On the Erdős-Faber-Lovász Conjecture

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Abstract

In 1972 Erdős, Faber and Lovász made the now famous conjecture: If a graph G consists of n copies of the complete graph K_n such that any two copies have at most one common vertex (such graphs are called EFL graphs), then G is n-colorable. In this paper we show that the conjecture is true for two different classes of EFL graphs. Furthermore, a new shorter proof of the conjecture is given for a third class of EFL graphs.

According to Jensen and Toft [6], in 1972 Erdős, Faber and Lovász made the following conjecture which first appeared in print in [4]: If a graph G consists of n copies of the complete graph K_n such that any two copies have at most one common vertex, then G can be n-colored. We call such a graph G an *EFL graph,* and call n the *rank* of G. Despite initially thinking that the conjecture would be easy to prove [2], they and everyone else who considered the conjecture have been unsuccessful in proving or disproving it. In 1981, Erdős [3] offered a prize of \$500 for settling the conjecture.

Most partial results up to now have given upper bounds on the number of colors required. Specifically, Mitchem [8], and independently Chang and Lawler [1], have shown that any EFL graph of rank n can be colored with $\lceil \frac{3n}{2} - 2 \rceil$ colors. Kahn [7] has an asymptotic result which won \$250 from Erdős [2]: Any rank n EFL graph can be colored with $n + o(n)$ colors. Recently, Jackson, Sethuraman and Whitehead [5] proved that EFL graphs of a certain class are n-colorable.

In this paper we focus on proving that certain additional classes of EFL graphs are n-colorable. We first introduce some definitions and notation.

In any rank n EFL graph G the n copies of K_n are denoted $H_1, H_2, \ldots, H_n$. The *special degree* of a vertex x, denoted sdeg(x), is the number of complete graphs H_j that contain x. An EFL graph is *s-uniform* if every vertex has special degree 1 or s. An s-uniform EFL graph of rank n is called *maximum* if there exists no other s-uniform rank n EFL graph with more vertices of special degree s. Let a and i be any two colors used on the vertices of G. Then $\langle a, i \rangle$ denotes the subgraph of G induced by all vertices colored a or i. Furthermore, if vertex x is colored a, then $\langle x, i \rangle$ denotes the component of $\langle a, i \rangle$ that contains x. A vertex x is *adjacent* to color i if x is adjacent to a vertex colored i.

We begin with a shorter proof of a result given in [8]. We follow that with our main results showing that two additional infinite classes of EFL graphs can be n-colored.

Theorem 1. *Given an EFL graph G of rank n such that each H_j has at most one vertex of special degree greater than 2, then G is n-colorable.*

Proof. Obviously special degree 1 vertices can always be colored. Thus we disregard them.

Case 1. n is even. We first $(n-1)$-color a 2-uniform maximum EFL graph G' of rank n. Let x be the vertex in H_j and H_k, $j < k$. If $k < n$, then color x with $j + k - 1 (\text{mod } n-1)$. If $k = n$, then color x with $2j - 1 (\text{mod } n-1)$. This results in an $(n-1)$-coloring of G', which we now use to color G: The special degree 2 vertices of G are colored the same as in G'. Since this uses at most $n-1$ colors, we use color n on all vertices with special degree larger than 2, which completes our n-coloring of G.

Case 2. n is odd. We first n-color a 2-uniform maximum EFL graph G' of rank n. Let x be the vertex that is in H_j and H_k. Color x with color $j + k - 1 (\text{mod}\, n)$. In this way no two vertices in H_j are colored the same, and all vertices are colored.

