Packing and Covering Complete Bipartite Multidigraphs with 6-circuits

Hung-Chih Lee*
Department of Information Technology
Ling Tung University
Taichung 40852, Taiwan
E-mail: birdy@teamail.ltu.edu.tw

Abstract

A k-circuit is a directed cycle of length k. In this paper, we completely solve the problem of finding maximum packings and minimum coverings of λ-fold complete bipartite symmetric digraphs with 6-circuits.

1 Introduction and preliminaries

For an integer $k \geq 2$, a *k-circuit* $\overrightarrow{C_k}$ is a directed cycle of length k. Let G be a multidigraph. A *$\overrightarrow{C_k}$-subdigraph* of G is a subdigraph of G which is isomorphic to $\overrightarrow{C_k}$. A *$\overrightarrow{C_k}$-decomposition* of G is a collection of $\overrightarrow{C_k}$-subdigraph of G which partition the arc set of G. If G has a $\overrightarrow{C_k}$-decomposition, we say $\overrightarrow{C_k}$ *decomposes* G, denoted by $\overrightarrow{C_k} \mid G$. The existence problems for $\overrightarrow{C_k}$-decomposition of the complete symmetric digraph and the complete bipartite symmetric digraph were solved by Alspach et al. [1] and Sotteau [18], respectively. When a multidigraph G can not be decomposed into k-circuits, two natural questions arise:

(1) What is the minimum number of arcs need to be removed from the arc set of G so that the resulting multidigraph can be decomposed into k-circuits, and what does the collection of removed arcs look like?

*This research was supported by NSC of R.O.C. under grant NSC 98-2115-M-275-002

(2) What is the minimum number of arcs need to be added into the arc set of G so that the resulting multidigraph can be decomposed into k-circuits, and what does the collection of added arcs look like?

These questions are respectively called the maximum packing and minimum covering problem with k-circuits.

For a multigraph H, we use the symbol H^* to denote the multidigraph obtained from H by replacing each edge e by two opposite arcs connecting the endvertices of e. Let r be a positive integer, rH denotes the multigraph obtained from H by replacing each edge e by r edges each of which has the same endvertices as e. For a multidigraph G, rG is similarly defined. Let $A(G)$ be the arc set of G. For $A \subseteq A(G)$, $G - A$ (resp. $G + A$) denotes the multidigraph obtained from G by removing (resp. adding) all arcs of A from (resp. to) G. Throughout the paper, any arc set mentioned may be a multiset.

Suppose that G is a multidigraph. A *$\overrightarrow{C_k}$-packing* of G with *leave* L is a $\overrightarrow{C_k}$-decomposition of $G - L$ where $L \subseteq E(G)$, and a *$\overrightarrow{C_k}$-covering* with *padding* P is a $\overrightarrow{C_k}$-decomposition of $G+P$ where $P \subseteq E(rG)$). When there is no confusion, we shall refer to a leave (resp. padding) of G as the digraph induced by the arcs of the leave (resp. padding). A packing $\mathscr{P}$ is *maximum* if the cardinality $|\mathscr{P}|$ is as large as possible, and the corresponding leave is referred to as a *minimum leave.* A covering $\mathscr{C}$ is *minimum* if the cardinality $|\mathscr{C}|$ is as small as possible, and the corresponding padding is called a *minimum padding.* Clearly, a $\overrightarrow{C_k}$-decomposition of G is a maximum $\overrightarrow{C_k}$-packing with leave the empty set, and also a minimum $\overrightarrow{C_k}$-covering with padding the empty set.

Maximum packings and minimum coverings of graphs with cycles or circuits have been and continue to be popular topics of research. For the complete graph K_n, the maximum C_k-packings have been found for $k \in \{3,4,5,6,8\}$ (see [9, 11, 13, 15, 16, 17]), and the minimum C_k-coverings have been found for $k \in \{3,4,6\}$ (see [7, 12, 17]). For the complete bipartite graph $K_{m,n}$, the maximum C_k-packing problem has been settled for $k \in \{4,6\}$ (see [3, 6]). For the balanced complete multipartite graph $K_{m(n)}$, the maximum C_k-packing problem was investigated for $k \in \{3,5,6\}$ (see [5, 8, 10]), and the minimum C_6-covering can be found in [8]. The result on maximum C_4-packing of λ-fold complete multipartite graph can be found in [3, 4]. For digraphs, the problem of finding maximum packings and minimum coverings of λK_n^* with 3- and 4-circuits was considered by Bennett and Yin [2], and the problem of finding maximum packings and minimum coverings of $\lambda K_{m,n}^*$ with 4-circuits was settled by Wu et al. [19].

In this paper, we completely solve the problem of finding maximum packings and minimum coverings of $\lambda K_{m,n}^*$ with 6-circuits, and give minimum leaves and minimum paddings explicitly.

2 Minimum possible leaves and paddings

In this section the minimum possible leaves and paddings are given. We start with some notations. Let P_k denote a path on k vertices and $\overrightarrow{C_2} \uplus \overrightarrow{C_2}$ denote the union of two disjoint copies of $\overrightarrow{C_2}$. For a vertex v of a multidigraph G, the *indegree* $d_G^-(v)$, or simply $d^-(v)$, of v is the number of arcs incident to v, and the *outdegree* $d_G^+(v)$, or simply $d^+(v)$, of v is the number of arcs incident from v. Note that every vertex v in $\lambda K_{m,n}^*$, $d^-(v) = d^+(v)$, and so does the vertex of $\overrightarrow{C_6}$ in $\lambda K_{m,n}^*$. Hence, we have the following.

