Maximum Packings of Complete Graphs with Octagons

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Abstract

The edge set of K_n cannot be decomposed into edge-disjoint octagons (or 8-cycles) when $n \not\equiv 1 \pmod{16}$. We consider the problem of removing edges from the edge set of K_n so that the remaining graph can be decomposed into edge-disjoint octagons. This paper gives the solution of finding maximum packings of complete graphs with edge-disjoint octagons and the minimum leaves are given.

1 Introduction and preliminaries

For $k \geq 3$, a *cycle* C_k is the graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and the edge set $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$ and it is denoted by $(v_1, v_2, \ldots, v_k)$; we also call it a *k-cycle.* An *octagon packing* of a graph G is a set P of edge-disjoint octagons (or 8-cycles) of G. A *leave* L of an octagon packing is a set of edges of G that occur in no octagon of the packing. When there is no chance of confusion, we also regard a leave L as the remaining graph obtained by removing an octagon packing from G. If P is a packing and $|P|$ is as large as possible (so that $|L|$ is as small as possible), then P is called a *maximum packing* and L a *minimum leave.* A *decomposition* of G is a packing of G with L the empty set. Throughout this paper we will refer to a maximum packing of K_n with octagons simply as a maximum packing, so does a minimum leave.

The existence problem for k-cycle decompositions of complete graphs K_n has been completely settled by Alspach, Gavlas [1], Šajna [8], and Hoffman, Lindner and Rodger [3]. A k-cycle decomposition of K_n may not exist, however, it is of interest to see just how "close" one can come to a

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k-cycle decomposition. Maximum k-cycle packings of K_n have been found for all values of n when $k \in \{3,4,5,6\}$ (see [2, 4, 6, 7, 9]). In this paper we solve the problem of finding a maximum octagon packing of K_n for all positive integers $n \geq 8$.

Consider two graphs $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$, for a set $A \subseteq E(G)$, the *edge addition* of A to G' is the graph $G' + A$ obtained from G' by adding all edges of A together with all endvertices of the edges in A; the *edge deletion* of A from G is the graph $G - A$ by removing all edges of A; the *union* of G and G' is the graph $G \cup G'$ with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. If the degree of any vertex in G is even (resp. odd) then G is called an *even* (resp. *odd*) *graph*. We use $\mathscr{E}_e$ (resp. $\mathscr{O}_e$) to denote the even (resp. odd) graph with e edges. Let $K_n[v_1, v_2, \ldots, v_n]$ be the complete graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and $K_{m,n}[a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n]$ the complete bipartite graph with bipartition $(\{a_1, a_2, \ldots, a_m\}, \{b_1, b_2, \ldots, b_n\})$, respectively. Let L_n be the minimum leave of K_n, furthermore, we denote by $L_{n\Box}$, where $\Box$ is an alphabet, the minimum leave of type $(n\Box)$ of K_n. For example, L_{12d} is the minimum leave of type $(12d)$ of K_{12}.

We have the following trivial decomposition.

Proposition 1.1. *For a positive integer n with $n \geq 4$, $K_n[1, 2, \ldots, n]$ can be decomposed into 3 subgraphs $K_4[1, 2, 3, 4]$, $K_{n-4}[5, 6, \ldots, n]$ and $K_{4,n-4}$ $[1, 2, 3, 4; 5, 6, \ldots, n]$.*

2 Complete graphs of odd orders

For a positive odd integer $n \geq 9$, K_n is an even graph and the degree of each vertex in an octagon is 2, so the leave must be an even subgraph of K_n. First, we have the following theorem.

Theorem 2.1. ([5]) *For positive integers k and q, K_{8kq+1} has a C_{4k}-decomposition.*

Next, D. Sotteau [10] obtained the following useful result on the cycle decomposition of complete bipartite graphs.

Theorem 2.2. ([10]) *$K_{m,n}$ has a C_{2k}-decomposition if and only if m and n are even, $m \geq k$, $n \geq k$ and $2k \mid mn$.*

On the other hand, J. A. Kennedy [6] mentioned an $(n + 12)$ MP construction, which we modify as an $(n + 16)$ *MP Construction* to suit our need in the proof.

The $(n+16)$ MP Construction. Let K_n be a complete graph of odd order $n \geq 9$ with vertex set $X \cup \{\infty\}$, K_{17} a complete graph with vertex set $Y \cup \{\infty\}$, and $K_{|X|,|Y|}$ a complete bipartite graph with bipartition $\{X, Y\}$. Let P be a maximum packing and L a minimum leave of K_n. By Theorems 2.1 and 2.2, we assume that K_{17} and $K_{|X|,|Y|}$ have the octagon decompositions H and B, respectively. Then $P \cup H \cup B$ is a maximum packing and L a minimum leave of K_{n+16} with vertex set $X \cup Y \cup \{\infty\}$.

There are eight cases to consider according to the residue classes of n modulo 16. We will give for the initial value of each case the method of the maximum packing. We then use the $(n+16)$ MP Construction to solve the problem of the maximum packing of K_n for every odd order n.

$n \equiv 1 \pmod{16}$

By Theorem 2.1, K_n has an octagon decomposition, hence the minimum leave is empty.

$n \equiv 9 \pmod{16}$

We assume that $n = 16m + 9$, $m \geq 0$. Since $\binom{n}{2} = \binom{16m+9}{2} = 128m^2 + 136m + 36$, the minimum possible leave is an even graph with 4 edges, that is, C_4 in view of the divisibility requirement for the number of edges in K_n. Arguing in the same way, we may summarize our results in Table 1. Note that there is no even graph with 2 edges or 1 edge, hence the minimum possible leaves are even graphs with size 10 and 9 whenever $n \equiv 5$ and 15 (mod 16), respectively.

