SYMMETRIC f BI-DERIVATIONS OF LATTICES

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Abstract. In this paper, we introduced the notion of symmetric f bi-derivations on lattices and investigated some related properties. We characterized the distributive lattice by symmetric f bi-derivations.

1. Introduction

The study of lattices is of great importance in several areas such as information theory, information retrieval, information access controls and cryptanalysis [2], [6], [7], [16].

The notion of derivation on a ring has an important role for the characterization of rings. As generalizations of derivations, α-derivations and symmetric bi-derivations on prime and semi-prime rings are studied by a lot of researchers [14], [9], [18], [19]. Firstly, the notion of derivation on a lattice was defined and some of its related properties were examined by Szasz, G. [15]. Luca Ferrari [8] pursued and completed the problems initiated by Szasz. Then, X. L. Xin, T. Y. Li, and J. H. Lu [20] went on studying the notion of derivation on a lattice and investigated some of its properties. Later, f-derivations and symmetric bi-derivations on a lattice and some properties related with these derivations were discussed by [5], [4] respectively.

In this paper we introduced the notion of symmetric f bi-derivations on a lattice and investigated some of its properties that were discussed in [4] for a symmetric f bi-derivation on a lattice. Additionally, we researched for a symmetric f bi-derivation of a lattice some properties that were stated in [18], [19] for a symmetric bi-derivation on prime and semi-prime rings. We characterized distributive and modular lattices with symmetric f bi-derivation under some conditions.

2. Preliminaries

Definition 2.1. [2] Let L be a nonempty set endowed with operations "$\wedge$" and "$\vee$". If $(L, \wedge, \vee)$ satisfies the following conditions: for all x, y, z in L

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(a) $x \wedge x = x$, $x \vee x = x$,
(b) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
(c) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,$(x \vee y) \vee z = x \vee (y \vee z)$,
(d)$(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$, then L is called as a lattice.

Definition 2.2. [2] Let $(L, \wedge, \vee)$ be a lattice. A binary relation " $\leq$ " is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Definition 2.3. [2] A lattice L is distributive if the following identities hold:
(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
In any lattice, the above conditions are equivalent.

Definition 2.4. [1] A lattice L is modular if the following identity holds: If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$

Definition 2.5. [3] An ideal is a non-void subset I of a lattice L with the properties
(1) $x \leq y$, $y \in \mathrm{I} \Rightarrow x \in \mathrm{I}$
(2) x, $y \in \mathrm{I} \Rightarrow x \vee y \in \mathrm{I}$

Lemma 2.6. *Let $(L, \wedge, \vee)$ be a lattice. Define the binary relation $x \leq y$ as Definition 2.2. Then $(L, \leq)$ is a poset and for any x, $y \in L$, $x \wedge y$ is the greatest lower bound of $\{x, y\}$, and $x \vee y$ is the lowest upper bound of $\{x, y\}$.*

Definition 2.7. Let L be a lattice. A mapping $D(.,.) : L \times L \rightarrow L$ is called symmetric if $D(x, y)= D(y, x)$ holds for all x, y in L.

Definition 2.8. Let L be a lattice. A mapping d: $L \rightarrow L$ defined by $d(x) = D(x, x)$ is called the trace of $D(.,.)$, where $D(.,.) : L \times L \rightarrow L$ is a symmetric mapping.

3. The Symmetric f Bi-Derivation on Lattices

The following definition introduces the notion of symmetric f bi-derivation for a lattice.

Definition 3.1. Let L be a lattice and $D : L \times L \rightarrow L$ be a symmetric mapping. We call D a symmetric f bi-derivation on L if there exists a function $f : L \rightarrow L$ such that $D(x \wedge y, z) = (D(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z))$ for all x, y, z in L.

It is clear that a symmetric f bi-derivation D on L satisfies the relation $D(x, y \wedge z)= (D(x, y) \wedge f(z)) \vee (f(y) \wedge D(x, z))$ for all x, y, z in L.

Example 3.1. *Let L be a lattice and define a mapping on L by $D(x, y) = f(x) \wedge f(y)$ for all x, y in L where f satisfies $f(x \wedge y) = f(x) \wedge f(y)$*

for all x, y in L. Then we can see that D is a symmetric f bi-derivation on L.

Example 3.2. *Let L be a lattice and $a \in L$. Define a mapping D on L by $D(x, y) = (f(x) \wedge f(y)) \wedge a$ where f satisfies $f(x \wedge y) = f(x) \wedge f(y)$ for all x, y in L. Then we can see that D is a symmetric f bi-derivation on L.*

Proposition 3.2. *Let L be a lattice and d be the trace of symmetric f bi-derivation D on L. Then the followings hold:*

i) $D(x, y) \leq f(x)$ and $D(x, y) \leq f(y)$

ii) $D(x, y) \leq f(x) \wedge f(y)$

iii) $d(x) \leq f(x)$

Proof: i) Since, $D(x, y) = D(x \wedge x, y) = (D(x, y) \wedge f(x)) \vee (f(x) \wedge D(x, y)) = f(x) \wedge D(x, y)$, then we get $D(x, y) \leq f(x)$. Similarly, we can get $D(x, y) \leq f(y)$.

ii) This is clear from i).

iii) Since $d(x) = D(x, x) = ((D(x, x) \wedge f(x)) \vee (f(x) \wedge D(x, x)) = f(x) \wedge d(x)$, hence we get $d(x) \leq f(x)$.