Lemma 2.1. *Suppose m and n are positive integers with $\min\{m,n\} \geq 3$. Then for each vertex v in a leave L (resp. padding P) of $\overrightarrow{C_6}$-packing (resp. $\overrightarrow{C_6}$-covering) of $\lambda K_{m,n}^*$, $d_L^-(v) = d_L^+(v)$ (resp. $d_P^-(v) = d_P^+(v)$).*

If $\lambda mn \equiv 0 \pmod 3$, then $|E(\lambda K_{m,n}^*)| = 2\lambda mn \equiv 0 \pmod 6$. Thus, $|E(\overrightarrow{C_6})|$ divides $|E(\lambda K_{m,n}^*)|$, which implies the minimum possible leave and minimum padding are both the empty set. If $\lambda mn \equiv 1 \pmod 3$, then $|E(\lambda K_{m,n}^*)| = 2\lambda mn \equiv 2 \pmod 6$. Thus, $|E(\overrightarrow{C_6})|$ divides $|E(\lambda K_{m,n}^*)| - 2$, which implies the size of the minimum possible leave is 2, and $|E(\overrightarrow{C_6})|$ divides $|E(\lambda K_{m,n}^*)| + 4$, which implies the size of the minimum possible padding is 4. By Lemma 2.1, the candidate for the minimum possible leave is only $\overrightarrow{C_2}$, and the candidates for the minimum possible padding are $P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}$ and $2\overrightarrow{C_2}$. If $\lambda mn \equiv 2 \pmod 3$, then $|E(\lambda K_{m,n}^*)| = 2\lambda mn \equiv 4 \pmod 6$. Thus, $|E(\overrightarrow{C_6})|$ divides $|E(\lambda K_{m,n}^*)| - 4$, which implies the size of the minimum possible padding is 4, and $|E(\overrightarrow{C_6})|$ divides $|E(\lambda K_{m,n}^*)| + 2$, which implies the size of the minimum possible padding is 2. Again by Lemma 2.1, the candidates for the minimum possible leave are $P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}$ if $\lambda = 1$ and $P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}$ together with $2\overrightarrow{C_2}$ if $\lambda \geq 2$, and the candidate for the minimum possible padding is only $\overrightarrow{C_2}$. We summarize the results discussed above in Table 1.

Table 1: The minimum possible leaves and paddings of $\lambda K_{m,n}^*$

λmn (mod 3)	0	1	2
Leave	$\emptyset$	$\overrightarrow{C_2}$	$P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}$ if $\lambda = 1$ $P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}$ if $\lambda \geq 2$
Padding	$\emptyset$	$P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}$	$\overrightarrow{C_2}$

3 Decompositions into circuits

In this section a necessary and sufficient condition on $\overrightarrow{C_6}$-decomposition of $\lambda K^*_{m,n}$ is given, which is useful for our discussions to follow. We begin with a criterion for decomposing the complete bipartite symmetric digraph into $2r$-circuits.

Proposition 3.1. (Sotteau [18]) *$K^*_{m,n}$ has a $\overrightarrow{C_{2r}}$-decomposition if and only if* $\min\{m,n\} \geq r$ *and* $mn \equiv 0 \pmod{r}$.

Let G be a multidigraph. If G can be decomposed into subdigraphs $G_1, G_2, \ldots, G_t$, we write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_t$. The following is trivial.

Lemma 3.2. *Suppose that $G, H_0, H_1, H_2, \ldots, H_n$ are multidigraphs and $G = H_0 \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_n$. If H_0 has a $\overrightarrow{C_k}$-packing (resp. $\overrightarrow{C_k}$-covering) with leave L (resp. padding P) and $\overrightarrow{C_k} \mid H_i$ for $i = 1, 2, \ldots, n$, then G has a $\overrightarrow{C_k}$-packing (resp. $\overrightarrow{C_k}$-covering) with leave L (resp. padding P).*

Since a $\overrightarrow{C_k}$-decomposition is a $\overrightarrow{C_k}$-packing with leave the empty set, next lemma follows immediately.

Lemma 3.3. *Suppose that $G, H_0, H_1, H_2, \ldots, H_n$ are multidigraphs. If $G = H_0 \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_n$ and $\overrightarrow{C_k} \mid H_i$ for each i, then $\overrightarrow{C_k} \mid G$.*

We need more terms and notations for our discussions. Suppose that G is a multigraph. Let x and y be distinct vertices of G. We use $\deg_G(x)$ to denote the number of edges incident with x, called the *degree* of x, and $e_G(x,y)$ to denote the number of edges joining x and y. A *star* with r edges, denoted by S_r, is a complete bipartite graph $K_{1,r}$. The vertex of degree r in S_r is called the *center* of S_r and any vertex of degree 1 is called an *endvertex* of S_r. A *multistar* is a star with multiple edges allowed. Before plunging into the circuit decomposition of the complete bipartite multidigraph, we will need a proposition due to Lin et al.

Proposition 3.4. ([14, Proposition 1.3]) *Suppose that H is a multistar. Then H has an S_r-decomposition if and only if there exists a nonnegative integer k such that $|E(H)| = rk$ and $e_H(w,x) \leq k$ where w is the center of H and x is any endvertex.*

Now we present a sufficient condition on decomposing a complete bipartite multidigraph into isomorphic circuits.

