $M\downarrow\}$.

Proof. Note that, except for the triangle, all the other cliques of $M\downarrow$ are K_2, so $\theta_e^*(M)=|E'(M)|-3s+s=|E'(M)|-2s$, and the lower bound $k^*(M)\ge|E'(M)|-|V(M)|-2s+2$ follows from Theorem 3.

To prove the upper bound $k^*(M)\le|E'(M)|-|V(M)|-s+2$, let $\{x,y,z\}$ be the vertex set of the triangle and we may assume that $\mu(x,y)=s$. Deleting the parallel edges between x and y from M, the resulting graph, denoted by $M-xy$, is a multitree, so Theorem 4 implies that $k^*(M-xy)=|E'(M-xy)|-|V(M-xy)|+2=|E'(M)|-|V(M)|-s+2$. Let D' be an acyclic digraph whose competition multigraph is $(M-xy)\cup I_k$, where $k=|E'(M)|-|V(M)|-s+2$. Since there are $\mu(x,z)$ edges joining x and z, and $\mu(y,z)$ edges joining y and z in $M-xy$, there are arcs (x,a_i), (z,a_i), (y,b_j) and (z,b_j) in D' for vertices a_i and b_j of D', where $i=1,2,\ldots,\mu(x,z)$ and $j=1,2,\ldots,\mu(y,z)$. Since there are no edges between x and y in $M-xy$, then $a_i\ne b_j$ for any $i\in\{1,2,\ldots,\mu(x,z)\}$ and any $j\in\{1,2,\ldots,\mu(y,z)\}$. By Lemma 5, there is an acyclic labelling π of D' such that whenever (u,v) is an arc of D', $\pi(v)<\pi(u)$. So we have $\pi(x)>\pi(a_i)$ and $\pi(y)>\pi(b_j)$ for any $i\in\{1,2,\ldots,\mu(x,z)\}$ and $j\in\{1,2,\ldots,\mu(y,z)\}$. Since $\pi(x)\ne\pi(y)$, we may assume that $\pi(x)<\pi(y)$. Add arcs (y,a_i) to D' and delete arcs (y,b_i) from D' for $i=1,2,\ldots,s$ to obtain a digraph D. The digraph D is acyclic because $\pi(y)>\pi(x)$ and $\pi(x)>\pi(a_i)$ imply $\pi(y)>\pi(a_i)$ for all $i\in\{1,2,\ldots,\mu(x,z)\}$. Therefore, $CM(D)=M\cup I_k$, where $k=|E'(M)|-|V(M)|-s+2$, and so $k^*(M)\le|E'(M)|-|V(M)|-s+2$. ∎

Theorem 8 *Suppose that a multigraph M is connected and $M\downarrow$ has at least two cycles, including at least one triangle. Then*

$$k^*(M)\le|E'(M)|-|V(M)|-2s+2,$$

where $s=\min\{\mu(u,v): uv$ is an edge in the triangle of $M\downarrow\}$.

Proof. Let $\{x, y, z\}$ be the vertex set of an triangle S which has at least one edge with multiplicity s. Let T be a spanning tree of $M \downarrow$ that has exactly two edges of S. Now we delete those two edges from T. Then the resulting graph is a forest with exactly three tree components, say T_1, T_2 and T_3. Clearly, each component contains exactly one of $\{x, y, z\}$. We may assume that x belongs to T_1, y belongs to T_2 and z belongs to T_3. Since there is one more cycle other than S, there is an edge $fg \in E(M \downarrow) - E(T)$. We may assume that T_1 does not contain f or g. Whether or not f and g belong to the same component, we may assume that g belongs to T_3. We use T_i' to denote the corresponding multitree of T_i and let $\ell_i = |E'(T_i')| - |V(T_i')| + 2$, where $i = 1, 2, 3$. Without loss of generality, we consider the following three cases.

Case 1. T_1 and T_2 are trivial, T_3 is nontrivial.

