FIBONACCI GRACEFUL GRAPHS

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Abstract

In this paper, we introduce two new types of labelings of graphs using Fibonacci numbers namely, Fibonacci graceful labelings and Super Fibonacci graceful labelings. We discuss the existence and non-existence of Fibonacci and Super Fibonacci graceful labelings for certain classes of graphs. Also we discuss the Fibonacci gracefulness of disjoint union of Super Fibonacci graceful graphs, pendant edge extension of Super Fibonacci graceful graphs and amalgamation of Super Fibonacci graceful graphs. Finally, we compare the graceful graphs with Fibonacci graceful graphs.

1.Introduction

Graph labelings, where the vertices are assigned values subject to certain conditions have often been motivated by practical problems.

Given a graph G (p,q), the set N of non-negative integers and a commutative binary operation $*$: N x N → N, every vertex function f: V (G) → N induces an edge function $\bar{f}$: E (G) → N such that $\bar{f}$ (uv) = f(u) $*$ f (v) for all uv ε E (G).

Graham and Sloane [5] defined a graph G of vertex set V (G) and Edge set E (G) with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label (f(x)+f(y)) (mod q), the resulting edge labels are distinct.

A generalization of harmonious labeling is felicitous labeling which was introduced by Lee, schmeichel, and Shee [9] as follows.

An injective function f from the vertices of a graph G with q edges to the set {0, 1, 2, … , q} is called felicitous if the edge labels induced by (f (x) + f(y)) (mod q) for each edge xy are distinct.

In 1981 Bange, Barkauskas and Slater [2] defined a graph G to be k-sequential if there is an injective function f: V(G) $\rightarrow$ {0,1,2,...,q-1} (q allowed if G is a tree) such that the induced edge labeling given by $\bar{f}$ (uv) = f(u) + f(v) has the property that the edge labels are k, k+1, k+2,..., k+q-1 for some integer k.

Rosa [10] introduced the concept of graceful labeling f of a (p, q) graph G as follows. f is a graceful labeling if f is an injection from V (G) to the set { 0, 1, 2, ... , q} such that, when each edge uv is assigned the label | f(u) – f(v) |, the resulting edge labels are distinct.

Acharya and Hegde [1] generalized graceful labeling to (k, d) graceful labeling by permitting the vertex labels to belong to { 0, 1, 2, ..., k + (q-1)d} and requiring the set of edge labels induced by the absolute difference of adjacent vertices to be {k, k+d, k+2d, ..., k+(q-1)d} where k and d are positive integers.

Kathiresan [7] introduced prime graceful labeling ϕ of the vertices of G with distinct integers from the set {0, 1, 2, ..., p_n} where p_n is the n^{th} prime, that is, p_1=2, p_2=3, p_3=5 etc., so that the induced edge labeling ϕ defined by $\bar{\phi}$ (uv)= | ϕ(u) - ϕ (v) | is a bijection onto the set { p_1, p_2, ..., p_n}.

In this paper, we are going to introduce two new types of labelings called Fibonacci graceful labeling and Super Fibonacci graceful labeling.

In labeling problems the induced labelings must be distinct. So to introduce Fibonacci graceful labelings we assume F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, ... as the sequence of Fibonacci numbers instead of 0, 1, 1, 2, 3, 5, ... [6].

Sethuraman and Elumalai [11] proved that certain pendant edge extensions of complete bipartite graphs are graceful. In this paper we prove that pendant edge extensions of Fibonacci graceful graphs are Fibonacci graceful. Also we prove that amalgamation and union of Super Fibonacci graceful graphs are Fibonacci graceful.

For further definitions and notations, we refer [3].

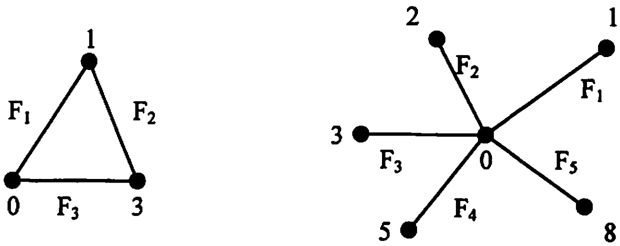
2. Fibonacci Graceful Graphs

In this section we introduce a new type of labeling namely Fibonacci graceful labeling.

