A Special Class of Antiautomorphisms of Directed Triple Systems

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Abstract

A transitive triple, (a,b,c), is defined to be the set $\{(a,b),(b,c),(a,c)\}$ of ordered pairs. A directed triple system of order v, DTS(v), is a pair (D,β), where D is a set of v points and β is a collection of transitive triples of pairwise distinct points of D such that any ordered pair of distinct points of D is contained in precisely one transitive triple of β. An antiautomorphism of a directed triple system, (D,β), is a permutation of D which maps β to β^{-1}, where $\beta^{-1}=\{(c,b,a)|(a,b,c)\in\beta\}$. In this paper we give necessary and sufficient conditions for the existence of a directed triple system of order v admitting an antiautomorphism consisting of two cycles of lengths M and $2M$, and one fixed point.

Keywords: antiautomorphism, bicyclic, directed triple system.

1 Introduction

A *Steiner triple system of order v*, STS(v), is a pair (S,β), where S is a set of v points and β is a collection of 3-element subsets of S, called *blocks*, such that any pair of distinct points of S is contained in precisely one block of β. Kirkman [9] showed that there is an STS(v) if and only if $v\equiv 1$ or 3 (mod 6) or $v=0$.

An *automorphism* of (S,β) is a permutation of S which maps β to itself. An automorphism, α, of (S,β) is called *cyclic* if the permutation defined by α consists of a single cycle of length v. Peltesohn [12] proved that an STS(v) having a cyclic automorphism exists if and only if $v\equiv 1$ or 3 (mod 6) and $v\neq 9$. An automorphism, α, of (S,β) is called *bicyclic* if the permutation defined by α consists of two cycles. Calahan-Zijlstra and Gardner [1] have shown that there exists an STS(v) admitting a bicyclic automorphism having cycles of length M and N, with $1 < M \leq N$, if and only if $M \equiv 1$ or 3 (mod 6), $M \neq 9$, $M|N$, and $M + N \equiv 1$ or 3 (mod 6).

A *transitive triple*, (a, b, c), is defined to be the set $\{(a, b), (b, c), (a, c)\}$ of ordered pairs. A *directed triple system of order v*, DTS(v), is a pair (D, β), where D is a set of v points and β is a collection of transitive triples of pairwise distinct points of D, called *triples*, such that any ordered pair of distinct points of D is contained in precisely one element of β. Hung and Mendelsohn [7] have shown that necessary and sufficient conditions for the existence of a DTS(v) are that $v \equiv 0$ or 1 (mod 3).

For a DTS(v), (D, β), we define β^{-1} by $\beta^{-1} = \{(c, b, a)|(a, b, c) \in \beta\}$. Then (D, β^{-1}) is a DTS(v) and is called the *converse* of (D, β). A DTS(v) which is isomorphic to its converse is said to be *self-converse*. Kang, Chang, and Yang [8] have shown that a self-converse DTS(v) exists if and only if $v \equiv 0$ or 1 (mod 3) and $v \neq 6$. An *automorphism* of (D, β) is a permutation of D which maps β to itself. A DTS(v) is called *cyclic* if there is an automorphism consisting of a single cycle of order v. Colbourn and Colbourn have shown that a cyclic DTS(v) exists if and only if $v \equiv 1$, 4, or 7 (mod 12) [6]. An *antiautomorphism* of (D, β) is a permutation of D which maps β to β^{-1}. Clearly, a DTS(v) is self-converse if and only if it admits an antiautomorphism.

An automorphism, α, on a DTS(v) is called *d-cyclic* if the permutation defined by α consists of a single cycle of length d and $v - d$ fixed points. Necessary and sufficient conditions for the existence of a DTS(v) admitting a d-cyclic automorphism have been given by Micale and Pennisi [11]. An automorphism, α, on a DTS(v) is called *f-bicyclic* if the permutation defined by α consists of two cycles and f fixed points. Micale and Pennisi [10] have given necessary and sufficient conditions for the existence of f-bicyclic directed triple systems.