We now use the coloring above to color G by first assigning all vertices of special degree 2 the same colors as in G'. Then let $x_1, x_2, \ldots, x_t$ be the vertices of special degree at least 3 and $\text{sdeg}(x_i) = s_i$, $1 \leq i \leq t$. Also let $p_1 = 0$, and for $2 \leq i \leq t$, let $p_i = \sum_{c=1}^{i-1} s_c$ so that p_i is the sum of the special degrees of the predecessors of x_i in the list. Furthermore, by relabeling if necessary, let $H_1, H_2, \ldots, H_{s_1}$ be the H_j's that contain x_1, and in general let $H_{p_i+1}, \ldots, H_{p_i+s_i}$ be the H_j's that contain x_i. We then color $x_1, x_2, \ldots, x_t$ by the following rule: For $1 \leq i \leq t$, color x_i with color $2p_i + s_i \pmod n$. Now, x_i is in both H_{p_i+1} and $H_{p_i+s_i}$, and the vertex of special degree 2 in G' that is in these two H_j's is colored $(p_i + 1) + (p_i + s_i) - 1 \pmod n \equiv 2p_i + s_i \pmod n$. Similarly the vertex of special degree 2 in G' that is in both H_{p_i+2} and $H_{p_i+s_i-1}$ also receives color $2p_i + s_i \pmod n$, and so forth. Thus the s_i H_j's that together contain x_i can be paired so that in G' the various x's in the pairs are all colored with $2p_i + s_i \pmod n$. Also, in the case where s_i is odd, there is an H_j that is not paired. However no vertex of this H_j in G' has color $2p_i + s_i \pmod n$. Thus no vertex except x_i in the s_i H_j's has color $2p_i + s_i \pmod n$. Hence we have a legitimate coloring of G with n colors. ■

Theorem 2. *Let n, s be integers such that $3 \leq s \leq n \leq s(s-1)(s-2)+1$. Then any s-uniform EFL graph G of rank n is n-colorable.*

Proof. In coloring G with n colors it is obvious that the vertices of special degree 1 can always be colored. Thus, we consider only the subgraph G' of G induced by the vertices of special degree s, and we let H_j' be the subgraph of H_j induced by the vertices of special degree s. Among all partial n-colorings of G' choose one that colors the maximum number of vertices. Assume there exists a vertex v that is uncolored. We need only show that we can find a partial coloring of G' that includes v and all vertices that are already colored. Clearly v must be adjacent to all colors i, $1 \leq i \leq n$. Let $\frac{n-1}{s(s-1)} = k$. Then $k+2 \leq s$. For $1 \leq j \leq n$, let r_j be the number of vertices in H_j'. Then $r_j \leq \lfloor \frac{n-1}{s-1} \rfloor = \lfloor ks \rfloor$. Thus the number of vertices of G' is $u \leq \lfloor \frac{n}{s} \lfloor ks \rfloor \rfloor$.

Lemma 1. *Given that G' is colored as indicated, then some color is used $m \leq \lceil k \rceil - 1$ times.*

Proof. On the contrary, assume that each color is used at least $\lceil k \rceil$ times. Then $n\lceil k \rceil + 1 \leq u$. However,

$$u \leq \left\lfloor \frac{n}{s} \lfloor ks \rfloor \right\rfloor \leq \left\lfloor \frac{n}{s} ks \right\rfloor \leq nk < n\lceil k \rceil + 1 \leq u,$$

a contradiction. □

Now if $k \leq 1$ then $m = 0$, and some color is unused on G'. Use that color on v, and the theorem is proved. Thus $k > 1$. Let 1 be a color that is used exactly $m \leq \lceil k \rceil - 1$ times. Let x_j, $1 \leq j \leq m$, be the vertices colored 1, and for $1 \le j \le s$, let H_j' contain v. In order to prove the theorem we use induction on m, and start with $m = 1$. Let $x_1 \in H_1'$ be adjacent to v. The number $b \ge 0$ of vertices that are adjacent to both x_1 and v is at most

$$\lfloor ks \rfloor - 2 + (s-1)^2 \le (ks - 2) + \left(s^2 - 2s + 1\right) = s^2 + (k-2)s - 1.$$