Theorem 3.5. *Suppose that λ, m and n are positive integers with $\min\{m,n\} \geq r$. Then $\lambda K^*_{m,n}$ has a $\overrightarrow{C_{2r}}$-decomposition if one of the following conditions holds:* (1) $mn \equiv 0 \pmod{r}$, (2) $\lambda n \equiv 0 \pmod{r}$, *or* (3) $\lambda m \equiv 0 \pmod{r}$.

Proof. The result for the condition $mn \equiv 0 \pmod r$ follows from Proposition 3.1 and Lemma 3.3. Since $\lambda K^*_{m,n}$ is isomorphic to $\lambda K^*_{n,m}$, we need only to show the result holds for the condition $\lambda n \equiv 0 \pmod r$.

Suppose that $\gcd(\lambda, r) = d$ where $\gcd(\lambda, r)$ denotes the greatest common divisor of λ and r. Let $\lambda = ds$. Since trivially $dK^*_{m,n} \mid dsK^*_{m,n}$ and by Proposition 3.1 $\overrightarrow{C_{2r}} \mid K^*_{m,r}$, it is sufficient to show that $K^*_{m,r} \mid dK^*_{m,n}$ by Lemma 3.3. We first show the connection between S_r-decomposition of dS_n and $K_{m,r}$-decomposition of $dK_{m,n}$. Consider a multistar dS_n with center a and endvertices $b_0, b_1, \ldots, b_{n-1}$. Replacing each edge ab_i in dS_n by S_m with center b_i and endvertices $a_0, a_1, \ldots, a_{m-1}$, we obtain $dK_{m,n}$, and any substar S_r of dS_n become a subgraph $K_{m,r}$ of $dK_{m,n}$. Thus, corresponding an S_r-decomposition of dS_n, there exists a $K_{m,r}$-decomposition of $dK_{m,n}$. Let $r = dt$. Then t and s are coprime since $\lambda = ds$ and $\gcd(\lambda, r) = d$. This implies $t \mid n$ by the condition $\lambda n \equiv 0 \pmod r$. Let $n = tk$. We have $d \le k$ from the assumption $r \le n$ and the fact $d = r/t$ and $k = n/t$. Note that $|E(dS_n)| = dn = dtk = rk$ and $e_{dS_n}(w, x) = d \le k$ where w is the center of H and x is any endvertex. By Proposition 3.4, dS_n has an S_r-decomposition. Hence, $dK_{m,n}$ has a $K_{m,r}$-decomposition. Replacing each edge in $dK_{m,n}$ by two arcs with opposite directions, we obtain $dK^*_{m,n}$, and any subgraph $K_{m,r}$ of $dK_{m,n}$ becomes a subdigraph $K^*_{m,r}$ of $dK^*_{m,n}$. Thus, we obtain a $K^*_{m,r}$-decomposition of $dK^*_{m,n}$. This completes the proof. □

Corollary 3.6. *For positive integers λ, m and n with $\min\{m, n\} \ge 3$, $\lambda K^*_{m,n}$ has a $\overrightarrow{C_6}$-decomposition if and only if $\lambda mn \equiv 0 \pmod 3$.*

Proof. The necessity follows immediately from applying counting arguments on arcs. For the sufficiency, $\lambda mn \equiv 0 \pmod 3$ implies $mn \equiv 0 \pmod 3$ or $\lambda \equiv 0 \pmod 3$. Thus, the result follows from Theorem 3.5. □

4 Small cases of packings and coverings

In this section, we give some necessary small cases of maximum packings and minimum coverings for the general construction to follow. Before that, we need more terms and notations. Let $\mathscr{R}$ be a set of subdigraphs of G and k a positive integer. Then $k\mathscr{R}$ denotes a multiset in which each element in $\mathscr{R}$ appears k times. For $k \ge 2$, we use $(v_1, v_2, \ldots, v_k)$ to denote the k-circuit consisting of distinct vertices $v_1, v_2, \ldots, v_k$ and arcs $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{k-1}v_k}, \overrightarrow{v_kv_1}$. In particular, (v_1, v_2) denotes the 2-circuit with arcs $\overrightarrow{v_1v_2}$ and $\overrightarrow{v_2v_1}$. Throughout the section, (A, B) denotes the bipartition of $\lambda K^*_{m,n}$ where $A = \{a_0, a_1, \ldots, a_{m-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$.

Lemma 4.1. *Suppose that $\mathscr{P}$ is a $\overrightarrow{C_k}$-packing of a multigraph G with leave L. If there exist $Q_1, Q_2, \ldots, Q_s \in \mathscr{P}$ and $P \subseteq A(rG)$ such that*

$Q_1 \cup Q_2 \cup \ldots \cup Q_s \cup L \cup P$ can be decomposed into k-circuits $R_1, R_2, \ldots, R_t$, then $\mathscr{P} - \{Q_1, Q_2, \ldots, Q_s\} \cup \{R_1, R_2, \ldots, R_t\}$ is a $\overrightarrow{C_k}$-covering of G with padding P.

Lemma 4.2. *Suppose that $t \in \{1, 2\}$.*
(a) *There exists a $\overrightarrow{C_6}$-packing of $tK^*_{4,4}$ with leave L where*

$$\begin{cases} L = \overrightarrow{C_2} & \text{if } t = 1, \\ L \in \{P^*_3, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } t = 2. \end{cases}$$

(b) *There exists a $\overrightarrow{C_6}$-covering of $tK^*_{4,4}$ with padding P where*

$$\begin{cases} P \in \{P^*_3, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } t = 1, \\ P = \overrightarrow{C_2} & \text{if } t = 2. \end{cases}$$

Proof. The proof is divided into two parts according to the value of t.
Case 1. $t = 1$.

