On flag-transitive $6-(v,k,\lambda)$ designs with $\lambda \leq 5$ *

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Abstract

Until now, all known simple $t-(v,k,\lambda)$ designs with $t \geq 6$ have $\lambda \geq 4$. On the other hand, P. J. Cameron and C. E. Praeger showed that there are no flag-transitive simple $7-(v,k,\lambda)$ designs. In the present paper we considered the flag-transitive simple $6-(v,k,\lambda)$ designs and proved that there are no non-trivial flag-transitive simple $6-(v,k,\lambda)$ designs with $\lambda \leq 5$.

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1 Introduction

A t-design $\mathcal{D} = (X, \mathcal{B})$ means a finite set X of points and a finite set B of blocks, with an incidence relation between points and blocks (for which we always use ordinary geometric terminology such as point on block, etc.) such that there are integers λ, t, k with $k \geq t, \lambda > 0$, for which:

(i) each block consists of exactly k points;

(ii) each set of t distinct points is on exactly λ common blocks.

If the number of points in X is v, then we shall say that $\mathcal{D}$ is a t-design denoted $t-(v,k,\lambda)$. We will use b to denote the number of blocks. A t-design is trivial if every t-subset of X is in fact a block; i.e., the number of blocks is $b = \binom{v}{k}$, the binomial coefficient. And a design is simple if no two blocks are identical. An automorphism of a t-design is a one-to-one mapping of points onto points, blocks onto blocks, which preserves incidence. An automorphism group of $\mathcal{D}$ will be called s-transitive if it is s-transitive when considered as a permutation group on the points. The flags of $\mathcal{D}$ are the order pairs (x, B), where x is a point and B is a block containing x. A group is flag-transitive if it is transitive on the set of flags; this is equivalent to the assertion that the group is transitive on blocks (points) and the subgroup fixing a block (point) is transitive on the points of that block (the blocks through that point). In this paper, we assume always that the designs are simple and nontrivial.

Constructing t-designs is an important task for the combinatorician. Until now, all known simple $t-(v,k,\lambda)$ designs with $t \geq 6$ have $\lambda \geq 4$. On the

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other hand, in 1993, P. J. Cameron and C. E. Praeger studied nontrivial block-transitive simple $t-(v,k,\lambda)$ designs for large t, and showed that there are no nontrivial block-transitive simple 8-designs, and no nontrivial flag-transitive simple 7-designs. As a continuation of their works, in this paper we consider nontrivial flag-transitive 6-designs, and show that there are no nontrivial flag-transitive simple $6-(v,k,\lambda)$ designs with $\lambda \leq 5$.

Theorem 1.1 *Let $\mathcal{D}=(X,B)$ be a nontrivial simple $6-(v,k,\lambda)$-design and $G \leq \mathrm{Aut}(\mathcal{D})$. If G is flag-transitive, then $\lambda \geq 5$.*

The second section describes the definitions and contains several preliminary results about flag-transitivity and $t-$designs. In the third section we give the proof of the Main Theorem.

2 Preliminary Results

Let $\mathcal{D}=(X,\mathcal{B})$ be a $t-(v,k,\lambda)$ design, and $G \leq Aut(\mathcal{D})$. Let b denote the number of blocks of $\mathcal{D}$, and r the number of blocks that is incident with a fixed point of $\mathcal{D}$. Now, we introduce the following results which play the role of important in the proof of Main Theorem.

Lemma 2.1 *Let $\mathcal{D}=(X,\mathcal{B})$ be a $t-(v,k,\lambda)$ design. Then the following holds:*

1. $bk=vr$;
2. $r=\frac{\lambda(v-1)\ldots(v-t+1)}{(k-1)\ldots(k-t+1)}$;
3. $b=\frac{\lambda v\ldots(v-t+1)}{k\ldots(k-t+1)}$.

Lemma 2.2 ([2]) *If $v \leq k+t$, then $t-(v,k,\lambda)$ is a trivial design.*

Lemma 2.3 (Fisher's Inequality) *If $\mathcal{D}$ is a $2-(v,k,\lambda)$ design, with b blocks, then $b \geq v$.*

By the results of Propositions 2.3 and 2.4 and Corollary 4.3 in [4], we have the following lemma:

Lemma 2.4 *Let $\mathcal{D}$ be a nontrivial simple $6-(v,k,\lambda)$ design admitting a flag-transitive automorphism group G. Then $G=AGL(d,2)$ and $v=2^d \geq 8$.*

3 The Proof of Main Theorem

First, we give a very useful lemma.