Table 1: The minimum possible leave of K_n for every odd order n

n mod 16:	1	3	5	7	9	11	13	15
Leave:	$\emptyset$	$\mathscr{E}_3$	$\mathscr{E}_{10}$	$\mathscr{E}_5$	$\mathscr{E}_4$	$\mathscr{E}_7$	$\mathscr{E}_6$	$\mathscr{E}_9$

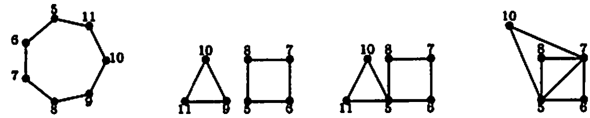
The initial value of n is 9 in this case. We first prove the following lemma.

Lemma 2.3. *There exists an octagon packing of K_9 with $L_9 \cong C_4$.*

Proof. This follows from the fact that $K_9[5, 6, \ldots, 13] - (5, 6, 7, 8)$ can be decomposed into 4 octagons: $(6, 9, 5, 7, 10, 11, 13, 12)$, $(6, 8, 9, 7, 11, 12, 10, 13)$, $(5, 10, 9, 13, 7, 12, 8, 11)$ and $(5, 12, 9, 11, 6, 10, 8, 13)$. □

$n \equiv 11 \pmod{16}$

The initial value of n is 11 in this case. The minimum possible leaves are even graphs with 7 edges listed in Figure 1.



(11*a*) (11*b*) (11*c*) (11*d*)

Figure 1: The minimum possible leaves of K_{11} ($\mathscr{E}_7$)

Lemma 2.4. *There exists an octagon packing of* K_{11} *with* $L_{11} \cong \mathscr{E}_7$.

Proof. The methods for the maximum packing of $K_{11}[5, 6, \ldots, 15]$ are listed in Table 2. □

Table 2: The maximum packings of K_{11}

maximum packing	type of minimum leave
(5, 8, 11, 6, 14, 7, 13, 12), (6, 9, 11, 7, 15, 5, 14, 13), (5, 7, 10, 8, 6, 12, 15, 13), (6, 10, 5, 9, 7, 12, 14, 15), (8, 12, 11, 15, 10, 14, 9, 13), (8, 14, 11, 13, 10, 12, 9, 15)	(11*a*)
(9, 8, 10, 5, 14, 6, 13, 12), (5, 7, 11, 6, 15, 9, 14, 13), (9, 6, 8, 11, 5, 12, 15, 13), (5, 9, 7, 10, 6, 12, 14, 15), (10, 12, 8, 15, 11, 14, 7, 13), (10, 14, 8, 13, 11, 12, 7, 15)	(11*b*)
(5, 9, 11, 6, 14, 7, 13, 12), (6, 8, 10, 7, 15, 5, 14, 13), (5, 7, 9, 10, 6, 12, 15, 13), (6, 9, 8, 11, 7, 12, 14, 15), (8, 12, 11, 15, 10, 14, 9, 13), (8, 14, 11, 13, 10, 12, 9, 15)	(11*c*)
(5, 9, 11, 6, 14, 7, 13, 12), (6, 8, 9, 7, 15, 5, 14, 13), (5, 11, 8, 10, 6, 12, 15, 13), (6, 9, 10, 11, 7, 12, 14, 15), (8, 12, 11, 15, 10, 14, 9, 13), (8, 14, 11, 13, 10, 12, 9, 15)	(11*d*)

Since there is no cycle with 2 edges, for convenience, we use (u, v) to denote an edge $\{uv\}$. In the next lemma we show that there exists a certain octagon packing of $K_{4,n-4}$ for every odd order $n \geq 13$.

Lemma 2.5. *For a positive odd integer* n *with* $n \geq 13$, $K_{4,n-4}[1, 2, 3, 4; 5, 6, \ldots, n] - \{(1, 11), (2, 11), (3, 9), (4, 9)\}$ *has an octagon decomposition. Moreover, one of these octagons is* $(5, 1, 6, 2, 7, 3, 8, 4)$.

Proof. Since n is odd, we will distinguish two cases to discuss.

Case 1: $n \equiv 1 \pmod 4$. We see that $K_{4,9}[1, 2, 3, 4; 5, 6, \ldots, 13] - \{(1, 11), (2, 11), (3, 9), (4, 9)\}$ can be decomposed into 4 octagons: $(5, 1, 6, 2, 7, 3, 8, 4)$, $(5, 2, 8, 1, 7, 4, 6, 3)$, $(9, 1, 13, 4, 12, 3, 10, 2)$, and $(10, 1, 12, 2, 13, 3, 11, 4)$. On the other hand, by Theorem 2.2, the graph $K_{4,n-13}$ obtained from $K_{4,n-4}$ by removing the edges of $K_{4,9}$ has an octagon decomposition. This completes Case 1.

Case 2: $n \equiv 3 \pmod 4$. We see that $K_{4,11}[1, 2, 3, 4; 5, 6, \ldots, 15] - \{(1, 11), (2, 11), (3, 9), (4, 9)\}$ can be decomposed into 5 octagons: $(5, 1, 6,$

2, 7, 3, 8, 4), (5, 2, 9, 1, 10, 4, 11, 3), (7, 1, 8, 2, 10, 3, 6, 4), (12, 1, 13, 2, 14, 3, 15, 4), and (14, 1, 15, 2, 12, 3, 13, 4). On the other hand, by Theorem 2.2, the graph $K_{4,n-15}$ obtained from $K_{4,n-4}$ by removing the edges of $K_{4,11}$ has an octagon decomposition. This completes Case 2. □

For a positive odd integer n with $n \geq 13$, define the subgraph $\mathscr{G}$ of $K_n[1, 2, \ldots, n]$ as follows.

$$\mathscr{G} = K_4[1, 2, 3, 4] \cup (5, 1, 6, 2, 7, 3, 8, 4) + \{(1, 11), (2, 11), (3, 9), (4, 9)\}$$

According to Proposition 1.1 and Lemma 2.5, the graph obtained from $K_n[1, 2, \ldots, n]$ by removing the edges of the union of $\mathscr{G}$ and L_{n-4} has an octagon decomposition, where L_{n-4} is the minimum leave of $K_{n-4}[5, 6, \ldots, n]$. Hence we will try to decompose the union of $\mathscr{G}$ and L_{n-4} into some octagons and L_n.