Let $\mathscr{P}_1 = \{(a_0, b_1, a_1, b_2, a_2, b_3), (a_0, b_2, a_1, b_3, a_3, b_1), (a_0, b_3, a_2, b_0, a_3, b_2), (b_0, a_1, b_1, a_2, b_2, a_3), (b_0, a_2, b_1, a_3, b_3, a_1)\}$. Then $\mathscr{P}_1$ is a $\overrightarrow{C_6}$-packing of $K^*_{4,4}$ with leave $\overrightarrow{C_2} : (a_0, b_0)$.

Now we use $\mathscr{P}_1$ to construct a $\overrightarrow{C_6}$-covering of $K^*_{4,4}$ with padding P for each $P \in \{P^*_3, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}\}$. By Lemma 4.1, it is easy to check that $\mathscr{P}_1 - \{(a_0, b_2, a_1, b_3, a_3, b_1)\} \bigcup \{(a_0, b_2, a_1, b_3, a_2, b_0), (a_0, b_0, a_2, b_3, a_3, b_1)\}$ is a covering with padding $P^*_3 : (a_2, b_0) \bigcup (a_2, b_3)$, $\mathscr{P}_1 - \{(a_0, b_2, a_1, b_3, a_3, b_1)\} \bigcup \{(a_0, b_2, a_1, b_3, a_2, b_0), (a_0, b_0, a_1, b_3, a_3, b_1)\}$ is a covering with padding $\overrightarrow{C_4} : (a_1, b_3, a_2, b_0)$, and $\mathscr{P}_1 - \{(a_0, b_3, a_2, b_0, a_3, b_2)\} \bigcup \{(a_0, b_3, a_3, b_2, a_2, b_0), (a_0, b_0, a_3, b_3, a_2, b_2)\}$ is a covering with padding $\overrightarrow{C_2} \uplus \overrightarrow{C_2} : (a_2, b_2) \bigcup (a_3, b_3)$. Moreover, $\{(a_0, b_0, a_1, b_1, a_2, b_2), (a_0, b_0, a_2, b_3, a_3, b_1), (a_0, b_0, a_3, b_2, a_1, b_3), (b_0, a_0, b_1, a_1, b_2, a_2), (b_0, a_0, b_2, a_3, b_3, a_1), (b_0, a_0, b_3, a_2, b_1, a_3)\}$ is a covering with padding $2\overrightarrow{C_2} : (a_0, b_0) \bigcup (a_0, b_0)$.
Case 2. $t = 2$.

First, we use $\mathscr{P}_1$ to construct the required packings of $2K^*_{4,4}$. Rename the vertices b_0, b_1 of the circuits in $\mathscr{P}_1$ to b_1, b_0, respectively. Then we obtain a packing $\mathscr{P}'_1$ of $K^*_{4,4}$ with leave (a_0, b_1). Thus $\mathscr{P}_2 = \mathscr{P}_1 \bigcup \mathscr{P}'_1$ is a packing of $2K^*_{4,4}$ with leave $P^*_3 : (a_0, b_0) \bigcup (a_0, b_1)$. Similarly, rename the vertices a_0, a_1, b_0, b_1 in $\mathscr{P}_1$ to a_1, a_0, b_1, b_0, respectively. Then we obtain a packing $\mathscr{P}''_1$ of $K^*_{4,4}$ with leave (a_1, b_1). Thus, $\mathscr{P}_1 \bigcup \mathscr{P}''_1$ is a packing of $2K^*_{4,4}$ with leave $\overrightarrow{C_2} \uplus \overrightarrow{C_2} : (a_0, b_0) \bigcup (a_1, b_1)$. Moreover, rearrange the arcs of the circuit $(b_1, a_2, b_0, a_3, b_3, a_1)$ and the leave of $\mathscr{P}_2$, we obtain a packing $\mathscr{P}_3 = \mathscr{P}_2 - \{(b_1, a_2, b_0, a_3, b_3, a_1)\} \bigcup \{(a_0, b_0, a_3, b_3, a_1, b_1)\}$ of $2K^*_{4,4}$ with leave $\overrightarrow{C_4} : (a_0, b_1, a_2, b_0)$. Finally, $2\mathscr{P}_1$ is trivially a packing of $2K^*_{4,4}$ with leave $2\overrightarrow{C_2} : (a_0, b_0) \bigcup (a_0, b_0)$.

Now we construct the required covering of $2K^*_{4,4}$. Note that $(a_0, b_0, a_1, b_2, a_2, b_3) \in \mathscr{P}_3$. By Lemma 4.1, it is easy to check that $\mathscr{P}_3 - \{(a_0, b_0, a_1, b_2, a_2, b_3)\} \bigcup \{(a_0, b_0, a_1, b_1, a_2, b_3), (a_0, b_1, a_1, b_2, a_2, b_0)\}$ is a covering of $2K^*_{4,4}$ with padding $\overrightarrow{C_2} : (a_1, b_1)$. □

Lemma 4.3. *Suppose that* $t \in \{1, 2\}$.
(a) *There exists a* $\overrightarrow{C_6}$*-packing of* $tK^*_{4,5}$ *with leave* L *where*

$$\begin{cases} L \in \{P^*_3, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}\} & \text{if } t = 1, \\ L = \overrightarrow{C_2} & \text{if } t = 2. \end{cases}$$

Moreover, there exists a $\overrightarrow{C_6}$*-packing of* $4K^*_{4,5}$ *with leave* L *where* $L \in \{P^*_3, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\}$.
(b) *There exists a* $\overrightarrow{C_6}$*-covering of* $tK^*_{4,5}$ *with padding* P *where*

$$\begin{cases} P = \overrightarrow{C_2} & \text{if } t = 1, \\ P \in \{P^*_3, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } t = 2. \end{cases}$$

Proof. We distinguish three cases.
Case 1. $t = 1$.