2.1. Definition:

Let G (p, q) be a graph. The function f : V (G)$\rightarrow$ {0, 1, 2, ..., F_q} where F_q is the q^{th} Fibonacci number, is said to be Fibonacci graceful if the induced edge labeling $\bar{f}$ (uv) = | f(u) – f(v) | is a bijection onto the set { F_1, F_2, ..., F_q } If a graph G (p, q) admits a Fibonacci graceful labeling, then G is called Fibonacci graceful graph.

2.2. Examples:



Fibonacci graceful labelings for K_3 and $K_{1,5}$

2.3. Observation:

The edge numbered F_q must have 0 and F_q as the vertex numbers. Also any vertex adjacent to a vertex labeled 0 must be labeled with a Fibonacci number.

2.4. Observation:

If $\{ a_1(=0), a_2,\ldots,a_m (= F_q)\}$ is a set of vertex labels of a Fibonacci graceful graph, then changing each label a_i to $F_q - a_i$ also gives a Fibonacci graceful labeling of the graph.

2.5. Theorem: K_n is Fibonacci graceful if and only if $n\leq 3$.

Proof:

Let $\{v_0, v_1, \ldots v_{n-1}\}$ be the vertex set.

In K_n, each v_i is adjacent to all other vertices. Let v_0 be labeled 0. By 2.3, all vertices must be labeled with Fibonacci numbers only.

Let v_0 and v_1 be labeled as 0 and F_q respectively, then v_2 must be given F_{q-1} or F_{q-2} so that the edge v_1v_2 will receive a Fibonacci number F_{q-2} or F_{q-1} otherwise v_1v_2 will not get a Fibonacci number. Without loss of generality let v_2 be labeled F_{q-1} so that v_1v_2 will get F_{q-2}. If we label v_3 as F_{q-2} then v_0v_3 will get F_{q-2} which is already received by v_1v_2. If we label v_3 with some other Fibonacci number (other than F_q, F_{q-1}, F_{q-2}) the edge v_1 v_3 will not receive Fibonacci number.

2.6. Theorem: If G is eulerian and Fibonacci graceful then $q \equiv 0 \pmod 3$.

Proof:

Since G is eulerian, it can be decomposed into edge disjoint cycles.

We have the result [12] "Suppose that integers not necessarily distinct are assigned to the vertices of a graph G and that each edge of G is given an edge number equal to the absolute difference of the vertex numbers at its end points. Then the sum of the edge numbers around any circuit of G is even".

By this result, we must have $F_1 + F_2 + \dots + F_q$ is even.

By the nature of our Fibonacci sequence, this is possible only when q is a multiple of 3.

Thus $q \equiv 0 \pmod 3$

2.7. Remark: The above condition is only a necessary condition but not a sufficient condition since K_7 is eulerian but it is not Fibonacci graceful.

2.8. Theorem: Every path P_n of length n is Fibonacci graceful.

Proof:

Let P_n be a path of length n.

Let $\{v_0, v_1, \dots, v_n\}$ be the vertex set and $\{e_1, e_2, \dots, e_n\}$ be the edge set where

$e_i = v_{i-1} v_i$ for $i = 1, 2, \dots, n$

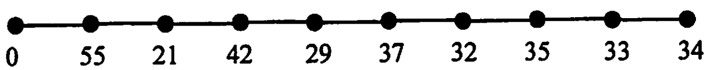
Define $f : V(P_n) \to \{0, 1, 2, \dots, F_n\}$ by

$f(v_0) = 0,$

$f(v_i) = F_n - F_{n-1} + F_{n-2} - \dots + (-1)^{i-1} F_{n-(i-1)}$ for $i = 1, 2, \dots, n$

Then we can easily verify that f is a Fibonacci graceful labeling.

The following figure shows the Fibonacci graceful labeling of P_9:



2.9. Theorem: P_n^2 is a Fibonacci graceful graph where P_n^k is the k^{th} power of P_n.

Proof:

Let $\{v_0, v_1, \dots, v_n\}$ be the vertex set and

$\{v_{i-1} v_i / i = 1,2,\dots,n\} \cup \{v_{i-2} v_i / i=2,3,\dots,n\}$ be the edge set of P_n^2, so that $|E(P_n^2)| = 2n-1$.