$n \equiv 13 \pmod{16}$

Let $K_{13}[1, 2, \ldots, 13]$ be the graph associated with the initial situation and the minimum possible leaves are even graphs with 6 edges listed in Figure 2. By Proposition 1.1 and Lemma 2.3, the graph $\mathscr{G} \cup (5, 6, 7, 8)$ can be decomposed into 2 octagons with minimum leaves L_{13}. We will summarize the decompositions in Table 3.

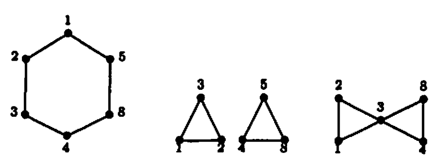


Figure 2: The minimum possible leaves of K_{13} ($\mathscr{E}_6$)

Table 3: The decompositions of $\mathscr{G} \cup (5, 6, 7, 8)$

octagons	type of minimum leave
(2, 11, 1, 6, 7, 3, 9, 4), (1, 4, 5, 6, 2, 7, 8, 3)	(13a)
(2, 11, 1, 6, 7, 3, 9, 4), (1, 4, 3, 8, 7, 2, 6, 5)	(13b)
(2, 11, 1, 6, 7, 8, 5, 4), (1, 4, 9, 3, 7, 2, 6, 5)	(13c)

$n \equiv 15 \pmod{16}$

Let $K_{15}[1, 2, \ldots, 15]$ be the graph associated with the initial situation and the minimum possible leaves are even graphs with 9 edges listed in Figure 3. By Proposition 1.1 and Lemma 2.4, the graph $\mathscr{G} \cup (5, 6, 7, 8, 9, 10, 11)$ can be decomposed into 2 octagons with minimum leaves L_{15}. We will summarize the decompositions in Table 4.

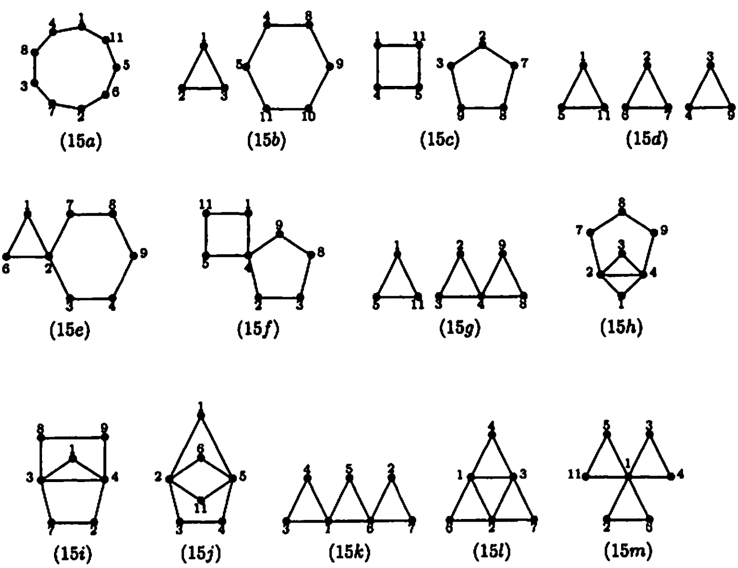


Figure 3: The minimum possible leaves of K_{15} ($\mathscr{E}_9$)

$n \equiv 5 \pmod{16}$

Let $K_{21}[1, 2, \ldots, 21]$ be the graph associated with the initial situation and the minimum possible leaves are even graphs with 10 edges listed in Figure 4.

By Theorem 2.1, we assume that one of these octagons in the octagon decomposition of $K_{17}[5, 6, \ldots, 21]$ is $(5, 6, 7, 8, 9, 10, 11, 12)$. We will decompose the graph $\mathscr{G} \cup (5, 6, 7, 8, 9, 10, 11, 12)$ into 2 octagons with minimum leaves L_{21} except $L_{21\alpha}$ and $L_{21\beta}$. We will summarize the decompositions in Table 5.

On the other hand, by Case 1 of Lemma 2.5, we will decompose the graph $\mathscr{G} \cup (5, 2, 8, 1, 7, 4, 6, 3)$ into 2 octagons with minimum leaves $L_{21\alpha}$, $L_{21\beta}$ and summarize the decompositions in Table 6.

Table 4: The decompositions of $\mathscr{G} \cup (5,6,7,8,9,10,11)$

octagons	type of minimum leave
(2,11,10,9,3,1,5,4), (2,1,6,7,8,9,4,3)	(15a)
(2,11,1,6,7,3,9,4), (2,6,5,1,4,3,8,7)	(15b)
(2,11,10,9,4,3,1,6), (2,1,5,6,7,3,8,4)	(15c)
(2,11,10,9,8,3,1,4), (2,1,6,5,4,8,7,3)	(15d)
(2,11,5,6,7,3,1,4), (1,5,4,8,3,9,10,11)	(15e)
(2,11,10,9,3,7,6,1), (2,6,5,1,3,4,8,7)	(15f)
(2,11,10,9,3,7,6,1), (2,6,5,4,1,3,8,7)	(15g)
(2,11,10,9,3,1,5,6), (1,6,7,3,8,4,5,11)	(15h)
(2,11,5,4,8,7,6,1), (2,3,9,10,11,1,5,6)	(15i)
(2,4,9,8,3,1,6,7), (1,4,8,7,3,9,10,11)	(15j)
(2,11,10,9,3,7,8,4), (2,1,11,5,4,9,8,3)	(15k)
(2,11,5,6,7,8,9,4), (1,5,4,8,3,9,10,11)	(15l)
(2,11,10,9,4,8,3,7), (2,3,9,8,7,6,5,4)	(15m)