Note that $K^*_{4,5} = K^*_{4,4} \oplus K^*_{4,1}$. We will use the packing $\mathscr{P}_1$ of $K^*_{4,4}$ to construct the required packing of $K^*_{4,5}$. Reconstruct 6-circuits from $(b_0, a_1, b_1, a_2, b_2, a_3)$, $(b_0, a_2, b_1, a_3, b_3, a_1)$, (a_1, b_4), (a_2, b_4) and (a_3, b_4). Then $\mathscr{P}_4 = \mathscr{P}_1 - \{(b_0, a_1, b_1, a_2, b_2, a_3), (b_0, a_2, b_1, a_3, b_3, a_1)\} \bigcup \{(b_0, a_1, b_4, a_2, b_2, a_3), (b_0, a_2, b_1, a_3, b_4, a_1), (a_1, b_1, a_2, b_4, a_3, b_3)\}$ is a packing of $K^*_{4,5}$ with leave $P^*_3 : (a_0, b_0) \bigcup (a_0, b_4)$. Similarly, reconstruct 6-circuits from $(a_0, b_1, a_1, b_2, a_2, b_3)$, $(a_0, b_2, a_1, b_3, a_3, b_1)$ (a_0, b_4), (a_1, b_4) and (a_2, b_4), then $\mathscr{P}_1 - \{(a_0, b_1, a_1, b_2, a_2, b_3), (a_0, b_2, a_1, b_3, a_3, b_1)\} \bigcup \{(a_0, b_1, a_1, b_2, a_2, b_4), (a_0, b_4, a_1, b_3, a_3, b_1), (a_0, b_2, a_1, b_4, a_2, b_3)\}$ is a packing of $K^*_{4,5}$ with leave $\overrightarrow{C_2} \uplus \overrightarrow{C_2} : (a_0, b_0) \bigcup (a_3, b_4)$. Moreover, rearrange the arcs of the circuit $(b_0, a_1, b_4, a_2, b_2, a_3)$ and the leave of $\mathscr{P}_4$, we obtain a packing $\mathscr{P}_5 = \mathscr{P}_4 - \{(b_0, a_1, b_4, a_2, b_2, a_3)\} \bigcup \{(a_0, b_4, a_2, b_2, a_3, b_0)\}$ of $K^*_{4,5}$ with leave $\overrightarrow{C_4} : (a_0, b_0, a_1, b_4)$.

Now we use $\mathscr{P}_4$ to construct the required covering of $K^*_{4,5}$. Note that $(a_1, b_1, a_2, b_4, a_3, b_3) \in \mathscr{P}_4$. By Lemma 4.1, it is easy to check $\mathscr{C}_1 = \mathscr{P}_4 - \{(a_1, b_1, a_2, b_4, a_3, b_3)\} \bigcup \{(a_0, b_0, a_1, b_1, a_2, b_4), (a_0, b_4, a_3, b_3, a_1, b_0)\}$ is a covering of $K^*_{4,5}$ with padding $\overrightarrow{C_2} : (a_1, b_0)$.
Case 2. $t = 2$.

First, we use $\mathscr{P}_5$, the packing of $K^*_{4,5}$ with leave (a_0, b_0, a_1, b_4), to construct the required packing of $2K^*_{4,5}$. Rename the vertices a_1, a_2, b_0, b_1, b_4 of the circuits in $\mathscr{P}_5$ to a_2, a_1, b_1, b_4, b_0, respectively. Then we obtain a packing $\mathscr{P}'_5$ of $K^*_{4,5}$ with leave (a_0, b_1, a_2, b_0). Since $(a_0, b_0, a_1, b_4) \bigcup (a_0, b_1, a_2, b_0)$

$= (a_0, b_1, a_2, b_0, a_1, b_4) \bigcup (a_0, b_0)$, $\mathscr{P}_6 = \mathscr{P}_5 \bigcup \mathscr{P}'_5 \bigcup \{(a_0, b_1, a_2, b_0, a_1, b_4)\}$ is a packing of $2K^*_{4,5}$ with leave $\overrightarrow{C_2} : (a_0, b_0)$.

Now we use $\mathscr{P}_6$ to construct the required coverings of $2K^*_{4,5}$. Note that $(a_0, b_1, a_1, b_2, a_2, b_3), (a_0, b_1, a_2, b_0, a_1, b_4) \in \mathscr{P}_6$. By Lemma 4.1, it is easy to check that $\mathscr{P}_6 - \{(a_0, b_1, a_1, b_2, a_2, b_3)\} \bigcup \{(a_0, b_1, a_1, b_2, a_3, b_0), (a_0, b_0, a_3, b_2, a_2, b_3)\}$ is a covering of $2K^*_{4,5}$ with padding $P^*_3 : (a_3, b_0) \bigcup (a_3, b_2)$, $\mathscr{P}_6 - \{(a_0, b_1, a_1, b_2, a_2, b_3)\} \bigcup \{(a_0, b_1, a_1, b_2, a_3, b_0), (a_0, b_0, a_1, b_2, a_2, b_3)\}$ is a covering with padding $\overrightarrow{C_4} : (a_1, b_2, a_3, b_0)$, and $\mathscr{P}_6 - \{(a_0, b_1, a_2, b_0, a_1, b_4)\} \bigcup \{(a_0, b_1, a_1, b_4, a_2, b_0), (a_0, b_0, a_1, b_1, a_2, b_4)\}$ is a covering with padding $\overrightarrow{C_2} \uplus \overrightarrow{C_2} : (a_1, b_1) \bigcup (a_2, b_4)$. Finally, $2\mathscr{C}_1$ is clearly a covering of $2K^*_{4,5}$ with padding $2\overrightarrow{C_2} : (a_1, b_0) \bigcup (a_1, b_0)$.