Note that in this case, either $f = y$ or f also belongs to T_3. By Lemma 6, there are an acyclic digraph D_3 and added vertices $u_{3,1}, u_{3,2}, \ldots, u_{3,\ell_3}$ so that the competition multigraph of D_3 is $T_3' \cup \{u_{3,1}, u_{3,2}, \ldots, u_{3,\ell_3}\}$. By the proof of Lemma 6, we may assume that each $u_{3,i}$ has no outgoing arcs in D_3. Let D' be a digraph whose vertex set is

$$V(M) \cup \bigcup_{i=1}^{\ell_3} \{u_{3,i}\} \cup \bigcup_{j=1}^{s} \{y_1^j\} \cup \bigcup_{k=1}^{\mu(f,g)-1} \{y_2^k\}$$

and whose arc set is

$$\begin{aligned} A(D_3) \cup \bigcup_{j=1}^{s} \{(x, y_1^j), (y, y_1^j), (z, y_1^j)\} \cup \{(f, x), (g, x)\} \cup \bigcup_{k=1}^{\mu(f,g)-1} \{(f, y_2^k), (g, y_2^k)\} \\ - \{(v, u_{3,1}) : (v, u_{3,1}) \in A(D_3)\} \cup \{(v, y) : (v, u_{3,1}) \in A(D_3)\}. \end{aligned}$$

We note that D' is acyclic, and $u_{3,1}$ has neither incoming nor outgoing arcs in D'. Let $D'' = D' - u_{3,1}$.

Case 2, T_1 is trivial, T_2 and T_3 are nontrivial.

By Lemma 6, there are acyclic digraphs D_i and added vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,\ell_i}$ so that the competition multigraph of D_i is $T_i' \cup \{u_{i,1}, u_{i,2}, \ldots, u_{i,\ell_i}\}$, where $i = 2, 3$. By the proof of Lemma 6, we may assume that each $u_{i,j}$ have no outgoing arcs. In each digraph D_i, the vertices v and v' of the highest and the second-highest indices, respectively, in an acyclic labeling can be assumed to have only outgoing arcs since the only possible incoming arc to either of these vertices is from v to v' and we can always delete this arc without changing the competition multigraph. Now let x_2 and y_2 be the vertices having only outgoing arcs in D_2. Let D' be a digraph whose vertex set is

$$V(M) \cup \bigcup_{i=2}^{3}\{u_{i,1}, u_{i,2}, \ldots, u_{i,\ell_i}\} \cup \bigcup_{j=1}^{s}\{y_1^j\}$$

and whose arc set is

$$A(D_2) \cup A(D_3) \cup \bigcup_{j=1}^{s}\{(x, y_1^j), (y, y_1^j), (z, y_1^j)\}$$
$$- \{(v, u_{2,1}) : (v, u_{2,1}) \in A(D_2)\} \cup \{(v, x) : (v, u_{2,1}) \in A(D_2)\}$$
$$- \{(v, u_{3,1}) : (v, u_{3,1}) \in A(D_3)\} \cup \{(v, x_2) : (v, u_{3,1}) \in A(D_3)\}.$$

We note that D' is acyclic, and $u_{2,1}$, $u_{3,1}$ have neither incoming nor outgoing arcs in D'. By Lemma 6, we may assume that $f = x_2$ if f belongs to T_2. Now we delete vertices $u_{2,1}$ and $u_{3,1}$ from D' and add vertices y_2^i and arcs $(f, y_2), (g, y_2), (f, y_2^i), (g, y_2^i)$ to D' for $i = 1, 2, \ldots, \mu(f, g) - 1$ to obtain D''. This still leaves an acyclic digraph since the arc (g, y_2) goes from D_3 to D_2 and the arc (f, y_2) goes from D_3 to D_2, or from a vertex with no incoming arcs in D_2.

Case 3, T_1, T_2 and T_3 are nontrivial.

By Lemma 6, there are acyclic digraphs D_i and added vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,\ell_i}$ so that the competition multigraph of D_i is $T_i' \cup \{u_{i,1}, u_{i,2}, \ldots, u_{i,\ell_i}\}$, where $i = 1, 2, 3$. By the proof of Lemma 6, we may assume that the $u_{i,j}$ have no outgoing arcs and x has only outgoing arcs in D_1. By the same case as in Case 2, let y_1 be another vertex that has only outgoing arcs in D_1 and let x_2 and y_2 be vertices having only outgoing arcs in D_2. Let D' be a digraph whose vertex set is

$$V(M) \cup \bigcup_{i=1}^{3}\{u_{i,1}, u_{i,2}, \ldots, u_{i,\ell_i}\} \cup \bigcup_{j=1}^{s-1}\{y_1^j\}$$

and whose arc set is

$$\bigcup_{i=1}^{3} A(D_i) \cup \{(x, y_1), (y, y_1), (z, y_1)\} \cup \bigcup_{j=1}^{s-1}\{(x, y_1^j), (y, y_1^j), (z, y_1^j)\}$$
$$- \{(v, u_{2,1}) : (v, u_{2,1}) \in A(D_2)\} \cup \{(v, x) : (v, u_{2,1}) \in A(D_2)\}$$
$$- \{(v, u_{3,1}) : (v, u_{3,1}) \in A(D_3)\} \cup \{(v, x_2) : (v, u_{3,1}) \in A(D_3)\}.$$