Table 5: The decompositions of $\mathscr{G} \cup (a,b,c,d,1,2,3,4)$

octagons	type of minimum leave
(2,11,10,9,3,1,5,4), (2,1,6,7,8,9,4,3)	(21a)
(2,11,1,6,7,3,9,4), (2,7,8,3,4,1,5,6)	(21b)
(2,11,10,9,8,3,1,4), (2,1,6,5,4,9,3,7)	(21c)
(2,11,10,9,4,3,1,6), (2,1,5,6,7,3,8,4)	(21d)
(2,11,10,9,8,3,1,4), (2,1,6,5,4,8,7,3)	(21e)
(2,11,10,9,3,7,6,1), (2,6,5,1,4,3,8,7)	(21f)
(2,11,10,9,4,3,7,6), (2,4,5,6,1,3,8,7)	(21g)
(2,11,10,9,3,7,6,1), (2,6,5,1,3,4,8,7)	(21h)
(2,11,10,9,3,7,6,1), (2,6,5,4,1,3,8,7)	(21i)
(2,11,1,6,7,8,3,4), (1,3,9,10,11,12,5,4)	(21j)
(2,11,1,5,4,8,3,7), (1,3,9,10,11,12,5,6)	(21k)
(2,11,10,9,3,4,5,1), (1,4,8,7,6,5,12,11)	(21l)
(2,1,5,12,11,10,9,4), (2,6,5,4,3,9,8,7)	(21m)
(2,4,9,8,3,1,6,7), (1,4,8,7,3,9,10,11)	(21n)
(2,11,1,4,9,8,3,7), (2,3,9,10,11,12,5,6)	(21o)
(2,11,12,5,4,1,3,7), (1,6,7,8,3,9,10,11)	(21p)
(2,11,10,9,3,8,7,6), (2,4,5,12,11,1,3,7)	(21q)
(2,3,1,5,4,8,7,6), (1,4,9,10,11,12,5,6)	(21r)
(2,3,9,8,4,5,6,7), (1,3,4,9,10,11,12,5)	(21s)
(2,4,9,10,11,12,5,6), (1,5,4,3,9,8,7,6)	(21t)
(2,1,5,6,7,8,9,4), (3,8,4,5,12,11,10,9)	(21u)
(2,3,1,5,4,9,8,7), (3,7,6,5,12,11,10,9)	(21v)
(2,11,1,5,4,3,8,7), (3,7,6,5,12,11,10,9)	(21w)
(2,11,10,9,4,8,3,7), (2,3,9,8,7,6,5,4)	(21x)
(1,5,6,7,8,9,4,3), (3,8,4,5,12,11,10,9)	(21y)
(2,6,1,5,4,9,3,7), (5,6,7,8,9,10,11,12)	(21z)

Table 6: The decompositions of $\mathscr{G} \cup (5,2,8,1,7,4,6,3)$

octagons	type of minimum leave
(2,5,3,6,1,7,4,8), (5,1,11,2,7,3,9,4)	(21α)
(2,6,3,9,4,7,1,8), (1,6,4,8,3,7,2,11)	(21β)

