Hyperstructures associated to arithmetic functions

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Abstract. In this paper, we introduce a hyperoperation associated to the set of all arithmetic functions and analyze the properties of this new hyperoperation. Several characterization theorems are obtained, especially in connection with multiplicative functions.

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1. Introduction

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty [7]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Hundreds of papers and several books have been written on this topic, see [3, 5, 6, 9]. A recent book on hyperstructures [5] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [6] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

Given a nonempty set H, a fuzzy subset of H (or a fuzzy set in H) is, by definition, an arbitrary mapping $\mu: H \longrightarrow [0,1]$ where [0,1] is the usual closed interval of real numbers. This important concept of a fuzzy sets has been introduced by Zadeh in [10]. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications (cf., for example, [2, 4, 5]).

The theory of numbers is one of the oldest branches of mathematics that many researchers have studied and developed it. Now, in this paper, we introduce a hyperoperation associated to the set of all arithmetic functions and analyze the properties of this new hyperoperation. Several characterization theorems are obtained, especially in connection with multiplicative functions. Finally, we introduce a property of fuzzy sets with arithmetic functions and hyperstructures.

2. Preliminaries

Let H be a nonempty set and let $\wp^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \wp^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*.

If A and B are nonempty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, we have $x \circ H = H \circ x = H$.

An element $e \in H$ is called an *identity* or *unit*, if $x \in x \circ e \cap e \circ x$, for all $x \in H$.

A hypergroupoid $(H, \circ)$ is called a *quasihypergroup*, if $x \circ H = H = H \circ x$, for all $x \in H$.

An element $x' \in H$ is called an *inverse* of $x \in H$, if there exists an identity $e \in H$, such that $e \in x \circ x' \cap x' \circ x$.

A hypergroup $(H, \circ)$ is called *canonical* [8] if the following conditions are satisfied:

((i) $x \circ y = y \circ x$, for all $x, y \in H$;
(ii) there exists $e \in H$ (unique) such that $e \circ x = x = x \circ e$, for all $x \in H$;
(iii) every element has a unique inverse;
(iv) it is reversible, that is, for all $x, y, z \in H$, $z \in x \circ y$ implies $x \in z \circ y'$ and $y \in z \circ x'$.

For any a and b of H, we denote the set $\{x \in H \mid a \in x \circ b\}$ by a/b.

A commutative hypergroup $(H, \circ)$ is called a *join space* if the following condition holds:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset.$$

A commutative hypergroup $(H, \circ)$ is canonical if and only if it is a join space with a scalar identity.

3. Arithmetic functions and hyperstructures

Any function $f : \mathbb{N}^* \longrightarrow \mathbb{R}$ whose domain of definition is the set of positive integers is said to be an *arithmetic* (or *number-theoretic) function.* Let m and n be two positive integers. An arithmetic function f is said to be *multiplicative* if

$$f(mn) = f(m)f(n)$$

whenever $gcd(m, n) = 1$.

Example 3.1. [1] Some important arithmetic functions are

(1) The classical Mobius function μ is an important multiplicative function in number theory and combinatorics. The definition is as follows:
$\mu(1) = 1$ and for $n > 1$ we write $n = p_1^{a_1} \dots p_k^{a_k}$, in this case

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \dots = a_k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(2) (Euler's Phi-Function): If $n \geq 1$ then $\phi(n)$ is the number of primes less than or equal to n.

(3) $\tau(n)$, the number of positive divisors of n, including 1 and n. This arithmetic function is multiplicative

Denote by $MF(G)$ the set of all multiplicative functions of G.

Now, we define a hyperoperation on the collection of all arithmetic functions. We will use a product notation to indicate this hyperoperation.

Definition 3.2. Let G be the set of all arithmetic functions, we can define a hyperstructure on G, as follows:

$$\circ : G \times G \longrightarrow \wp^*(G)$$

$$(\alpha, \beta) \longmapsto \alpha \circ \beta$$

such that

$$(\alpha \circ \beta)(n) = \left\{ \alpha(d)\beta(\frac{n}{d}) : \ d|n \right\} = \bigcup_{d|n} \alpha(d)\beta(\frac{n}{d})$$

for all $(\alpha, \beta) \in G^2$ and d runs through the positive divisor of n.