Also, there are $n - 1 = k\left(s^2 - s\right)$ colors different from 1. Applying $s > k + 1$ gives

$$(k-1)s^2 > (k-1)(k+1)s = \left(k^2 - 1\right)s > (2k-2)s - 1,$$

which can be rewritten as

$$ks^2 - ks > s^2 + (k-2)s - 1 \ge b.$$

Therefore, there exists color i such that $\langle x_1, i \rangle$ contains no neighbor of v. Interchange colors on $\langle x_1, i \rangle$; then use color 1 on v. This creates a larger n-coloring of G', and thus the theorem is proved for $m = 1$.

Now assume the theorem is true if color 1 is used exactly $1, 2, \ldots, \text{or}\,(m-1)$ times, $m \ge 2$. To complete the inductive proof let color 1 be used exactly m times. Let q be the number of vertices colored 1 that are adjacent to v, and let $x_j \in H_j'$, $1 \le j \le q$, be those vertices. Note $q \le m \le \lceil k \rceil - 1 < k < s - 1$. Now in order to prove the general case for m we use induction on q. Consider $q = 1$. As previously, x_1 has $b \le s^2 + (k-2)s - 1$ neighbors that are also neighbors of v. Also, for $2 \le i \le n$, $\langle x_1, i \rangle$ must contain a vertex colored i that is a neighbor of v; otherwise, interchange colors on $\langle x_1, i \rangle$, which results in v having no neighbors colored 1. Then color v with 1, increasing the number of colored vertices.

For $2 \le i \le n$, $\langle x_1, i \rangle$ has a shortest path from x_1 to a vertex colored i that is adjacent to v. This shortest path is of length 1 or of length larger than 1. There are $b \le s^2 + (k-2)s - 1$ of the former colors, and $n - 1 - b$ of the latter colors. For these latter colors, each of the paths must go through at least one x_j, $j > 1$. Furthermore, each x_j is adjacent to each color $i > 1$, for otherwise, recolor x_j with the missing color and then there are only $m - 1$ vertices colored 1. By our inductive assumption the theorem is proved. Each x_j, $j > 1$, is adjacent to at least one vertex of each of the b colors above. From the preceding discussion each x_j is adjacent to at least one vertex of each of the $n - 1 - b$ colors, and for each of these colors one x_j is adjacent to at least two vertices of that color. Thus,

$$\sum_{j=2}^{m} \deg(x_j) \geq (m-1)b + m(n-1-b) = m(n-1) - b. \tag{1}$$

However, each vertex of G' has degree at most $s(\lfloor ks \rfloor - 1) \leq ks^2 - s$. Thus,

$$\sum_{j=2}^{m} \deg(x_j) \leq (m-1)(ks^2 - s). \tag{2}$$

In order to complete this part of the proof we only need to show that the right-hand side of (1) is larger than the right-hand side of (2). Now $m < k < s-1$, and $k > 1$. First note that

$$\begin{aligned}(k-1)s + (m+1) - k(m+1) &= (k-1)s + (m+1)(1-k) \\ &> (k-1)(k+1) - (m+1)(k-1) \\ &= (k-1)(k-m) > 0.\end{aligned}$$

It follows that

$$\begin{aligned}&(k-1)s^2 + (m-1-mk-k+2)s + 1 \\ &= [(k-1)s + m + 1 - mk - k]s + 1 > 0.\end{aligned} \tag{3}$$

Thus,

$$\begin{aligned}m(n-1) - b &= m[k(s^2 - s)] - b \\ &\geq mk(s^2 - s) - [s^2 + (k-2)s - 1] \\ &= (mk-1)s^2 - (mk+k-2)s + 1 \\ &> (mk-k)s^2 - (m-1)s,\end{aligned}$$

where the last inequality follows from (3). Hence, the right-hand side of (1) is larger than the right-hand side of (2) and the theorem is proved for arbitrary m when $q = 1$.