We note that D' is acyclic, and $u_{2,1}$, $u_{3,1}$ have neither incoming nor outgoing arcs in D'. By Lemma 6, we may assume that $f = x_2$ if f belongs to T_2. Now we delete vertices $u_{2,1}$ and $u_{3,1}$ from D', add vertices y_2^i, arcs $(f, y_2), (g, y_2), (f, y_2^i), (g, y_2^i)$ to D' to obtain D'', where $i = 1, 2, \ldots, \mu(f, g) - 1$. This still leaves an acyclic digraph.

Let $\overline{E} = E(M\downarrow) - \bigcup_{1\le i\le 3} E(T_i) - \{xy, xz, yz, fg\}$. Now we construct D from the D'' constructed in each case above, by adding $\mu(u,v)$ new isolated vertices to D'', corresponding to each edge $uv \in \overline{E}$, and arcs from the end vertices of uv to each of these vertices. For each edge ab in the triangle whose multiplicity is bigger than s, adding $\mu(a,b) - s$ vertices and arcs from the end vertices of ab to each of these vertices. It is easy to check that D is still acyclic and its competition multigraph is $M \cup I_{|V(D)|-|V(M)|}$. Now we count $|V(D)| - |V(M)|$ by summing the number of vertex added to $V(M)$ in each step (we just count it for the D that was gotten from D'' constructed in Case 3, and the other two cases are similar).

$$
\begin{aligned}
&|V(D)| - |V(M)| \\
&= \sum_{i=1}^{3} \ell_i + (s-1) + (\mu(f,g)-1) - 2 + \sum_{uv\in\overline{E}} \mu(u,v) + \sum_{uv\in\{xy,yz,xz\}} (\mu(u,v) - s) \\
&= \sum_{i=1}^{3} (|E'(T_i')| - |V(T_i')| + 2) + \mu(f,g) + \sum_{uv\in\overline{E}\cup\{xy,yz,xz\}} \mu(u,v) - 2s - 4 \\
&= \sum_{i=1}^{3} |E'(T_i')| + \mu(f,g) + \sum_{uv\in\overline{E}\cup\{xy,yz,xz\}} \mu(u,v) - \sum_{i=1}^{3} |V(T_i')| - 2s + 2 \\
&= |E'(M)| - |V(M)| - 2s + 2.
\end{aligned}
$$

Therefore, $CM(D) = M \cup I_{|E'(M)|-|V(M)|-2s+2}$ and we have $k^*(M) \le |E'(M)| - |V(M)| - 2s + 2$. ∎

By Theorems 7 and 8, we immediately have the following result.

Corollary 9 *Suppose that a multigraph M is connected and $M\downarrow$ has exactly one triangle. Let $s = \min\{\mu(u,v) : uv$ is an edge in the triangle in $M\downarrow\}$. If $M\downarrow$ has a cycle of length at least 4, then*

$$k^*(M) = |E'(M)| - |V(M)| - 2s + 2.$$

Otherwise

$$|E'(M)| - |V(M)| - 2s + 2 \le k^*(M) \le |E'(M)| - |V(M)| - s + 2.$$

3 Multigraphs of type 2

In this section, we study the multicompetition number for a class of multigraphs whose underlying graphs include at least one clique of size at least 3, and any two different maximal cliques of the underlying graph have at most one common vertex. Note that the subgraph of $G = (V, E)$ induced by $E_1 \subset E$ is denoted by $G[E_1]$.

Theorem 10 *Suppose that a multigraph M is connected, all the maximal cliques of size at least 3 of $M\downarrow$ are $K_{n_1}, K_{n_2}, \ldots, K_{n_t}$, and $s_i = min\{\mu(u,v) : uv \in E(K_{n_i})\}$, where $t \geq 1$ and $n_i \geq 3$ for $i = 1, 2, \ldots, t$. If any two different maximal cliques of $M\downarrow$ have at most one common vertex and $K_{n_i}[\{uv : \mu(u,v) > s_i, uv \in E(K_{n_i})\}]$ is triangle-free for each $i \in \{1, 2, \ldots, t\}$, then*

$$|E'(M)| - |V(M)| - \sum_{i=1}^{t}\left(\binom{n_i}{2} - 1\right)s_i + 2 \leq k^*(M)$$
$$\leq |E'(M)| - |V(M)| - \sum_{i=1}^{t}\binom{n_i - 1}{2}s_i + 2.$$