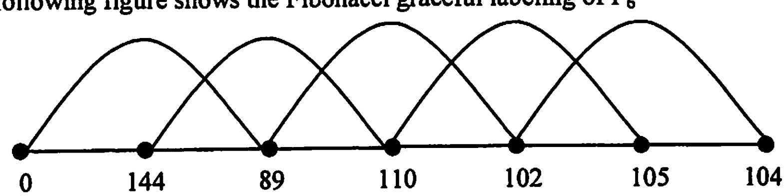
Define $f : V \to \{0, 1, 2, \dots, F_{2n-1}\}$ by

$f(v_0) = 0$

$f(v_i) = F_q - F_{q-2} + F_{q-4} - \dots + (-1)^{i-1} F_{q-2(i-1)}$ for $i=1, 2, \dots, n$ where $q = 2n-1$

Then f is a Fibonacci graceful labeling.

The following figure shows the Fibonacci graceful labeling of P_6^2



2.10. Theorem: Caterpillars are Fibonacci graceful.

Proof:

Let G be a caterpillar. Choose a longest path in G and call its vertices $v_0, v_1, v_2, \ldots, v_k$

Let $v_{i1}, v_{i2}, \ldots, v_{ij}$ be the end points adjacent to v_i, $i=1, 2, \ldots, k-1$.
Let n be the number of edges of G.
Define f on the vertices of G as follows:

$f(v_0) = 0$
$f(v_1) = F_n$
$f(v_{11}) = F_n - F_{n-1}$
$f(v_{12}) = F_n - F_{n-2}$
.
.
.
$f(v_{ij}) = F_n - F_{n-j}$
$f(v_2) = f(v_1) - F_{n-j-1}$
$f(v_{21}) = f(v_2) + F_{n-j-2}$
$f(v_{22}) = f(v_2) + F_{n-j-3}$
.
.
.
$f(v_{2r}) = f(v_2) + F_{n-j-r-1}$
$f(v_3) = f(v_2) + F_{n-j-r-2}$
$f(v_{31}) = f(v_3) - F_{n-j-r-3}$
.
.
.
$f(v_{3s}) = f(v_3) - F_{n-j-r-s-2}$
$f(v_4) = f(v_3) - F_{n-j-r-s-3}$
....

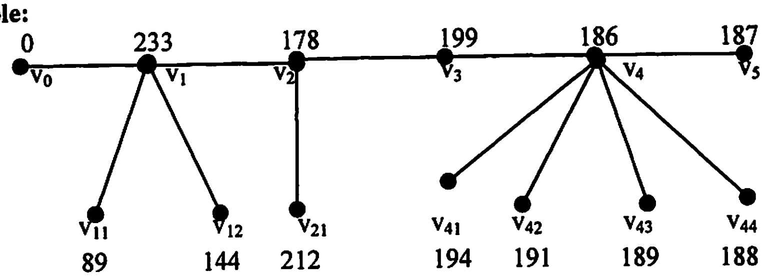
Proceed to label the neighbours of v_4 by adding the appropriate sequence of Fibonacci numbers from $f(v_4)$.

Continue in this manner, adding the appropriate sequence of Fibonacci number to $f(v_4)$, $f(v_6)$, ... to get the label for neighbours of v_4, v_6, ... and subtracting the suitable Fibonacci numbers from $f(v_5)$, $f(v_7)$, ... to get the label for neighbours of v_5, v_7, ...

By this construction, the edges of G receive the values $F_1, F_2, \ldots, F_n$.

Hence the caterpillars are Fibonacci graceful.

Example:



2.11. Corollary: The bistar $B_{m,n}$ is Fibonacci graceful.

3. Super Fibonacci graceful

In Fibonacci graceful labeling we assign the integers from $[0, F_q]$ to the vertices. The vertex number need not be a Fibonacci number. In this section we assign only Fibonacci numbers together with 0 to the vertices. Hence we have the following definition.

3.1. Definition:

Let G (p, q) be a graph. The function $f : V(G) \rightarrow \{0, F_1, F_2, \ldots, F_q\}$ where F_q is the q^{th} Fibonacci number, is said to be a Super Fibonacci graceful labeling if the induced edge labeling $\bar{f}(uv) = | f(u) - f(v) |$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$.