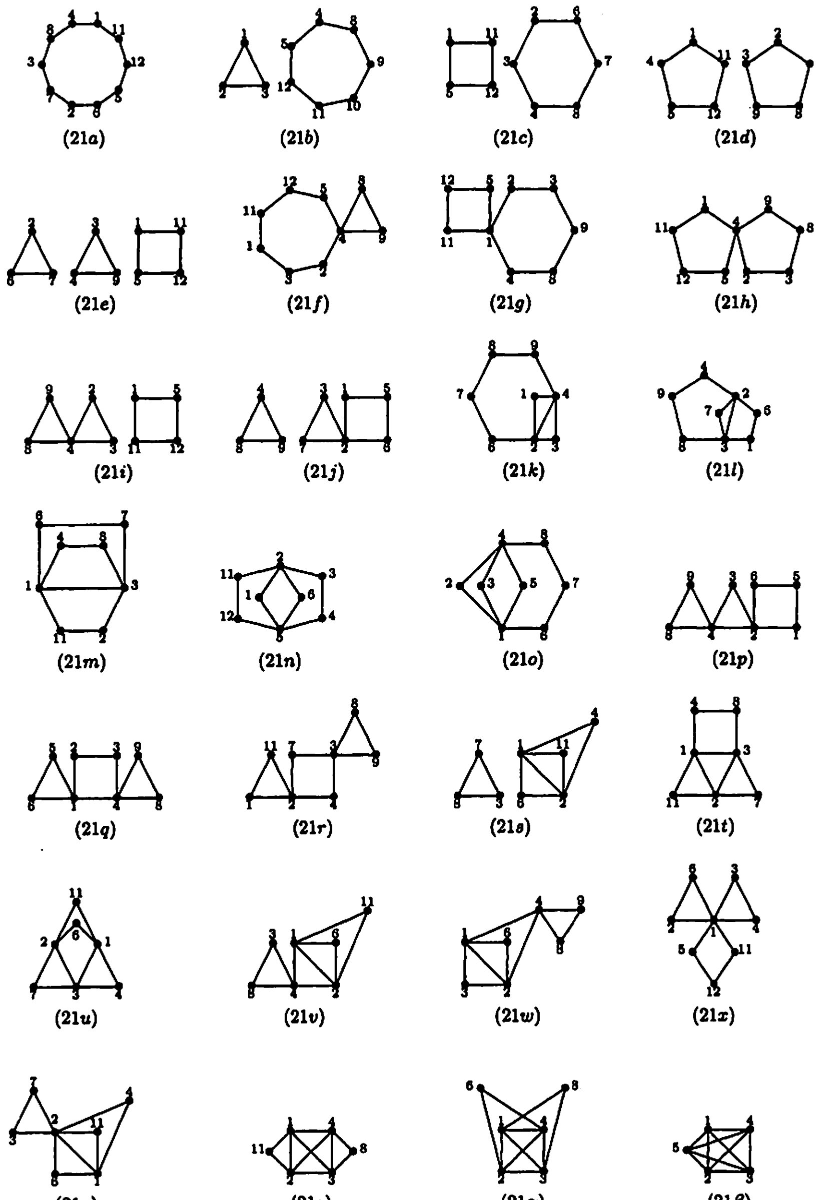


Figure 4: The minimum possible leaves of K_{21} ($\mathscr{E}_{10}$)

$n \equiv 3 \pmod{16}$

Let $K_{19}[1,2,\ldots,19]$ be the graph associated with the initial situation and the minimum possible leave is a 3-cycle. By the result of the case K_{15}, we assume that the minimum leave L_{15} is a 9-cycle: $(5,6,7,8,9,10,11,12,13)$ of $K_{15}[5,6,\ldots,19]$. We will decompose the graph $\mathcal{G} \cup (5,6,7,8,9,10,11,12,13)$ into 3 octagons $(2,11,12,13,5,4,8,3)$, $(2,1,5,6,7,3,9,4)$, $(2,6,1,11,10,9,8,7)$ and the minimum leave $L_{19} : (1,3,4)$.

$n \equiv 7 \pmod{16}$

Let $K_{23}[1,2,\ldots,23]$ be the graph associated with the initial situation and the minimum possible leave is a 5-cycle. By the result of the case K_{19}, we assume that the minimum leave L_{19} is a 3-cycle: $(5,6,7)$ of $K_{19}[5,6,\ldots,23]$. We will decompose the graph $\mathcal{G} \cup (5,6,7)$ into 2 octagons $(2,1,5,6,7,3,9,4)$, $(2,6,1,3,8,4,5,7)$ and the minimum leave L_{23} : $(1,4,3,2,11)$.

The results for the minimum leaves of K_n, when n is odd, now follow from the above discussion, and they are summarized in the following theorem.

Theorem 2.6. *Let n be a positive odd integer with $n \geq 9$.*

1. *If $n \equiv 1 \pmod{16}$, then the minimum leave is empty.*
2. *If $n \equiv 3 \pmod{16}$, then the minimum leave is a 3-cycle.*
3. *If $n \equiv 5 \pmod{16}$, then the minimum leaves are those in Types $(21a)$–$(21z)$, (21α) and (21β).*
4. *If $n \equiv 7 \pmod{16}$, then the minimum leave is a 5-cycle.*
5. *If $n \equiv 9 \pmod{16}$, then the minimum leave is a 4-cycle.*
6. *If $n \equiv 11 \pmod{16}$, then the minimum leaves are those in Types $(11a)$–$(11d)$.*
7. *If $n \equiv 13 \pmod{16}$, then the minimum leaves are those in Types $(13a)$–$(13c)$.*
8. *If $n \equiv 15 \pmod{16}$, then the minimum leaves are those in Types $(15a)$–$(15m)$.*

Proof. Starting with any one of the maximum packings in the initial cases of this section, the $(n+16)$ MP Construction yields a maximum packing and a minimum leave for every odd order $n \geq 9$. □

3 Complete graphs of even orders

For a positive even integer $n \geq 8$, K_n is an odd graph and the degree of each vertex in an octagon is 2, so the leave must be an odd spanning subgraph of K_n and the size not less than $n/2$.

We have the following well-known theorem.

Theorem 3.1. ([1]) *For positive even integers m and n with $4 \leq m \leq n$, the graph $K_n - I$ can be decomposed into cycles of length m if and only if the number of edges in $K_n - I$ is a multiple of m, where I is a 1-factor in K_n.*

$n \equiv 0, 2 \pmod 8$

Note that $8 \mid [\binom{n}{2} - \frac{n}{2}]$ for $n \geq 8$. By Theorem 3.1, the minimum leave is a 1-factor, which is the smallest spanning subgraph of K_n whenever $n \equiv 0, 2$ (mod 8).

There are four cases remains to consider according to the residue classes of n modulo 16. However, if $n \equiv 4, 6, 12, 14 \pmod{16}$, then the divisibility requirement for the number of edges in K_n, $\binom{n}{2} - (n/2+4)$ is divisible by 8, hence a minimum possible leave has $n/2+4$ edges. We may summarize the minimum possible leaves of the initial cases in Table 7. Note that such a leave is an odd spanning subgraph of K_n. Accordingly, the only possible degree sequences for such a leave with order n and size $n/2+4$ are: $(9, 1, \ldots, 1)$, $(7, 3, 1, \ldots, 1)$, $(5, 5, 1, \ldots, 1)$, $(5, 3, 3, 1, \ldots, 1)$, and $(3, 3, 3, 3, 1, \ldots, 1)$.

Table 7: The minimum possible leaves of K_n for every odd order n (odd spanning subgraph of K_n)

K_n:	12	14	20	22
Leave:	$\mathcal{O}_{10}$	$\mathcal{O}_{11}$	$\mathcal{O}_{14}$	$\mathcal{O}_{15}$

Similar to the previous section, we modify the $(n+8)$ *Construction* as follows.