Now inductively assume that if color 1 is used exactly m times, and if it occurs less than q times, $q \geq 2$, on H'_j, $1 \leq j \leq s$, then G' is n-colorable. Now assume that G' has m vertices colored 1, and q of them are in H'_j, $1 \leq j \leq s$. We will show inductively that we can color v, increasing the number of colored vertices of G'. This contradiction will complete our double induction proof of the theorem. We will either reduce the number of vertices colored 1, in which case we apply our inductive assumption on m, or we will keep the number of vertices colored 1 at m and reduce their number q in H'_j, $1 \leq j \leq s$, in which case we apply our inductive assumption on q.

For $1 \leq j \leq q$, let $x_j \in H_j'$. If $\langle x_1, i \rangle$ has only one vertex colored i, and it is not adjacent to v, then interchange colors on it. In so doing either the number of vertices colored 1 is reduced to less than m, or that number remains m and the number of vertices colored 1 that are adjacent to v is reduced to less than q. Either way we use induction to complete the proof. Thus, every $\langle x_1, i \rangle$ with only a single vertex colored i must have that vertex adjacent to v. Let there be b such $\langle x_1, i \rangle$. Then $b \leq s^2 + (k-2)s - 1$. There are $n - 1 - b$ other colors, and for each of them $\langle x_1, i \rangle$ has at least two vertices colored i. Hence, for each such color i there is at least one x_j, $1 \leq j \leq m$, in $\langle x_1, i \rangle$ that is adjacent to two vertices colored i. Furthermore, each x_j, $1 \leq j \leq m$, is adjacent with all colors i, $2 \leq i \leq n$; otherwise recolor x_j with the missing color and again apply the induction because the number of vertices colored 1 is less than m. By hypothesis

$$(ks - s) = (k-1)s \geq (k-1)(k+2) = k^2 + k - 2.$$

Hence,

$$\begin{aligned}(ks + m - mk - k)s &= [k(s-1) - m(k-1)]s \\ &> [k(s-1) - k(k-1)]s \\ &= (ks - k^2)s \\ &> (k + s - 2)s - 1 \geq b.\end{aligned} \tag{4}$$

Each of the b colors is adjacent to each x_j, and for each of the $n - 1 - b$ colors one x_j is adjacent to two vertices of that color. Hence,

$$\begin{aligned}\sum_{j=1}^{m} \deg(x_j) &\geq mb + (m+1)(n-1-b) \\ &= (m+1)(n-1) - b \\ &= (m+1)(ks^2 - ks) - b \\ &> m(ks^2 - s) \\ &\geq \sum_{j=1}^{m} \deg(x_j).\end{aligned}$$

The strict inequality follows from (4). We thus have a contradiction which completes the proof. ■

Before we prove our last theorem we need to introduce the relationship between certain block designs and s-uniform EFL graphs. For example, a *Steiner triple system* (one type of block design) consists of 3-element subsets (called *blocks* or *triples*) of a set of n elements such that each pair of elements appears in exactly one triple.

Now consider any Steiner triple system where the n elements are complete graphs $H_1, H_2, \ldots, H_n$ each with n vertices. Form an EFL graph by letting each triple $\{H_{j_1}, H_{j_2}, H_{j_3}\}$ correspond to the unique vertex that is in the intersection of H_{j_1}, H_{j_2}, and H_{j_3}. The fact that each pair of complete graphs appears in exactly one triple implies that each pair of H_j intersects exactly once. Then each pair of H_j has exactly one vertex in common and the resulting graph corresponding to the Steiner triple system is a rank n, maximum, 3-uniform EFL graph.