3.2 Result: The bistar $B_{n,n}$ is Fibonacci graceful but not Super Fibonacci graceful for $n \geq 5$.

From the definition of Fibonacci numbers we observe that for a given n,

$| F_n - F_m |$ is a Fibonacci number for atmost 4 values of m. ------------------(1)

In $B_{n,n}$ there are two center vertices and 2n pendant vertices. When we assign numbers to the vertices, 0 can be assigned either to one of the center vertex or to the end vertex. Wherever it may be, atleast one center vertex must receive only Fibonacci number. From the center we have atleast 5 pendant vertices (because $n \geq 5$). By (1) it is not possible to assign Fibonacci numbers to all the pendant vertices.

3.3 Theorem: C_n is Super Fibonacci graceful if and only if $n \equiv 0 \pmod 3$

Proof:

By theorem 2.6, If C_n is Super Fibonacci graceful then $n \equiv 0 \pmod 3$.

Suppose $n \equiv 0 \pmod 3$ and $n = 3m$.

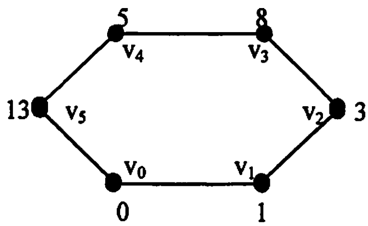
Let $\{v_0, v_1, \ldots, v_{3m-1}\}$ be the vertex set and $\{e_1, \ldots, e_{3m}\}$ be the edge set of C_{3m}, where $e_i = v_{i-1} v_i$ for $i = 1, 2, \ldots, 3m-1$ and $e_{3m} = v_{3m-1} v_0$.

Define $f : V \rightarrow \{0, 1, 2, \ldots, F_{3m}\}$ by
$f(v_0) = 0$

$$f(v_i) = \begin{cases} F_i \text{ if } i \equiv 1 \pmod 3 \\ F_{i+1} \text{ if } i \equiv 2 \pmod 3 \\ F_{i+2} \text{ if } i \equiv 0 \pmod 3 \end{cases}$$

We can easily verify that f is a Super Fibonacci graceful labeling.

Example:



3.4. Theorem: Every fan f_n is a Super Fibonacci graceful graph.
Proof:
Let f_n be a fan with vertex set $\{v_0, v_1, \ldots, v_n\}$ and edge set $\{ v_0 v_i, v_{i-1} v_i /$ $i=1, 2, \ldots, n \}$ so that

$| V(f_n) | = n+1$ and $| E(f_n) | = 2n-1$.

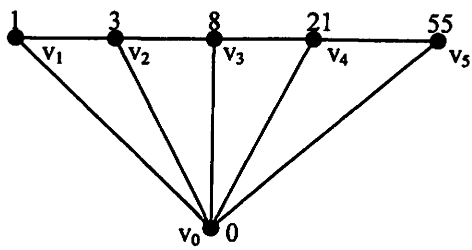
Define: $f : V(f_n) \rightarrow \{0, F_1, F_2, \ldots, F_{2n-1}\}$ by

$f(v_i) = F_{2i-1}$ for $i = 1, 2, \ldots, n$

$f(v_0) = 0$

It is a routine matter to verify that f is a Super Fibonacci graceful labeling of f_n

Example:



3.5. Observation:

Obviously, stars $K_{1,n}$ are Super Fibonacci graceful.

If we give 0 to the central vertex and Fibonacci numbers $F_1, F_2, \ldots, F_n$ to the end vertices, then we get a Super Fibonacci graceful labeling for $K_{1,n}$.

4. Further results on Fibonacci (Super) graceful graphs

In this section we form new families of Fibonacci (Super) graceful graphs from the known families of Fibonacci (Super) graceful graphs.

4.1. Definition:

Pendant edge extension G* of a graph G (p, q) is obtained from the graph G by adjoining a pendant edge to each vertex of G. So G* is a (2p, p+q) graph.