The $(n+8)$ Construction. Let K_n be a complete graph of even order $n \geq 8$ with vertex set X, K_8 a complete graph with vertex set Y, and $K_{|X|,|Y|}$ a complete bipartite graph with bipartition $\{X, Y\}$. Let P_1 be a maximum octagon packing, L_1 a minimum leave of K_n; P_2 be a maximum octagon packing, L_2 a minimum leave of K_8. By Theorems 2.2, we assume that $K_{|X|,|Y|}$ has the octagon decomposition B. Then $P_1 \cup P_2 \cup B$ is a maximum octagon packing and $L_1 \cup L_2$ a minimum leave of K_{n+8} with vertex set $X \cup Y$.

We will give for the initial value of n in each case the method of the maximum octagon packing. We then use the $(n+8)$ Construction to solve the problem of the maximum packing of K_n for every even order n.

For later use in our proof, the following lemmas give specific methods for maximum packings of K_8, K_{10} and the octagon decomposition of $K_{4,n-4}$ for every even order $n \geq 12$.

Lemma 3.2. *There exists an octagon packing of K_8 such that L_8 is a 1-factor of K_8.*

Proof. This follows from the fact that $K_8[5,6,\ldots,12]-\{(5,7),(6,8),(9,11),(10,12)\}$ can be decomposed into 3 octagons: $(5,6,7,8,9,10,11,12)$, $(5,8,12,9,7,10,6,11)$, and $(5,9,6,12,7,11,8,10)$. □

Lemma 3.3. *There exists an octagon packing of K_{10} such that L_{10} is a 1-factor of K_{10}.*

Proof. This follows from the fact that $K_{10}[5,6,\ldots,14]-\{(5,7),(6,8),(9,11),(10,12),(13,14)\}$ can be decomposed into 5 octagons: $(5,6,7,8,9,10,11,12)$, $(5,14,8,11,7,12,6,13)$, $(11,6,10,7,9,14,12,13)$, $(5,10,8,13,7,14,6,9)$ and $(5,8,12,9,13,10,14,11)$. □

Lemma 3.4. *For a positive even integer n with $n \geq 12$, $K_{4,n-4}[1,2,3,4;5,6,\ldots,n]$ has an octagon decomposition. Moreover, two of these octagons are $(5,1,6,2,7,3,8,4)$ and $(1,12,2,11,3,10,4,9)$.*

Proof. Since n is even, we will distinguish two cases to discuss.

Case 1: $n \equiv 0 \pmod 4$. We see that $K_{4,8}[1,2,3,4;5,6,\ldots,12]$ can be decomposed into 4 octagons: $(5,1,6,2,7,3,8,4)$, $(1,12,2,11,3,10,4,9)$, $(5,2,8,1,7,4,6,3)$, and $(9,2,10,1,11,4,12,3)$. On the other hand, by Theorem 2.2, the graph $K_{4,n-12}$ obtained from $K_{4,n-4}$ by removing the edges of $K_{4,8}$ has an octagon decomposition. This completes Case 1.

Case 2: $n \equiv 2 \pmod 4$. We see that $K_{4,10}[1,2,3,4;5,6,\ldots,14]$ can be decomposed into 5 octagons: $(5,1,6,2,7,3,8,4)$, $(1,12,2,11,3,10,4,9)$, $(5,2,8,1,7,4,6,3)$, $(12,3,14,2,13,1,11,4)$, and $(9,2,10,1,14,4,13,3)$. On the other hand, by Theorem 2.2, the graph $K_{4,n-14}$ obtained from $K_{4,n-4}$ by removing the edges of $K_{4,10}$ has an octagon decomposition. This completes Case 2. □

For a positive even integer $n \geq 12$, define the subgraph $\mathscr{H}$ of $K_n[1,2,\ldots,n]$ as follows.

$$\mathscr{H} = K_4[1,2,3,4] \cup (5,6,7,8,9,10,11,12) + \{(5,7),(6,8),(9,11),(10,12)\} \cup (5,1,6,2,7,3,8,4) \cup (1,12,2,11,3,10,4,9)$$

According to Proposition 1.1, Lemmas 3.2, 3.3 and 3.4, we have $K_{12}[1,2,\ldots,12]-E(\mathscr{H})$ and $K_{14}[1,2,\ldots,14]-E(\mathscr{H})-(13,14)$ have octagon decompositions, respectively. Hence we will try to decompose $\mathscr{H}$ (or $\mathscr{H}$+(13,14)) into some octagons and L_n.

$n \equiv 12 \pmod{16}$

Let $K_{12}[1,2,\ldots,12]$ be the graph associated with the initial situation and the minimum possible leaves are odd graphs with order 12 and size 10 listed in Figure 5.

By Lemma 3.4, there is an octagon $(5,2,8,1,7,4,6,3)$ of the octagon decomposition of $K_{4,8}[1,2,3,4;5,6,\ldots,,12]$. We will decompose the graph $\mathscr{H} \cup (5,2,8,1,7,4,6,3)$ into 4 octagons and the minimum leaves L_{12a}, L_{12b}; decompose the graph $\mathscr{H}$ into 3 octagons and the minimum leaves in types $(12c)$ through $(12s)$. We will summarize the decompositions in Tables 8 and 9.

$n \equiv 14 \pmod{16}$

Let $K_{14}[1,2,\ldots,14]$ be the graph associated with the initial situation and the minimum possible leaves are odd graphs with order 14 and size 11. They are composed of a minimum leave of $K_{12}[1,2,\ldots,12]$ together with a disjoint edge $(13,14)$ to form minimum possible leaves in types $(14a)$ through $(14s)$. We also list in Figure 6 the minimum possible leaves L_{14t} and L_{14u}. By the result of K_{12}, it suffices to show the existence of the minimum possible leaves L_{14t} and L_{14u}. Here we give another octagon packing of K_{10} for our discussions.