Corollary 3.3. *Let L be a lattice and D be a symmetric f bi-derivation on L and L have a least element 0 and a greatest element 1 and $f(0) = 0$, then $D(0, x) = 0$ and $D(1, x) \leq f(x)$ for all x in L.*

Proof: It is trivial from Proposition 3.2 i).

Definition 3.4. Let L be a lattice and $D : L \times L \to L$ be a symmetric mapping. We call D a joinitive mapping if it satisfies $D(x \vee y, z) = D(x, z) \vee D(y, z)$ for all x, y, z in L.

Proposition 3.5. *Let L be a lattice and d be the trace of joinitive symmetric f bi-derivation D on L. Then $d(x \vee y) = d(x) \vee d(y) \vee D(x, y)$ and $d(x) \vee d(y) \leq d(x \vee y)$ for all x, y in L.*

Proof: $d(x \vee y) = D(x \vee y, x \vee y) = D(x, x \vee y) \vee D(y, x \vee y) = D(x, x) \vee D(x, y) \vee D(y, x) \vee D(y, y) = d(x) \vee d(y) \vee D(x, y)$ and $d(x) \vee d(y) \leq d(x, y)$ for all x, y in L.

Theorem 3.6. *Let L be a lattice and D be a symmetric f bi-derivation on L where f satisfies $f(x \wedge y) = f(x) \wedge f(y)$ for all x, y in L and let d be the trace of symmetric f bi-derivation D on L. Then $d(x \wedge y) = D(x, y) \vee (f(x) \wedge d(y)) \vee (f(y) \wedge d(x))$ for all x, y in L.*

Proof: Using Proposition 3.2 i) and iii) we get

$$\begin{aligned} d(x \wedge y) &= D(x \wedge y, x \wedge y) \\ &= (D(x \wedge y, x) \wedge f(y)) \vee (f(x) \wedge D(x \wedge y, y)) \\ &= D(x \wedge y, x) \vee D(x \wedge y, y) \end{aligned}$$

$= (D(x, x) \wedge f(y)) \vee (f(x) \wedge D(y, x)) \vee (D(x, y) \wedge f(y)) \vee (f(x) \wedge D(y, y))$

$= (d(x) \wedge f(y)) \vee (f(x) \wedge D(y, x)) \vee (f(y) \wedge D(x, y)) \vee (d(y) \wedge f(x))$

$= (d(x) \wedge f(y)) \vee (d(y) \wedge f(x)) \vee D(x, y)$

Corollary 3.7. *Let L be a lattice and D be a symmetric f bi-derivation on L where f satisfies $f(x \wedge y) = f(x) \wedge f(y)$ for all x, y in L and let d be the trace of symmetric f bi-derivation D on L. Then the followings hold:*

i) $D(x, y) \leq d(x \wedge y)$

ii) $f(x) \wedge d(y) \leq d(x \wedge y)$

iii) $d(x) \wedge d(y) \leq d(x \wedge y)$

Proof: i) and ii) are trivial from Theorem 3.6 and iii) can be easily proven using by ii) and Proposition 3.2 iii).

From Corollary 3.7 i) and ii) we have $D(x, 1) \leq d(x)$ and $f(x) \wedge d(1) \leq d(x)$.

Corollary 3.8. *Let L be a lattice and D be a symmetric f bi-derivation on L where f satisfies $f(x \wedge y) = f(x) \wedge f(y)$ for all x, y in L, let L have a least element 0 and a greatest element 1, let d be the trace of symmetric f bi-derivation D on L. Then we have*

i) If $f(x) \geq d(1)$, then $d(x) \geq d(1)$.

ii) If $f(x) \leq d(1)$, then $d(x) = f(x)$.

iii) If $x \leq y$ and f is an increasing function and $d(y) = f(y)$, then $d(x) = f(x)$.

Proof: i) If $f(x) \geq d(1)$, then we get $d(1) = d(1) \wedge f(x) \leq d(x \wedge 1) = d(x)$, then we get $d(x) \geq d(1)$.

ii) If $f(x) \leq d(1)$, then we have $f(x) = f(x) \wedge d(1) \leq d(x \wedge 1) = d(x)$ so we have $d(x) = f(x)$.

iii) Let $x \leq y$ and f be an increasing function and $d(y) = f(y)$. Since $f(x) \wedge f(y) = f(x)$, $d(y) = f(y)$ and $D(x, y) \leq f(x)$ and $d(x) \leq f(x) \leq f(y)$ we have $d(x) = d(x \wedge y) = D(x, y) \vee (f(x) \wedge d(y)) \vee (f(y) \wedge d(x)) = D(x,y) \vee (f(x) \vee d(x)) = f(x)$.