Proof. Suppose that any two different maximal cliques of $M\downarrow$ have at most one common vertex and $K_{n_i}[\{uv : \mu(u,v) > s_i, uv \in E(K_{n_i})\}]$ is triangle-free for each $i \in \{1, 2, \ldots, t\}$. By the condition of the theorem, each clique $K_{n_i}, i = 1, 2, \ldots, t$, has $\binom{n_i}{2}$ edges, and the other cliques of $M\downarrow$ are all K_2, so we have

$$\theta_e^*(M) = |E'(M)| - \sum_{i=1}^{t}\binom{n_i}{2}s_i + \sum_{i=1}^{t}s_i = |E'(M)| - \sum_{i=1}^{t}\left(\binom{n_i}{2} - 1\right)s_i.$$

By Theorem 3,

$$k^*(M) \geq \theta_e^*(M) - |V(M)| + 2 = |E'(M)| - \sum_{i=1}^{t}\left(\binom{n_i}{2} - 1\right)s_i - |V(M)| + 2. \quad (1)$$

In order to prove the upper bound, Let $\{v_1^r, v_2^r, \ldots, v_{n_r}^r\}$ be the vertex set of K_{n_r}, where $r = 1, 2, \ldots, t$. Let $E_r = \bigcup_{2 \leq i < j \leq n_r}\{v_i^r v_j^r\}$, where $r = 1, 2, \ldots, t$. Deleting the edges in $\bigcup_{r=1}^{t} E_r$ from $M\downarrow$, the resulting graph $M\downarrow - \bigcup_{r=1}^{t} E_r$ is connected and triangle-free. Let M' denote the corresponding multigraph of $M\downarrow - \bigcup_{r=1}^{t} E_r$. Theorem 4 implies that $k^*(M') = |E'(M')| - |V(M')| + 2 = |E'(M')| - |V(M)| + 2$. Let D' be an acyclic digraph whose competition multigraph is $M' \cup I_{|E'(M')| - |V(M)| + 2}$. For each $r \in \{1, 2, \ldots, t\}$, since $\{v_1^r v_2^r, v_1^r v_3^r, \ldots, v_1^r v_{n_r}^r\}$ is a subset of $E(M\downarrow - \bigcup_{r=1}^{t} E_r)$, there are arcs $(v_1^r, a_{i,j}^r)$, $(v_i^r, a_{i,j}^r)$ in D' for a vertex $a_{i,j}^r$ of D', where $i = 2, 3, \ldots, n_r$ and $j = 1, 2, \ldots, \mu(v_1^r, v_i^r)$. For each $r \in \{1, 2, \ldots, t\}$ and any different $i_1, i_2 \in \{2, 3, \ldots, n_r\}$, since $v_{i_1}^r v_{i_2}^r \notin E(M\downarrow - \bigcup_{r=1}^{t} E_r)$, then $a_{i,j}^r$ is distinct for each $i \in \{2, 3, \ldots, n_r\}$ and each $j \in \{1, 2, \ldots, \mu(v_1^r, v_i^r)\}$. By Lemma 5, there is an acyclic labeling π of D' such that $\pi(y) < \pi(x)$ whenever (x, y) is an arc of D'. Without loss of generality, we may assume that $\pi(v_2^r) < \pi(v_i^r)$ for any $i \in \{3, 4, \ldots, n_r\}$ and $r \in \{1, 2, \ldots, t\}$. Let D be the digraph obtained from D' by adding arcs $(v_i^r, a_{2,j}^r)$ to D', deleting arcs $(v_i^r, a_{i,j}^r)$ from D', and for each edge $uv \in E_r$ adding $\mu(u,v) - s_r$ vertices to D' and adding arcs from the end vertices of uv to each of these vertices, where $i = 3, 4, \ldots, n_r$, $j = 1, 2, \ldots, s_r$ and $r = 1, 2, \ldots, t$. Note that the number of the vertices adding to D' in this step is

$$\sum_{r=1}^{t} \sum_{uv \in E_r} (\mu(u,v) - s_r) = \sum_{r=1}^{t} \left(\sum_{uv \in E_r} \mu(u,v) - \binom{n_r - 1}{2} s_r \right).$$

It is easy to see that D is acyclic. Therefore, the competition multigraph of D is

$$M \cup I_{|E'(M')|-|V(M)|+2} \cup I_{\sum_{r=1}^{t}\left(\sum_{uv\in E_r} \mu(u,v) - \binom{n_r-1}{2} s_r\right)} = M \cup I_{|E'(M)|-|V(M)|-\sum_{r=1}^{t}\binom{n_r-1}{2}s_r+2}.$$

Hence

$$k^*(M) \le |E'(M)| - |V(M)| - \sum_{r=1}^{t} \binom{n_r - 1}{2} s_r + 2. \tag{2}$$

Combining (1) and (2), the conclusion follows. ∎

The following corollary is the special case of the theorem above when $t = 1$ and $n_1 = m$.