Lemma 3.5. *There exists an octagon packing of K_{10} such that L_{10} is a 1-factor of K_{10}.*

Proof. This follows from the fact that $K_{10}[5,6,\ldots,14]-\{(5,7),(6,8),(9,12),(10,13),(11,14)\}$ can be decomposed into 5 octagons: $(5,6,7,8,9,10,11,12)$, $(5,14,8,11,7,12,6,13)$, $(11,6,10,7,9,14,12,13)$, $(5,10,8,13,7,14,6,9)$ and $(5,8,12,10,14,13,9,11)$. □

Then the graph $\mathscr{H} + \{(9,12),(10,13),(11,14)\} - \{(9,11),(10,12)\}$ can be decomposed into 3 octagons and the minimum leaves L_{14t}, L_{14u}. We will summarize the decompositions in Table 10.

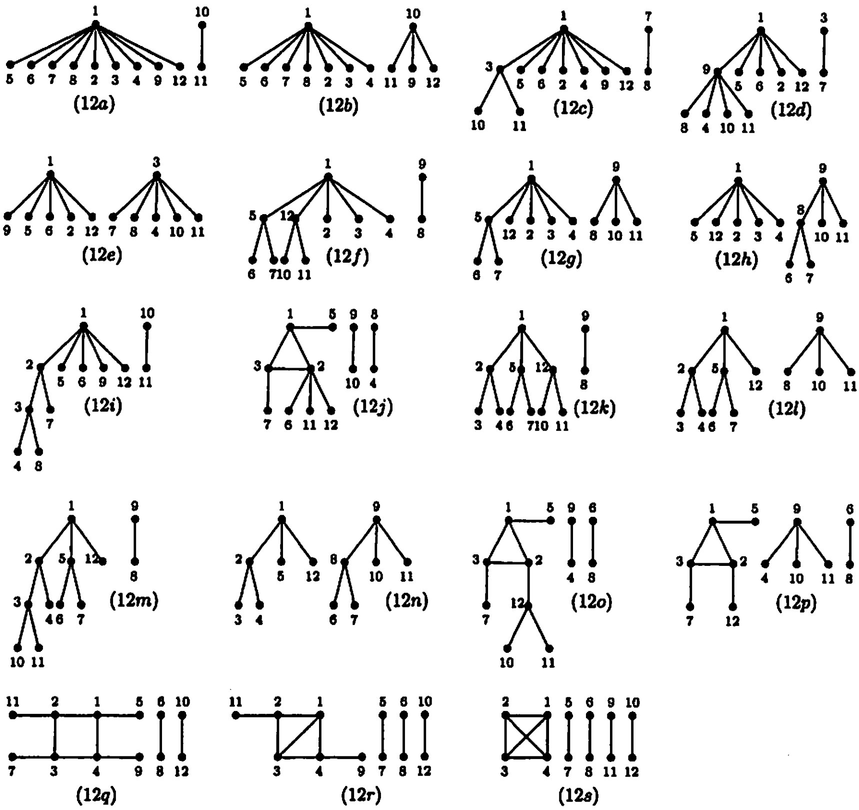


Figure 5: The minimum possible leaves of K_{12}

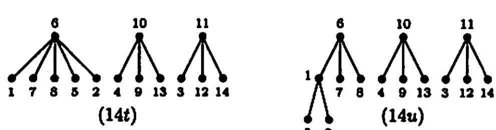


Figure 6: The minimum possible leaves L_{14t} and L_{14u}

Table 8: The decompositions of $\mathscr{H} \cup (5,2,8,1,7,4,6,3)$

octagons	type of minimum leave
(2,11,9,8,4,6,3,5), (2,12,5,6,8,7,3,4) (3,10,9,4,5,7,6,2), (3,11,12,10,4,7,2,8)	(12*a*)
(2,11,9,8,4,6,3,5), (2,12,5,6,8,7,3,4) (3,10,4,5,7,6,2,8), (3,11,12,1,9,4,7,2)	(12*b*)

Table 9: The decompositions of $\mathscr{H}$

octagons	type of minimum leave
(2,12,10,11,9,4,8,3), (2,6,8,9,10,4,5,7), (2,11,12,5,6,7,3,4)	(12c)
(2,11,10,3,1,4,5,7), (2,12,5,6,7,8,4,3), (2,6,8,3,11,12,10,4)	(12d)
(2,11,9,8,4,5,7,6), (2,12,11,10,9,4,1,3), (2,4,10,12,5,6,8,7)	(12e)
(2,11,9,1,6,8,4,3), (2,4,10,11,3,8,7,6), (2,12,5,4,9,10,3,7)	(12f)
(2,4,9,1,6,7,8,3), (2,12,5,4,10,11,3,7), (2,11,12,10,3,4,8,6)	(12g)
(2,12,10,4,5,6,7,3), (2,4,3,10,11,12,5,7), (2,11,3,8,4,9,1,6)	(12h)
(2,12,11,9,8,6,5,4), (2,11,3,10,4,8,7,6), (1,4,9,10,12,5,7,3)	(12i)
(2,4,3,10,12,1,6,7), (1,4,5,7,8,3,11,9), (4,9,8,6,5,12,11,10)	(12j)
(2,11,9,1,3,4,8,6), (2,12,5,4,9,10,3,7), (1,4,10,11,3,8,7,6)	(12k)
(2,11,12,10,3,4,8,6), (2,12,5,4,9,1,3,7), (1,4,10,11,3,8,7,6)	(12l)
(2,11,10,4,1,6,8,7), (2,12,11,9,4,3,7,6), (1,3,8,4,5,12,10,9)	(12m)
(2,11,3,1,4,5,6,7), (2,12,10,3,4,9,1,6), (4,10,11,12,5,7,3,8)	(12n)
(2,11,10,3,4,1,6,7), (2,4,8,9,1,12,5,6), (3,11,9,10,4,5,7,8)	(12o)
(2,11,10,3,4,1,6,7), (2,4,8,9,1,12,5,6), (3,11,12,10,4,5,7,8)	(12p)
(2,12,11,3,10,9,1,6), (2,4,8,3,1,12,5,7), (4,10,11,9,8,7,6,5)	(12q)
(2,12,1,9,10,3,8,4), (2,6,1,5,12,11,3,7), (4,10,11,9,8,7,6,5)	(12r)
(2,11,3,10,4,9,1,12), (2,6,1,5,4,8,3,7), (5,6,7,8,9,10,11,12)	(12s)