4.2. Definition:

A graph G with a fixed vertex u will be denoted by the ordered pair (G, u). Given two pairs (G, u) and (H, v), one can form a new graph by amalgamation, from the disjoint union of G and H and identifying u and v. The resulting graph will be denoted by (G, u) o (H, v). [8]

4.3. Theorem: If G is Fibonacci or Super Fibonacci graceful then its pendant edge extension G* is Fibonacci graceful.

Proof:

Let G (p, q) be a Fibonacci or Super Fibonacci graceful graph with respect to the labeling f.

Let the vertex set of G be $\{v_0, v_1, \ldots, v_{p-1}\}$ where $f(v_0) < f(v_1) < \ldots < f(vp-1)$ and let the vertex set of G* be $\{v_0, v_1, v_2, \ldots, v_{p-1}, u_0, u_1, \ldots, u_{p-1}\}$ where $u_0, u_1, \ldots, u_{p-1}$ are pendant vertices joined respectively with $v_0, v_1, \ldots, v_{p-1}$.

Define $\varphi : G^* \rightarrow \{0, 1, 2, \ldots, F_{p+q}\}$ by

$\varphi(v_i) = f(v_i)$

$\varphi(u_i) = f(v_i) + F_{p+q-i}$ for $i=0,1,2,\ldots,p-1$

Clearly φ is a Fibonacci graceful labeling.

4.4. Theorem: If G_1 and G_2 are Super Fibonacci graceful in which no two adjacent vertices have the labeling 1 and 2, then their union $G_1 \cup G_2$ is Fibonacci graceful.

Proof:

Let G_1 (p_1, q_1) and G_2 (p_2, q_2) be two Super Fibonacci graceful graphs with respect to the labelings f_1 and f_2 respectively.

Let $\{v_0, v_1, v_2, \ldots, V_{p_1-1}, u_0, u_1, \ldots, u_{p_2-1}\}$ be the vertex set of $G_1 \cup G_2$.

Define $\varphi : V(G_1 \cup G_2) \rightarrow \{0,1,2,\ldots, F_{q_1+q_2}\}$ by

$\varphi(v_i) = f_1(v_i) + 1$ for $i=0,1,2,\ldots,p_1-1$ and

$$\varphi(u_i) = \begin{cases} 0 \text{ if } f_2(u_i) = 0 \\ F_{r_i+q_1} \text{ if } f_2(ui) = F_{r_i} \end{cases}$$

so that the edges of G_1 will receive $F_1, F_2, \ldots, F_{q_1}$ and edges of G_2 will receive $F_{q_1+1}, F_{q_1+2}, \ldots, F_{q_1+q_2}$. Then φ is a Fibonacci graceful labeling.

4.5. Theorem: Amalgamation of Super Fibonacci graceful graphs $G_1(p_1, q_1)$, $G_2(p_2, q_2), \ldots, G_n(p_n, q_n)$ with the functions $f_1, f_2, \ldots, f_n$ by identifying the vertices with 0 label, is also Super Fibonacci graceful where in any G_i, no two adjacent vertices have the labeling 1 and 2.

Proof:

Let u_0 be the central vertex, $u_{i_1}, u_{i_2}, \ldots, u_{i_{p_i-1}}$ be the vertices of $G_i(p_i, q_i)$ for $i=0,1,2,\ldots,n$

Define $\varphi : V \rightarrow \{0, F_1, F_2, \ldots, F_{q_1+q_2+\ldots+q_n}\}$ by

$\varphi(u_0) = 0$

$\varphi(u_{ij}) = F_{r_j^i+q_1+q_2+\ldots+q_{i-1}}$ for $i=0,1,2,\ldots,n$, $j = 1, 2, \ldots, p_i-1$

where $F_{r_j^i} = f_i(u_{ij})$

Then φ is a Super Fibonacci graceful labeling.

5. Comparison of graceful and Fibonacci graceful graphs

In this section we compare the concept of graceful and Fibonacci graceful graphs with the help of examples.

1. C_3 is both graceful [4] and Fibonacci graceful (Theorem 3.3).
2. K_4 is graceful [12] but not Fibonacci graceful (Theorem 2.5).
3. C_6 is Fibonacci graceful (Theorem 3.3) but not graceful [4].
4. K_n $(n \geq 5)$ is neither Fibonacci graceful (Theorem 2.5) nor graceful [12].

Thus we conclude that the graceful labeling and Fibonacci graceful

Acknowledgement

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