Case 3. $t = 4$.

Since $4K^*_{4,5} = 2K^*_{4,5} \oplus 2K^*_{4,5}$ and $\mathscr{P}_6$ is a $\overrightarrow{C_6}$-packring of $2K^*_{4,5}$ with leave (a_0, b_0), $2\mathscr{P}_6$ is a $\overrightarrow{C_6}$-packing of $4K^*_{4,5}$ with leave $2\overrightarrow{C_2} : (a_0, b_0) \bigcup (a_0, b_0)$. On the other hand, $4K^*_{4,5} = K^*_{4,5} \oplus 3K^*_{4,5}$, the result follows from Lemma 3.2, Corollary 3.6 and the case 1 of this lemma. □

Lemma 4.4. *Suppose that* $t \in \{1, 2\}$.

(a) *There exists a* $\overrightarrow{C_6}$*-packing of* $tK^*_{5,5}$ *with leave* L *where*

$$\begin{cases} L = \overrightarrow{C_2} & \text{if } t = 1, \\ L \in \{P^*_3, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } t = 2. \end{cases}$$

(b) *There exists a* $\overrightarrow{C_6}$*-covering of* $tK^*_{5,5}$ *with padding* P *where*

$$\begin{cases} P \in \{P^*_3, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } t = 1, \\ P = \overrightarrow{C_2} & \text{if } t = 2. \end{cases}$$

Proof. The proof is divided into two parts according to the value of t.

Case 1. $t = 1$.

Let $\mathscr{P}_7 = \{(a_0, b_1, a_1, b_2, a_2, b_4), (a_0, b_4, a_1, b_3, a_3, b_1), (a_0, b_3, a_4, b_4, a_3, b_2), (b_0, a_1, b_1, a_2, b_2, a_4), (b_0, a_4, b_1, a_3, b_3, a_1), (a_0, b_2, a_1, b_4, a_2, b_3), (b_0, a_2, b_1, a_4, b_2, a_3), (b_0, a_3, b_4, a_4, b_3, a_2)\}$. Then $\mathscr{P}_7$ is a $\overrightarrow{C_6}$-packing of $K^*_{5,5}$ with leave $\overrightarrow{C_2} : (a_0, b_0)$.

Now we use $\mathscr{P}_7$ to construct required coverings of $K^*_{5,5}$. By Lemma 4.1, it is easy to check that $\mathscr{P}_7 - \{(a_0, b_3, a_4, b_4, a_3, b_2)\} \bigcup \{(a_0, b_3, a_4, b_4, a_2, b_0), (a_0, b_0, a_2, b_4, a_3, b_2)\}$ is a covering of $K^*_{5,5}$ with padding $P^*_3 : (a_2, b_0) \bigcup (a_2, b_4)$, $\mathscr{P}_7 - \{(a_0, b_3, a_4, b_4, a_3, b_2)\} \bigcup \{(a_0, b_3, a_4, b_4, a_2, b_0), (a_0, b_0, a_1, b_4, a_3, b_2)\}$ is a covering with padding $\overrightarrow{C_4} : (a_1, b_4, a_2, b_0)$, and $\mathscr{P}_7 - \{(a_0, b_3, a_4, b_4, a_3, b_2), (b_0, a_3, b_4, a_4, b_3, a_2)\} \bigcup \{(a_0, b_3, a_4, b_4, a_2, b_0), (a_0, b_0, a_3, b_4, a_4, b_2), (a_2, b_4, a_3, b_2, a_4, b_3)\}$ is a covering with padding $\overrightarrow{C_2} \uplus \overrightarrow{C_2} : (a_2, b_4) \bigcup (a_4, b_2)$. Finally, $\{(a_i, b_{4+i}, a_{1+i}, b_{3+i}, a_{2+i}, b_{2+i}) : i = 0, 1, 2, 3, 4\} \bigcup \{(a_0, b_1, a_1, b_4, a_4, b_2), (a_0, b_2, a_3, b_1, a_2, b_0), (a_0, b_2, a_2, b_3, a_3, b_4), (a_0, b_3, a_4, b_0, a_1, b_2)\}$, where the subscripts of a_i and b_j are taken modulo 5, is a covering of $K^*_{5,5}$ with padding $2\overrightarrow{C_2} : (a_0, b_2) \bigcup (a_0, b_2)$.

Case 2. $t = 2$.