Lemma 3.1 *If $\mathcal{D}$ is a $t-(v,k,\lambda)$ design, then*

$$\lambda(v-t+1) \geq (k-t+1)(k-t+2).$$

Proof. The well known that we can get a $2-(v-t+2, k-t+2, \lambda)$ design from $\mathcal{D}$. Thus by Lemmas 2.1 and 2.3 we get

$$\frac{\lambda(v-t+2)(v-t+1)}{(k-t+2)(k-t+1)} \geq v-t+2,$$

that is

$$\lambda(v-t+1) \geq (k-t+2)(k-t+1).$$

Now we may prove our Theorem 1.1 occurring in Introduction. Suppose that $\mathcal{D} = (X, \mathcal{B})$ is a non-trivial simple $6-(v,k,\lambda)$ design with $\lambda \leq 5$, and $G \leq Aut(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$. By Lemma 2.4, $G = AGL(d,2)$ and $v = 2^d \geq 8$. By Lemma 2.2, $6 < k < v-6 = 2^d - 6$. This yields that $d \geq 4$. Let e_i denote the i-th standard basis vector of the vector space $V(d,2)$ and $\langle e_i \rangle$ the 1-dimensional vector subspace spanned by e_i. Then any six distinct vectors are non-coplanar and hence generate a subspace of dimension at least 3. Let $\Phi = \langle e_1, e_2, e_3 \rangle$ denote the 3-dimensional vector subspace spanned by e_1, e_2, e_3. Thus $SL(d,2)_\Phi$ acts point-transitively on $V(d,2)\backslash\Phi$. Let $\Psi = \{0, e_1, e_2, e_3, e_1+e_2, e_1+e_2+e_3\}$. By definition of design, there exist exactly λ blocks $B_1, \ldots, B_\lambda$, such that $\Psi \subseteq B_1 \cap \ldots \cap B_\lambda$. If B_1 contains a vector $\alpha \in V(d,2) \setminus \Phi$, then $V(d,2) \setminus \Phi \subseteq B_1 \cup \ldots \cup B_\lambda$ as $SL(d,2)_\Phi$ acts point-transitively on $V(d,2)\backslash\Phi$. It follows that $2^d - 8 \leq \lambda(k-6)$ and so $v \leq \lambda(k-6)+8$. By Lemma 3.1, we get that

$$k^2 - (9+\lambda^2)k + 6\lambda^2 - 3\lambda + 20 \leq 0. \tag{1}$$

If $\lambda = 1$, then by inequality (1) we have $k < 7$, which conflicts with $k \geq 7$ (ref. [7]). If $\lambda = 2$, then $k = 7$ or 8, if $\lambda = 3$, then $7 \leq k \leq 13$, and if $\lambda = 4$, then $7 \leq k \leq 19$. But, on the other hand, since $\mathcal{D}$ is also a 5-design admitting $G \leq \mathrm{Aut}(\mathcal{D})$, we have $2^d - 3$ must divide $\binom{k}{4}$ by [1] or [4]. Note that $\binom{19}{4} = 3876$ and $2^{12} = 4096$. Thus $d \leq 11$. We recall that $d \geq 4$. Thus when $4 \leq d \leq 11$ and $7 \leq k \leq 19$, if $2^d - 3$ divides $\binom{k}{4}$, then $d = 4$ and $13 \leq k \leq 16$. But, by Lemma 2.2, $k < v - t = 10$. Hence we get a contradiction. If $\lambda = 5$, by (1) we get $7 \leq k \leq 28$. Since $2^{15} > \binom{28}{4}$ and $(2^d - 3) | \binom{28}{4}$, we have $d \leq 14$. As above we have $d \neq 4$. If $5 \leq d \leq 14$ then it is not hard to check that the only parameters d and k satisfying the divisibility are $d = 8$ and $k = 23, 24$ or 25. However, by the inequality $2^d \leq \lambda(k-6)+8$ we get $k \geq 556$, a contradiction. So $B_1 \subset \Phi$, and we know by the flag-transitivity of G that each block must be contained in a 3-dimensional affine subspaces. Thus $k \leq 8$, and we can obtain a contradiction as above. Therefore, we complete the proof of Theorem 1.1 occurring in Introduction.

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