Given a Steiner triple system, a *parallel class* of triples is a collection of triples in which each of the n elements appears in precisely one triple. From the point of view of coloring the corresponding EFL graph this means that the $\frac{n}{3}$ vertices that correspond to the $\frac{n}{3}$ blocks can all be colored with the same color. A Steiner triple system in which the triples can be partitioned into $\frac{n-1}{2}$ parallel classes is called *resolvable*. Thus, a resolvable Steiner triple system on n elements corresponds to a 3-uniform EFL graph of rank n that can be colored with $\frac{n-1}{2}$ colors (one color for the vertices corresponding to the triples in each parallel class). These resolvable Steiner triple systems are called *Kirkman triple systems* after the famous "Kirkman's schoolgirl problem."

In contrast to the fact that all EFL graphs that correspond to Kirkman triple systems can be colored with $\frac{n-1}{2}$ colors, it has not been shown that all EFL graphs corresponding to Steiner triple systems can be n-colored.

Suppose we have a set of $n = st$ elements where $s \geq 3$ and t are integers. Partition the set into subsets $S_1, S_2, \ldots, S_s$ of size t. Suppose also that we can form new subsets (called *blocks*) $V_1, V_2, \ldots, V_{t^2}$ such that each V_i has exactly one element from each S_j, and each pair of elements from different S_j appears exactly once among the various V_i. The result is called a *transversal design*.

Similarly to Steiner triple systems, a transversal design is called *resolvable* if the blocks $V_1, V_2, \ldots, V_{t^2}$ can be partitioned into $\frac{n}{s} = t$ parallel classes. In this case a *parallel class* is a collection of $\frac{n}{s}$ of the V_i such that each of the n original elements appears in exactly one of the V_i.

Lemma 2. *Corresponding to each resolvable transversal design on $n = st$ elements is an s-uniform EFL graph G of rank n that can be t-colored.*

Proof. Let the n elements of the resolvable transversal design be n complete graphs of order n, $H_1, H_2, \ldots, H_n$. Form graph G by letting each block V_i correspond to a vertex v_i of special degree s that is the intersection of the s complete graphs H_j that are in V_i. That each pair of H_j from different S_i occurs exactly once among the various V_i implies that in G each pair of H_j intersect at most once. So G is an s-uniform EFL graph. No H_j occurs in two different V_i in the same parallel class. Thus the vertices v_i that correspond to blocks V_i in a given parallel class can all receive the same color. Since there are t parallel classes, it follows that G can be t-colored. □

Theorem 3. *Let G be an s-uniform EFL graph of rank n where $s \geq 3$ and t are integers. Suppose G contains a subgraph G' that corresponds to a resolvable transversal design that has s subsets $S_1, S_2, \ldots, S_s$ of size t. Then G is n-colorable.*

Proof. Using Lemma 2 we color G' with $t = \frac{n}{s}$ colors. We then have $n - t$ remaining colors. By the definition of resolvable transversal design each pair of complete graphs H_j that come from different S_i intersects at a vertex in G'. Thus any vertex in $G - G'$ must be in the intersection of H_j's that are all in one S_i. These are called S_i-vertices. If we can color all S_1-vertices with $n - t$ colors we can similarly color all S_i-vertices, $2 \leq i \leq s$, with the same $n - t$ colors, yielding an n-coloring of G.

Thus to complete the proof it is sufficient to show that all S_1-vertices can be $n - t$ colored. The maximum number of S_1-vertices is

$$\left\lfloor \frac{t}{s} \left\lfloor \frac{t-1}{s-1} \right\rfloor \right\rfloor \leq \frac{t^2 - t}{s^2 - s}.$$

The subgraph of $G - G'$ induced by all S_1-vertices is formed by t copies of K_{st}. However, this can be viewed as t copies of K_t. (We just disregard many vertices of special degree 1.) Now $t = \frac{n}{s} \leq \frac{n}{3}$. The number of colors available for the S_1-vertices is $n - t \geq \frac{2}{3}n \geq 2t$. By applying the Mitchem/Chang-Lawler Theorem (as stated in the second paragraph of this paper) the S_1-vertices can be colored with $n - t$ colors, and the theorem is proved. ■

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