Corollary 11 *Suppose that a multigraph M is connected, $M \downarrow$ has a clique K_m, every triangle of $M \downarrow$ is included in the clique K_m, and $s = min\{\mu(u,v) : uv \in E(K_m)\}$, where $3 \le m \le |V(M)|$. If $K_m[\{uv : \mu(u,v) > s, uv \in E(K_m)\}]$ is triangle-free, then*

$$|E'(M)| - |V(M)| - \left(\binom{m}{2} - 1\right) s + 2 \le k^*(M) \le |E'(M)| - |V(M)| - \binom{m-1}{2} s + 2.$$

On the other hand, if $n_i = 3$ for $i = 1, 2, \ldots, t$, then the following corollary follows from Theorem 10.

Corollary 12 *Suppose that a multigraph M is connected, all the maximal cliques of size at least 3 of $M \downarrow$ are triangles $\Delta_1, \Delta_2, \ldots, \Delta_t$, and $s_i = min\{\mu(u,v) : uv \in E(\Delta_i)\}$, where $i = 1, 2, \ldots, t$. If any two different triangles of $M \downarrow$ have no common edge, then*

$$|E'(M)| - |V(M)| - 2\sum_{i=1}^{t} s_i + 2 \le k^*(M) \le |E'(M)| - |V(M)| - \sum_{i=1}^{t} s_i + 2.$$

It is easy to see that Theorem 7 follows from Corollary 11 when $m = 3$, or from Corollary 12 when $t = 1$.

4 Conclusions

Theorems 7, 8 and 10 generalize Theorems 5, 6 in [9] and 2.4 in [18], respectively. The following three corollaries show the details. Note that we may regard a simple graph G as a multigraph such that each edge with multiplicity one. So we have $E'(G) = E(G)$ and $|E'(G)| = |E(G)|$.

Corollary 13 (Kim and Roberts [9]) *If a graph G is connected and has exactly one triangle, then $k(G) = |E(G)|-|V(G)|$ or $|E(G)|-|V(G)|+1$.*

Proof. Since G has exactly one triangle, then $\theta_e(G) = |E(G)| - 3 + 1 = |E(G)|-2$. By Theorem 1, $k(G) \geq |E(G)|-2-|V(G)|+2 = |E(G)|-|V(G)|$. By Theorems 2 and 7, $k(G) \leq k^*(G) \leq |E'(G)| - |V(G)| - 1 + 2 = |E(G)| - |V(G)| + 1$. ∎

Corollary 14 (Kim and Roberts [9]) *Suppose that a graph G is connected and has at least two cycles, including at least one triangle. Then $k(G) \leq |E(G)| - |V(G)|$.*

Proof. By Theorems 2 and 8, $k(G) \leq k^*(G) \leq |E'(G)| - |V(G)| - 2 + 2 = |E(G)| - |V(G)|$. ∎

Corollary 15 (Zhao and He [18]) *Suppose that all the maximal cliques of size at least 3 of a connected graph G are $K_{n_1}, K_{n_2}, \ldots, K_{n_t}$, where $t \geq 1$ and $n_i \geq 3$ for $i = 1, 2, \ldots, t$. If any two different maximal cliques of G have at most one common vertex, then*

$$|E(G)|-|V(G)|-\sum_{i=1}^{t}\binom{n_i}{2}+t+2 \leq k(G) \leq |E(G)|-|V(G)|-\sum_{i=1}^{t}\binom{n_i-1}{2}+2.$$

Proof. Suppose any two different maximal cliques of G have at most one common vertex. By the condition of the corollary, $\theta_e(G) = |E(G)| - \sum_{i=1}^{t}\binom{n_i}{2} + t$. By Theorem 1, $k(G) \geq \theta_e(G) - |V(G)| + 2 = |E(G)| - |V(G)| - \sum_{i=1}^{t}\binom{n_i}{2} + t + 2$. By Theorems 2 and 10, $k(G) \leq k^*(G) \leq |E'(G)| - |V(G)| - \sum_{r=1}^{t}\binom{n_r-1}{2} + 2 = |E(G)| - |V(G)| - \sum_{r=1}^{t}\binom{n_r-1}{2} + 2$. ∎

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