Remark 3.1. *Let L be a lattice and let d be the trace of symmetric f bi-derivation D on L where f is an increasing function from L to L that satisfies $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for all x, y in L. Denote $Fix_d(L) = \{ x \in L \mid d(x) = f(x)\}$. By Corollary 3.8 iii) we can see that $Fix_d(L)$ is a down-closed set, that is $y \in Fix_d(L)$ and $x \leq y$ imply $x \in Fix_d(L)$. In fact, it is obvious that $Fix_d(L)$ satisfies the condition 1) of Definition 2.5. For the condition 2) let D be a joinitive mapping and consider $x,y \in Fix_d(L)$. By Proposition 3.5 we have $f(x \vee y) = f(x) \vee f(y) = d(x) \vee d(y) \leq d(x \vee y)$. By Proposition 3.2 iii) we* *have also* $d(x \vee y) \leq f(x \vee y)$. *This means that every* $Fix_d(L)$ *does not satisfies Definition* 2.5. *If the mapping* f *satisfies the conditions* $f(x \vee y) = f(x) \vee f(y)$ *and* $f(x \wedge y) = f(x) \wedge f(y)$ *for all* x, y *in* L *and* f *is an increasing function, then* $Fix_d(L)$ *satisfies the Definition 2.5.*

Theorem 3.9. *Let* L *be a lattice. If there exists a symmetric* f *bi-derivation on* L *that is joinitive where* f *is an epimorphism, then* L *is a distributive lattice.*

Proof: From Example 3.1, we know that $D(x, z) = f(x) \wedge f(z)$ for all x, z in L where f is a homomorphism is a symmetric f bi-derivation on L. Also suppose that f is onto and D is joinitive mapping. Then for all a, b, c in L there exist x, y, z in L such that $f(x) = a$, $f(y) = b$, $f(z) = c$. Hence

$$\begin{aligned}(a \vee b) \wedge c &= (f(x) \vee f(y)) \wedge f(z)\\ &= D(x \vee y, z)\\ &= D(x, z) \vee D(y, z)\\ &= (f(x) \wedge f(z)) \vee (f(y) \wedge f(z))\\ &=(a \wedge b) \vee (b \wedge c)\end{aligned}$$

Since every distributive lattice is a modular lattice we can state the following corollary.

Corollary 3.10. *Let* L *be a lattice. If there exists a symmetric* f *bi-derivation on* L *that is joinitive where* f *is an epimorphism, then* L *is a modular lattice.*

Proof: Since a distributive lattice is a modular lattice, the proof is clear.

Proposition 3.11. *Let* L *be a lattice,* I *be an ideal of* L. *If* f *is a function from* L *to* L *that satisfies* $f(I) \subset I$ *then* $D(I, I) \subset I$.

Proof: Let $y \in D(I, I)$, then $y = D(x, z)$ for some $x, z \in I$. By Proposition 3.2 we have $D(x, z) \leq f(x)$ and similarly $D(x, z) \leq f(z)$ where $f(x) \in I$ and $f(z) \in I$. Since I is an ideal of L we have $D(x, z) \in I$, this means that $D(I, I) \subset I$.

Proposition 3.12. *Let* L *be a lattice and* d *be a trace of a symmetric* f *bi-derivation* D *on* L. *If* f *is a decreasing onto function from* L *to* L *then* $D(d(x), f(x)) \geq d(f(x))$ *for all* $x \in L$. *In particular, if* L *has a least element* 0 *and* $D(d(x), f(x)) = 0$ *for all* $x \in L$, *then we have* d.

Proof: Let L be a lattice and D be a symmetric f bi-derivation on L and f be a decreasing function from L to L, then by Proposition 3.2 i) and ii) we have

$$D(d(x), f(x)) = D(d(x) \wedge f(x), f(x))$$

$= (D(d(x),\ f(x)) \wedge f(f(x))) \vee (f(d(x)) \wedge$
$D(f(x),\ f(x)))$
$= D(d(x),\ f(x)) \vee (f(d(x)) \wedge D(f(x),\ f(x)))$
$= D(d(x),\ f(x)) \vee d(f(x)).$

Hence we have $D(d(x),\ f(x)) \geq d(f(x))$. If L has a least element 0 and $D(d(x),\ f(x)) = 0$ for all $x \in L$, then we have $0 \leq d(f(x)) \leq D(d(x),\ f(x)) = 0$ hence we have $d(f(x)) = 0$ for all $x \in L$, i.e; $d = 0$.

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