First, we use $\mathscr{P}_7$ to construct the required packings of $2K^*_{5,5}$. Rename the vertices b_0, b_1 of the circuits in $\mathscr{P}_7$ to b_1, b_0, respectively. Then we obtain a packing $\mathscr{P}'_7$ of $K^*_{5,5}$ with leave (a_0, b_1). Thus $\mathscr{P}_8 = \mathscr{P}_7 \bigcup \mathscr{P}'_7$ is a packing of $2K^*_{5,5}$ with leave $P^*_3 : (a_0, b_0) \bigcup (a_0, b_1)$. Similarly, rename the vertices a_0, a_1, b_0, b_1 of the circuits in $\mathscr{P}_7$ to a_1, a_0, b_1, b_0, respectively. Then we obtain a packing $\mathscr{P}''_7$ of $K^*_{5,5}$ with leave (a_1, b_1). Thus $\mathscr{P}_7 \bigcup \mathscr{P}''_7$ is a packing of $2K^*_{5,5}$ with leave $\overrightarrow{C_2} \uplus \overrightarrow{C_2} : (a_0, b_0) \bigcup (a_1, b_1)$. Moreover, rearrange the arcs of the circuit $(b_1, a_2, b_0, a_4, b_2, a_3)$ and the leave of $\mathscr{P}_8$, then $\mathscr{P}_8 - \{(b_1, a_2, b_0, a_4, b_2, a_3)\} \bigcup \{(a_0, b_0, a_4, b_2, a_3, b_1)\}$ is a packing of $2K^*_{5,5}$ with leave $\overrightarrow{C_4} : (a_0, b_1, a_2, b_0)$. Finally, $2\mathscr{P}_7$ is clearly a packing of $2K^*_{5,5}$ with leave $2\overrightarrow{C_2} : (a_0, b_0) \bigcup (a_0, b_0)$.

Now we use $\mathscr{P}_8$, the packing of $2K^*_{5,5}$ with leave $(a_0, b_0) \bigcup (a_0, b_1)$, to construct the required covering of $2K^*_{5,5}$. Note that $(b_1, a_3, b_4, a_4, b_3, a_2) \in \mathscr{P}_8$. By Lemma 4.1, $\mathscr{P}_8 - \{(b_1, a_3, b_4, a_4, b_3, a_2)\} \bigcup \{(a_0, b_1, a_3, b_4, a_4, b_0), (a_0, b_0, a_4, b_3, a_2, b_1)\}$ is a covering of $2K^*_{5,5}$ with padding $\overrightarrow{C_2} : (a_4, b_0)$. □

5 Maximum packings and minimum coverings

In this section maximum packings and minimum coverings will be constructed according to leaves and paddings, respectively. We begin with a lemma which is useful for the discussions to follow.

Lemma 5.1. *Suppose that p, q are nonnegative integers and s, t are positive integers with $\min\{s, t\} \geq 3$. If $\lambda K^*_{s,t}$ has a $\overrightarrow{C_6}$-packing (resp. $\overrightarrow{C_6}$-covering) with leave L (resp. padding P), then $\lambda K^*_{s+3p,t+3q}$ also has a $\overrightarrow{C_6}$-packing (resp. $\overrightarrow{C_6}$-covering) with leave L (resp. padding P).*

Proof. Note that

$$\lambda K^*_{s+3p,t+3q} = \begin{cases} \lambda K^*_{s,t} & \text{if } p = q = 0, \\ \lambda K^*_{s,t} \oplus \lambda K^*_{3p,t} & \text{if } p > 0,\ q = 0, \\ \lambda K^*_{s,t} \oplus \lambda K^*_{s,3q} & \text{if } p = 0,\ q > 0, \\ \lambda K^*_{s,t} \oplus \lambda K^*_{3p,t} \oplus \lambda K^*_{s,3q} \oplus \lambda K^*_{3p,3q} & \text{if } p > 0,\ q > 0. \end{cases}$$

For positive integers p and q, $\lambda K^*_{3p,t}$, $\lambda K^*_{s,3q}$ and $\lambda K^*_{3p,3q}$ have 6-circuit decompositions by Corollary 3.6. Then the result follows from Lemma 3.2. □

Lemma 5.2. *Suppose that m and n are positive integers with* $\min\{m,n\} \geq 3$ *and* $mn \equiv s \pmod 3$ *where* $s \in \{1,2\}$. *Let* $t \in \{1,2\}$.
(a) *There exists a* $\overrightarrow{C_6}$*-packing of* $tK^*_{m,n}$ *with leave* L *where*

$$\begin{cases} L = \overrightarrow{C_2} & \text{if } s=t=1 \text{ or } s=t=2, \\ L \in \{P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}\} & \text{if } s=2, t=1, \\ L \in \{P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } s=1, t=2. \end{cases}$$

Moreover, if $s=2$, *there exists a* $\overrightarrow{C_6}$*-packing* $4K^*_{m,n}$ *with leave* L *where* $L \in \{P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\}$.
(b) *There exists a* $\overrightarrow{C_6}$*-covering of* $tK^*_{m,n}$ *with padding* P *where*

$$\begin{cases} P \in \{P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } s=t=1 \text{ or } s=t=2, \\ P = \overrightarrow{C_2} & \text{if } s=1, t=2 \text{ or } s=2, t=1. \end{cases}$$

Proof. Let $r \in \{1,2\}$. Then $mn \equiv s \pmod 3$ if and only if $m \equiv r \pmod 3$ and $n \equiv (-1)^{s+1}r \pmod 3$. Let $m = a + 3p$ and $n = b + 3q$ where $a = b = 3 + r$ for $s = 1$ and $a = 3 + r, b = 6 - r$ for $s = 2$, and p and q are nonnegative integers. Then the result follows from Lemmas 4.2–4.4, 5.1, and the fact that $tK^*_{a,b}$ is isomorphic to $tK^*_{b,a}$. □

Now, we are ready for the main result.