An antiautomorphism, α, on a DTS(v) is called *d-cyclic* if the permutation defined by α consists of a single cycle of length d and $v - d$ fixed points. Necessary and sufficient conditions for the existence of a DTS(v) admitting a d-cyclic antiautomorphism have been given by Carnes, Dye, and Reed [2]. A *bicyclic* antiautomorphism of a DTS(v) is an antiautomorphism, α, which consists of two cycles of length M and N respectively, where $v = M + N$. Carnes, Dye, and Reed [3, 4, 5] have given necessary and sufficient conditions for a DTS(v) to admit a bicyclic antiautomorphism with cycles of length M and N, $1 < M \leq N$. We call an antiautomorphism, α, on a DTS(v) *f-bicyclic* if the permutation defined by α consists of two cycles and f fixed points. In this paper we consider 1-bicyclic antiautomorphisms with cycles of lengths M and $2M$.

2 Preliminaries

If K is the length of a cycle, $K \in \{M, 2M\}$, we let the cycles be $(0_i, 1_i, 2_i, \ldots, (K-1)_i)$, $i \in \{0,1\}$ and let ∞ be the fixed point. Let $\Delta = \{0, 1, 2, \ldots, (K-1)\}$. We shall use all additions modulo K in the triples. For $a_i, b_j, c_k \in D - \{\infty\}$, i, j, $k \in \{0,1\}$, $(a_i, b_j, c_k) \in \beta$, let the *orbit* of (a_i, b_j, c_k) be $\{((a+t)_i, (b+t)_j, (c+t)_k) | t \in \Delta, t \text{ even}\} \cup \{((c+t)_k, (b+t)_j, (a+t)_i) | t \in \Delta, t \text{ odd}\}$. For $a_i, b_j \in D - \{\infty\}$, i, $j \in \{0,1\}$, $(a_i, \infty, b_j) \in \beta$, let the *orbit* of (a_i, ∞, b_j) be $\{((a+t)_i, \infty, (b+t)_j) | t \in \Delta, t \text{ even}\} \cup \{((b+t)_j, \infty, (a+t)_i) | t \in \Delta, t \text{ odd}\}$ Clearly the orbits of the elements of β yield a partition of β.

We say that a collection of triples, $\bar{\beta}$, is a collection of *base triples* of a DTS(v) under α if the orbits of the triples of $\bar{\beta}$ produce β and exactly one triple of each orbit occurs in $\bar{\beta}$. Also, we say that the *reverse* of the transitive triple (a, b, c) is the transitive triple (c, b, a).

3 Necessary Conditions

Lemma 3.1 *If a DTS(v) admits a 1-bicyclic antiautomorphism with cycles of length M and $2M$, then $M \equiv 2 \pmod 6$.*

Proof: Assume M is odd. The ordered pair $(\infty, 0_0)$ must occur in a triple. If the third vertex in the triple is from the cycle of length M, the triple must be of the form $(\infty, 0_0, a_0)$, $(\infty, a_0, 0_0)$, or $(a_0, \infty, 0_0)$.

We first consider the triple $(\infty, 0_0, a_0)$. If a is even, then $\alpha^a((\infty, 0_0, a_0)) = (\infty, a_0, 2a_0)$, a contradiction, since the edge (∞, a_0) then occurs in two distinct triples. If a is odd, then $\alpha^{M-a}((\infty, 0_0, a_0)) = (\infty, (M-a)_0, 0_0)$, again leading to a contradiction.

Next we consider the triple $(\infty, a_0, 0_0)$. For a even, $\alpha^a((\infty, a_0, 0_0)) = (\infty, 2a_0, a_0)$, a contradiction. For a odd, $\alpha^{M-a}((\infty, a_0, 0_0)) = (\infty, 0_0, (M-a)_0)$, a contradiction.

Finally we consider the triple $(a_0, \infty, 0_0)$. For a even, $\alpha^{M-a}((a_0, \infty, 0_0)) = ((M-a)_0, \infty, 0_0)$, a contradiction. For a odd, $\alpha^a((a_0, \infty, 0_0)) = (a_0, \infty, (2a)_0)$, a contradiction.

Hence the third vertex in the triple must be from the cycle of length $2M$, so a triple of the form $(\infty, 0_0, b_1)$, $(\infty, b_1, 0_0)$, or $(b_1, \infty, 0_0)$ must occur. In each case, the orbit of the triple will include all of the ordered pairs of the form (∞, a_0) and (a_0, ∞), for $a \in \{0, 1, \ldots, (M-1)\}$. However, exactly half of the ordered pairs of the form (∞, b_1) and (b_1, ∞) for $b \in \{0, 1, \ldots, (2M-1)\}$ will be included.