Example 3.3. We have

$$(\phi \circ \tau)(1) = \{\phi(1)\tau(1)\} = \{1\} = 1,$$
$$(\phi \circ \tau)(2) = \{\phi(1)\tau(2), \phi(2)\tau(1)\} = \{1, 2\},$$
$$(\phi \circ \tau)(3) = \{\phi(1)\tau(3), \phi(3)\tau(1)\} = \{2\} = 2.$$

If p be a prime number, we have

$$(\phi \circ \tau)(p) = \{\phi(1)\tau(p), \phi(p)\tau(1)\} = \{2, p-1\}.$$

The use of this hyperoperation leads to a convenient calculus of arithmetic functions. We will establish several useful properties of multiplicative function that will enable us to manipulate these functions with comparative ease.

We will next prove that the arithmetic functions with above hyperoperation are commutative and associative.

Proposition 3.4. *Let G be the set of all arithmetic functions and $(\alpha, \beta) \in G^2$. Then $\alpha \circ \beta = \beta \circ \alpha$.*

Proof. Let n be a positive integer. As d ranges over the divisors of n, so does $d|n$. Let $d_1 = \frac{n}{d}$. Then

$$\begin{aligned}(\alpha \circ \beta)(n) &= \bigcup_{d|n} \alpha(d)\beta(\frac{n}{d}) \\ &= \bigcup_{d_1|n} \beta(d_1)\alpha(\frac{n}{d_1}) \\ &= (\beta \circ \alpha)(n),\end{aligned}$$

for each positive integer n, so the functions $\alpha \circ \beta$ and $\beta \circ \alpha$ are equal. □

Theorem 3.5. *Let G be the set of all arithmetic functions. Then $(G, \circ)$ is a semihypergroup.*

Proof. We must show that if n is a positive integer, then $(\alpha \circ (\beta \circ \gamma))(n) = ((\alpha \circ \beta) \circ \gamma)(n)$, for all $\alpha, \beta, \gamma \in G$. If we consider the first of these expressions, we see that

$$\begin{aligned}(\alpha \circ (\beta \circ \gamma))(n) &= \bigcup_{d|n} \alpha(d)(\beta \circ \gamma)(\frac{n}{d}) \\ &= \bigcup_{d|n} \alpha(d)(\bigcup_{m|\frac{n}{d}} \beta(m)\gamma(\frac{n}{md})) \\ &= \bigcup_{d|n} \bigcup_{m|\frac{n}{d}} \alpha(d)\beta(m)\gamma(\frac{n}{md})\end{aligned}$$

This double set indicates that we consider all positive integers m and d such that $m|\frac{n}{d}$; that is, such that $md|n$. Thus, we can replace the double set by a single set and obtain

$$(\alpha \circ (\beta \circ \gamma))(n) = \bigcup_{md|n} \alpha(d)\beta(m)\gamma(\frac{n}{md})$$

We will now consider the expression $((\alpha\circ\beta)\circ\gamma)(n)$. We write

$$\begin{aligned}((\alpha\circ\beta)\circ\gamma)(n) &= \bigcup_{d|n}(\alpha\circ\beta)(d)\gamma(\frac{n}{d})\\ &= \bigcup_{d|n}(\bigcup_{s|d}\alpha(s)\beta(\frac{d}{s}))\gamma(\frac{n}{d})\\ &= \bigcup_{d|n}\bigcup_{s|d}\alpha(s)\beta(\frac{d}{s})\gamma(\frac{n}{d}).\end{aligned}$$

Since $s|d$, then exists a positive integer t such that $d = st$. Thus, the above set is equal to the single set

$$((\alpha\circ\beta)\circ\gamma)(n) = \bigcup_{st|n}\alpha(s)\beta(t)\gamma(\frac{n}{st}).$$

Then, $\alpha\circ(\beta\circ\gamma)$ and $(\alpha\circ\beta)\circ\gamma$ have the same value for each positive integer n, the two functions are equal. □

Instead of $\alpha\circ(\beta\circ\gamma)$ or $(\alpha\circ\beta)\circ\gamma$, we will write $\alpha\circ\beta\circ\gamma$.