Theorem 5.3. *Suppose that* λ, m *and* n *are positive integers with* $\min\{m,n\} \geq 3$.
(a) *There exists a maximum* $\overrightarrow{C_6}$*-packing of* $\lambda K^*_{m,n}$ *with leave* L *if and only if*

$$\begin{cases} L = \emptyset & \text{if } \lambda mn \equiv 0 \pmod 3, \\ L = \overrightarrow{C_2} & \text{if } \lambda mn \equiv 1 \pmod 3, \\ L \in \{P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}\} & \text{if } \lambda mn \equiv 2 \pmod 3 \text{ and } \lambda = 1, \\ L \in \{P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } \lambda mn \equiv 2 \pmod 3 \text{ and } \lambda \geq 2. \end{cases}$$

(b) *There exists a minimum* $\overrightarrow{C_6}$*-covering of* $\lambda K^*_{m,n}$ *with padding* P *if and only if*

$$\begin{cases} P = \emptyset & \text{if } \lambda mn \equiv 0 \pmod 3, \\ P \in \{P_3^*, \overrightarrow{C_4}, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, 2\overrightarrow{C_2}\} & \text{if } \lambda mn \equiv 1 \pmod 3, \\ P = \overrightarrow{C_2} & \text{if } \lambda mn \equiv 2 \pmod 3. \end{cases}$$

Proof. The necessity follows from the arguments about minimum possible leaves and paddings in Section 2. It is sufficient to show that $\lambda K^*_{m,n}$ has required packings and coverings. The result for $\lambda mn \equiv 0 \pmod 3$ follows from Corollary 3.6 immediately. So it remains to consider the following cases: $\lambda mn \equiv 1 \pmod 3$ and $\lambda mn \equiv 2 \pmod 3$.

Let $r \in \{1,2\}$. Then $\lambda mn \equiv 1 \pmod 3$ if and only if $\lambda \equiv mn \equiv r \pmod 3$, and $\lambda mn \equiv 2 \pmod 3$ if and only if $\lambda \equiv r \pmod 3$ and $mn \equiv 3-$

$r \pmod 3$. Note that $\lambda K^*_{m,n} = rK^*_{m,n} \oplus (\lambda - r)K^*_{m,n}$. Since $\overrightarrow{C_6}$ decomposes $(\lambda - r)K^*_{m,n}$ for $\lambda \equiv r \pmod 3$ by Corollary 3.6, the result follows from Lemmas 3.2 and 5.2. □

Acknowledgment
The author is grateful to the referee for the valuable comments.

References

[1] B. Alspach, H. Gavlas, M. Šajna, and H. Verrall, Cycle decompositions IV: complete directed graphs and fixed length directed cycles, J. Combin. Theory, Ser. A 103 (2003), 165–208.

[2] F. E. Bennett and J. Yin, Packings and coverings of the complete directed multigraph with 3- and 4-circuits, Discrete. Math. 162 (1996), 23–29.

[3] E. J. Billington, H.-L. Fu and C. A. Rodger, Packing complete multipartite graphs with 4-cycles, J. Combin. Des. 9 (2001), 107–127.

[4] E. J. Billington, H.-L. Fu and C. A. Rodger, Packing λ-fold complete multipartite graphs with 4-cycles, Graphs Combin. 21 (2005), 169–185.

[5] E. J. Billington and C. C. Lindner, Maximum packings of uniform group divisible triple systems, J. Combin. Des. 4 (1996), 397–404

[6] L. Brown, G. Coker, R. Gardner and J. Kennedy, Packing the complete bipartite graph with hexagons, Congr. Num. 174 (2005), 97–106.

[7] M. K. Fort, Jr. and G. A. Hedlund, Minimal coverings of pairs by triples, Pacific J. Math. 8 (1958), 709–719.

[8] H.-L. Fu and M.-H. Huang, Packing balanced complete multipartite graphs with hexagons, Ars Combin. 71 (2004), 49–64.

[9] D. G. Hoffman and W. D. Wallis, Packing complete graphs with squares, Bull. ICA 1 (1991), 89–92.

[10] M.-H. Huang, C.-M. Fu and H.-L. Fu, Packing 5-cycles into balanced complete m-partite graphs for odd m, J. Comb. Optim. 14 (2007), 323–329.

[11] J. A. Kennedy, Maximum packings of K_n with hexagons, Australas. J. Combin. 7 (1993), 101–110; Corrigendum: ibid 10 (1994), 293.

[12] J. A. Kennedy, Minimum coverings of K_n with hexagons, Australas. J. Combin. 16 (1997), 295–303.

[13] J.-J. Lin, Maximum packings of complete graphs with octagons, Ars Combin., to appear.

[14] C. Lin, J.-J. Lin and T.-W. Shyu, Isomorphic star decompositions of multicrowns and the power of cycles, Ars Combin. 53 (1999), 249–256.

[15] A. Rosa and S. Znám, Packing pentagons into complete graphs: how clumsy can you get?, Discrete Math. 128 (1994), 305–316.

[16] J. Schönheim, On maximal systems of k-tuples, Studia. Sci. Math. Hung. 1 (1966), 363–368.

[17] J. Schönheim and A. Bialostocki, Packing and covering of the complete graph with 4-cycles, Canad. Math. Bull. 18 (1975), 703–708.

[18] D. Sotteau, Decomposition of $K_{m,n}$ ($K^*_{m,n}$) into cycles (circuits) of length $2k$, J. Combin. Theory, Ser. B 30 (1981), 75–81.

[19] W.-C. Wu, H.-C. Lee and J.-J. Lin, Packing and covering complete bipartite multidigraphs with 4-circuits, Ling Tung J. 24 (2008), 223–231.