We must have a triple of the form (∞, a_1, b_1), (a_1, ∞, b_1), or (a_1, b_1, ∞). In any of these cases, if the orbit of the triple is of length $2M$, there are $4M$ ordered pairs containing ∞ and b_1, a contradiction, since there are only $2M$ such ordered pairs left. Therefore, the orbit of the triple must have length

M. The only way for this to happen is to have $\{a, b\} = \{0, M\}$. Hence, the orbit of the triple will have length M. We note that all other triples will have orbits of length $2M$. So, there is exactly one orbit of length M. The number of directed edges is $(3M+1)(3M)$, making the number of triples $(3M+1)(M)$. Since M is odd, $3M+1$ is even, so there must be an even number of orbits of length M, which is impossible. Therefore, M is even.

For M even, if the ordered pair $(\infty, 0_0)$ is used in a triple which contains a vertex from the cycle of length $2M$, a contradiction is easily reached by considering the image of the triple under α^M. Hence the ordered pair $(\infty, 0_0)$ must be used in a triple of the form $(\infty, 0_0, a_0)$, $(\infty, a_0, 0_0)$, or $(a_0, \infty, 0_0)$. There are $M(M-1)$ directed edges in the cycle of length M. Exactly M of them will be used in the triples containing ∞. Also, no edge of the cycle of length M can be used in a triple containing a vertex from the cycle of length $2M$. There are then $M^2 - 2M$ edges left in the cycle of length M. Each triple uses three edges and has orbit of length M, so there will be $\frac{M^2-2M}{3M}$ orbits within the cycle of length M. Therefore $\frac{M^2-2M}{3M} = k$ for some integer k. Hence, $k = \frac{M}{3} - \frac{2}{3}$, so $M = 3k+2$. Since M is even, we have $M = 6t+2$, or $M \equiv 2 \pmod 6$. □

4 Sufficient Conditions

Lemma 4.1 *If $M \equiv 2 \pmod 6$, then there exists a directed triple system of order $v = 3M+1$ admitting a 1-bicyclic antiautomorphism with cycles of lengths M and $2M$.*

Proof: We consider the cases modulo 24. The collection of base triples for each case will be denoted by β.

Let $M = 24t+2$.

We define the following sets.

$\beta_1 = \{(0_1, \infty, (24t+2)_1)\}$.

$\beta_2 = \{(0_0, \infty, (12t+1)_0)\} \cup \{(0_0, s_1, (24t-s+1)_1) | s = 0, 1, \ldots, 12t\}$.

$\beta_3 = \{(0_0, (6t+1)_0, (10t)_0), (0_0, (5t)_0, (5t+1)_0), (0_1, (12t+2)_1, (20t)_1), (0_1, (10t)_1, (10t+2)_1)\} \cup \{(0_0, (8t+s)_0, (12t-s)_0) | s = 0, 1, \ldots, 2t-1\} \cup \{(0_1, (16t+2s)_1, (24t-2s)_1) | s = 0, 1, \ldots, 2t-1\}$.

$\beta_4 = \{(0_0, (6t)_0, (8t-1)_0), (0_1, (12t)_1, (16t-2)_1)\} \cup \{(0_0, (4t+s+1)_0, (8t-s-2)_0) | s = 0, 1, \ldots, t-2\} \cup \{(0_1, (8t+2s+2)_1, (16t-2s-4)_1) | s = 0, 1, \ldots, t-2\}$.

$\beta_5 = \{(0_0, (5t+s+2)_0, (7t-s-1)_0) | s = 0, 1, \ldots, t-3\} \cup \{(0_1, (10t+2s+4)_1, (14t-2s-2)_1) | s = 0, 1, \ldots, t-3\}$.

For $t = 0$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1}$.

For $t = 1$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_3 \cup \beta_3^{-1}$.

For $t = 2$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_3 \cup \beta_3^{-1} \cup \beta_4 \cup \beta_4^{-1}$.
For $t \geq 3$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_3 \cup \beta_3^{-1} \cup \beta_4 \cup \beta_4^{-1} \cup \beta_5 \cup \beta_5^{-1}$.

Let $M = 24t + 8$.

We define the following sets.