Table 10: The decompositions of $\mathscr{H} + \{(9,12),(10,13),(11,14)\} - \{(9,11),(10,12)\}$

octagons	type of minimum leave
(2,11,10,3,8,4,1,12),(2,1,9,12,5,4,3,7),(2,3,1,5,7,8,9,4)	(14t)
(2,11,10,3,8,4,1,12),(2,3,7,8,9,4,5,6),(2,4,3,1,9,12,5,7)	(14u)

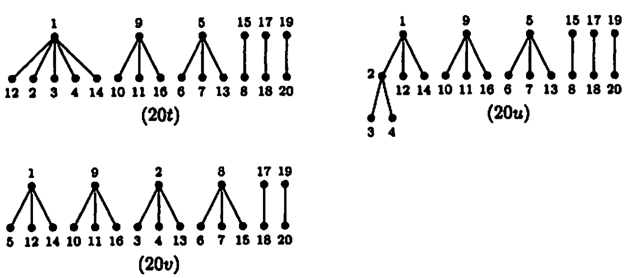


Figure 7: The minimum possible leaves L_{20t}, L_{20u} and L_{20v}

$n \equiv 4 \pmod{16}$

Let $K_{20}[1, 2, \ldots, 20]$ be the graph associated with the initial situation and the minimum possible leaves are odd graphs with order 20 and size 14. They are composed of a minimum leave of $K_{12}[1, 2, \ldots, 12]$ together with four disjoint edges $\{(13, 14), (15, 16), (17, 18), (19, 20)\}$ to form minimum possible leaves in types $(20a)$ through $(20s)$. We also list in Figure 7 the minimum possible leaves L_{20t}, L_{20u} and L_{20v}. By the result of K_{12}, it suffices to show the existence of the minimum possible leaves L_{20t}, L_{20u} and L_{20v}.

By Theorems 2.2, without loss of generality, we assume that one of the octagons in $K_{12,8}[1, 2, \ldots, 12; 13, 14, \ldots, 20]$ is $(13, 1, 14, 8, 15, 9, 16, 5)$. Since $L_{20g} = L_{12g} + \{(13, 14), (15, 16), (17, 18), (19, 20)\}$ and $L_{20l} = L_{12l} + \{(13, 14), (15, 16), (17, 18), (19, 20)\}$, we have

$$L_{20g} \cup (13, 1, 14, 8, 15, 9, 16, 5) = L_{20t} \cup (13, 1, 5, 16, 15, 9, 8, 14)\ ,$$

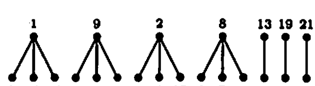
$$L_{20l} \cup (13, 1, 14, 8, 15, 9, 16, 5) = L_{20u} \cup (13, 1, 5, 16, 15, 9, 8, 14).$$

To obtain the minimum leave of type $(20v)$, we assume that one of the octagons in $K_{12,8}[1, 2, \ldots, 12; 13, 14, \ldots, 20]$ is $(13, 1, 14, 8, 15, 9, 16, 2)$. Since $L_{20n} = L_{12n} + \{(13, 14), (15, 16), (17, 18), (19, 20)\}$, we have

$$L_{20n} \cup (13, 1, 14, 8, 15, 9, 16, 2) = L_{20v} \cup (13, 1, 2, 16, 15, 9, 8, 14).$$

$n \equiv 6 \pmod{16}$

Let $K_{22}[1, 2, \ldots, 22]$ be the graph associated with the initial situation and the minimum possible leaves are odd graphs with order 22 and size 15. They are composed of a minimum leave of $K_{14}[1, 2, \ldots, 14]$ together with four disjoint edges $\{(15, 16), (17, 18), (19, 20), (21, 22)\}$ to form minimum possible leaves in types $(22a)$ through $(22u)$. We also list in Figure 8 the minimum possible leave L_{22v}. By the result of K_{14}, it suffices to show the existence of L_{22v}.



(22v)

Figure 8: The minimum possible leave L_{22v}

By Theorems 2.2, without loss of generality, we assume that one of the octagons in $K_{14,8}[1, 2, \ldots, 14; 15, 16, \ldots, 22]$ is $(15, 1, 16, 8, 17, 9, 18, 2)$. Since $L_{22n} = L_{14n} + \{(15, 16), (17, 18), (19, 20), (21, 22)\}$, we have

$$L_{22n} \cup (15, 1, 16, 8, 17, 9, 18, 2) = L_{22v} \cup (15, 1, 2, 18, 17, 9, 8, 16).$$

The results for the minimum leaves of K_n, when n is even, now follow from the above discussion, and they are summarized in the following theorem.

Theorem 3.6. *Let n be a positive even integer with $n \geq 12$.*

1. *If $n \equiv 4 \pmod 8$, then the minimum leaves are those in Types (12a)-(12s) for $n = 12$. For $n \geq 20$ the leave is one of those in Types (20a)-(20v) plus a disjoint 1-factor of K_{n-20}.*
2. *If $n \equiv 6 \pmod 8$, then the minimum leaves are those in Types (14a)-(14u) for $n = 14$. For $n \geq 22$ the leave is one of those in Types (22a)-(22v) plus a disjoint 1-factor of K_{n-22}.*

Proof. Starting with any one of the maximum packings in the initial cases of this section, the $(n+8)$ Construction yields a maximum packing and a minimum leave for every even order $n \equiv 4, 6 \pmod 8 \geq 12$. □

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