$\beta_1 = \{(0_0, \infty, (12t+4)_0,), (0_1, \infty, (24t+8)_1,)\}$.

$\beta_2 = \{(0_0, (6t+2)_0, (10t+3)_0), (0_1, (12t+4)_1, (20t+6)_1)\} \cup \{(0_0, s_1, (24t-s+7)_1) | s = 0, 1, \ldots, 12t+3\}$.

$\beta_3 = \{(0_0, 6_0, 7_0), (0_0, 9_0, 11_0), (0_0, 12_0, 15_0), (0_0, 10_0, 14_0), (0_1, 12_1, 14_1), (0_1, 18_1, 22_1), (0_1, 24_1, 30_1), (0_1, 20_1, 28_1)\}$.

$\beta_4 = \{(0_0, (8t+3)_0, (10t+2)_0), (0_0, (11t+2)_0, (11t+3)_0), (0_1, (16t+6)_1, (20t+4)_1), (0_1, (22t+4)_1, (22t+6)_1)\} \cup \{(0_0, (4t+s+2)_0, (8t-s+2)_0) | s = 0, 1, \ldots, 2t-1\} \cup \{(0_0, (8t+s+4)_0, (12t-s+3)_0) | s = 0, 1, \ldots, t-1\} \cup \{(0_1, (8t+2s+4)_1, (16t-2s+4)_1) | s = 0, 1, \ldots, 2t-1\} \cup \{(0_1, (16t+2s+8)_1, (24t-2s+6)_1) | s = 0, 1, \ldots, t-1\}$.

$\beta_5 = \{(0_0, (9t+s+4)_0, (11t-s+1)_0) | s = 0, 1, \ldots, t-3\} \cup \{(0_1, (18t+2s+8)_1, (22t-2s+2)_1) | s = 0, 1, \ldots, t-3\}$.

For $t = 0$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1}$.
For $t = 1$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_3 \cup \beta_3^{-1}$.
For $t = 2$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_4 \cup \beta_4^{-1}$.
For $t \geq 3$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_4 \cup \beta_4^{-1} \cup \beta_5 \cup \beta_5^{-1}$.

Let $M = 24t + 14$.

We define the following sets.

$\beta_1 = \{(0_1, \infty, (24t+14)_1)\}$.

$\beta_2 = \{(0_0, \infty, (12t+7)_0,)\} \cup \{(0_0, s_1, (24t-s+13)_1) | s = 0, 1, \ldots, 12t+6\}$.

$\beta_3 = \{(0_0, 1_0, 2_0), (0_0, 6_0, 4_0), (0_0, 11_0, 8_0), (0_0, 5_0, 10_0), (0_1, 4_1, 2_1), (0_1, 24_1, 8_1), (0_1, 16_1, 22_1), (0_1, 20_1, 10_1)\}$.

$\beta_4 = \{(0_0, (12t+5)_0, (24t+10)_0), (0_0, 4_0, 2_0), (0_1, (24t+18)_1, (24t+10)_1), (0_1, 8_1, 4_1)\}$.

$\beta_5 = \{(0_0, 10_0, 11_0), (0_0, 13_0, 16_0), (0_0, 9_0, 14_0), (0_0, 12_0, 18_0), (0_0, 8_0, 15_0), (0_1, 20_1, 22_1), (0_1, 26_1, 32_1), (0_1, 18_1, 28_1), (0_1, 24_1, 36_1), (0_1, 16_1, 30_1)\}$.

$\beta_6 = \{(0_0, (10t+5)_0, (10t+6)_0), (0_0, (6t+2)_0, (6t+5)_0), (0_0, (6t+4)_0, (8t+5)_0), (0_0, (6t+3)_0, (10t+3)_0), (0_0, (7t+4)_0, (11t+5)_0), (0_0, (8t+4)_0, (12t+6)_0), (0_0, (6t+1)_0, (10t+4)_0), (0_1, (20t+10)_1, (20t+12)_1), (0_1, (12t+4)_1, (12t+10)_1), (0_1, (12t+8)_1, (16t+10)_1), (0_1, (12t+6)_1, (20t+6)_1), (0_1, (14t+8)_1, (22t+10)_1), (0_1, (16t+8)_1, (24t+12)_1), (0_1, (12t+2)_1, (20t+8)_1)\} \cup \{(0_0, (4t+s+4)_0, (8t-s+3)_0) | s = 0, 1, \ldots, t-2\} \cup \{(0_0, (8t+s+6)_0, (12t-s+4)_0) | s = 0, 1, \ldots, t-2\} \cup \{(0_1, (8t+2s+8)_1, (16t-2s+6)_1) | s = 0, 1, \ldots, t-2\} \cup \{(0_1, (16t+2s+12)_1, (24t-2s+8)_1) | s = 0, 1, \ldots, t-2\}$.

$\beta_7 = \{(0_0, (5t+s+3)_0, (7t-s+3)_0) | s = 0, 1, \ldots, t-3\} \cup$
$\{(0_0, (9t+s+5)_0, (11t-s+4)_0) | s = 0, 1, \ldots, t-3\} \cup$
$\{(0_1, (10t+2s+6)_1, (14t-2s+6)_1) | s = 0, 1, \ldots, t-3\} \cup$
$\{(0_1, (18t+2s+10)_1, (22t-2s+8)_1) | s = 0, 1, \ldots, t-3\}$.

For $t = 0$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_3$.
For $t = 1$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_4 \cup \beta_5 \cup \beta_5^{-1}$.
For $t = 2$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_4 \cup \beta_6 \cup \beta_6^{-1}$.
For $t \geq 3$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_4 \cup \beta_6 \cup \beta_6^{-1} \cup \beta_7 \cup \beta_7^{-1}$.

Let $M = 24t + 20$.

We define the following sets.
$\beta_1 = \{(0_0, \infty, (12t+8)_0), (0_1, \infty, (24t+16)_1), (0_0, (6t+5)_0, (12t+10)_0),$
$(0_0, (6t+4)_0, (18t+16)_0), (0_1, (24t+20)_1, (12t+10)_1),$
$(0_1, (12t+8)_1, (36t+32)_1)\}$.
$\beta_2 = \{(0_0, s_1, (24t-s+19)_1) | s = 0, 1, \ldots, 12t+9\}$.
$\beta_3 = \{(0_0, 1_0, 2_0), (0_0, 3_0, 6_0), (0_0, 7_0, 14_0), (0_0, 9_0, 18_0), (0_1, 4_1, 2_1),$
$(0_1, 12_1, 6_1), (0_1, 28_1, 14_1), (0_1, 36_1, 18_1)\}$.
$\beta_4 = \{(0_0, (6t+3)_0, (10t+4)_0), (0_0, (11t+5)_0, (11t+6)_0),$
$(0_0, (10t+5)_0, (12t+6)_0), (0_0, (12t+7)_0, (12t+9)_0),$
$(0_1, (12t+6)_1, (20t+8)_1), (0_1, (22t+10)_1, (22t+12)_1),$
$(0_1, (20t+10)_1, (24t+12)_1), (0_1, (24t+14)_1, (24t+18)_1)\} \cup$
$\{(0_0, (4t+s+3)_0, (8t-s+5)_0) | s = 0, 1, \ldots, 2t-1\} \cup$
$\{(0_1, (8t+2s+6)_1, (16t-2s+10)_1) | s = 0, 1, \ldots, 2t-1\}$.
$\beta_5 = \{(0_0, (8t+s+6)_0, (12t-s+5)_0) | s = 0, 1, \ldots, t-2\} \cup$
$\{(0_0, (9t+s+5)_0, (11t-s+4)_0) | s = 0, 1, \ldots, t-2\} \cup$
$\{(0_1, (16t+2s+12)_1, (24t-2s+10)_1) | s = 0, 1, \ldots, t-2\} \cup$
$\{(0_1, (18t+2s+10)_1, (22t-2s+8)_1) | s = 0, 1, \ldots, t-2\}$.

For $t = 0$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_3$.
For $t = 1$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_4 \cup \beta_4^{-1}$.
For $t \geq 2$, $\beta = \beta_1 \cup \beta_2 \cup \beta_2^{-1} \cup \beta_4 \cup \beta_4^{-1} \cup \beta_5 \cup \beta_5^{-1}$.

□

5 Conclusion

By the lemmas in the previous sections, we have the following theorem.

Theorem 5.1 *There is a DTS(v) which admits a 1-bicyclic antiautomorphism with cycles of length M and 2M if and only if $M \equiv 2 \pmod 6$.*

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