Example 3.6. Let $\sigma(n)$ denote the sum of positive divisors of positive integer n, i.e., $\sigma(n) = \sum_{d|n} d$. Then

$$\begin{aligned}(\phi\circ\sigma\circ\tau)(8) &= \{\phi(1)\sigma(8)\tau(1), \phi(1)\sigma(4)\tau(2), \phi(2)\sigma(4)\tau(1), \phi(1)\sigma(2)\tau(4),\\ &\quad \phi(2)\sigma(2)\tau(2), \phi(4)\sigma(2)\tau(1), \phi(8)\sigma(1)\tau(1), \phi(4)\sigma(1)\tau(2),\\ &\quad \phi(2)\sigma(1)\tau(4), \phi(1)\sigma(1)\tau(8)\}\\ &= \{15, 14, 12, 7, 6, 4\}.\end{aligned}$$

If p is a prime number, we have

$$(\phi\circ\sigma\circ\tau)(p) = \{\phi(p)\sigma(1)\tau(1), \phi(1)\sigma(p)\tau(1), \phi(1)\sigma(1)\tau(p)\} = \{p-1, p+1, 2\}.$$

Definition 3.7. Let G be the set of all arithmetic functions. We can define a map '$\bullet$' on $G\circ G$, as follows:

$$\bullet : (G\circ G)\times(G\circ G) \longrightarrow \wp^*(G)$$

$$((\alpha_1\circ\beta_1), (\alpha_2\circ\beta_2)) \longmapsto (\alpha_1\circ\beta_1)\bullet(\alpha_2\circ\beta_2)$$

such that

$$((\alpha_1\circ\beta_1)\bullet(\alpha_2\circ\beta_2))(m,n) = \bigcup_{\alpha\in(\alpha_1\circ\beta_1)(m),\ \beta\in(\alpha_2\circ\beta_2)(n)} \alpha\beta,$$

where $m, n \in \mathbb{N}^*$.

In view of the above definition, it is not difficult to prove the following corollary.

Corollary 3.8. *Let G be the set of all arithmetic functions. Then*
(i) $((\alpha_1 \circ \beta_1) \bullet (\alpha_2 \circ \beta_2))(m,n) = ((\alpha_2 \circ \beta_2) \bullet (\alpha_1 \circ \beta_1))(n,m)$, *for all* $\alpha_1, \beta_1, \alpha_2, \beta_2 \in G$, *and for all* $m, n \in \mathbb{N}^*$.
(ii) $(G \circ G, \bullet)$ *is associative.*

Definition 3.9. Let $\alpha, \beta \in G$. Then $\alpha \circ \beta$ is called a multiplicative function in $G \circ G$ if the following condition holds:

$$(\alpha \circ \beta)(mn) = (\alpha \circ \beta)(m) \bullet (\alpha \circ \beta)(n),$$

whenever $gcd(m,n) = 1$.
Denote by $MF(G \circ G)$ the set of all multiplicative functions in $G \circ G$.

Example 3.10. If m, n are two distinct positive primes, then by Example 3.3, we have

$$\begin{aligned}(\phi \circ \tau)(m) \bullet (\phi \circ \tau)(n) &= \{2, m-1\} \bullet \{2, n-1\} \\ &= \{4, 2(n-1), 2(m-1), (m-1)(n-1)\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\phi \circ \tau)(mn) &= \{\phi(1)\tau(mn), \phi(n)\tau(m), \phi(m)\tau(n), \phi(mn)\tau(1)\} \\ &= \{4, 2(n-1), 2(m-1), (m-1)(n-1)\}.\end{aligned}$$

Thus $(\phi \circ \tau)(mn) = (\phi \circ \tau)(m) \bullet (\phi \circ \tau)(n)$, which implies $(\phi \circ \tau) \in MF(G \circ G)$.

Lemma 3.11. *Let $\alpha, \beta \in G$ be two arithmetic functions and m, n be two positive integers. Then*

$$\bigcup_{d|m, D|n} \alpha(d)\beta(D) = \bigcup_{d|m} \alpha(d) \bullet \bigcup_{D|n} \beta(D).$$

Proof. Let $d_1, ..., d_s$ and $D_1, ..., D_t$ be the positive divisors of m and n, respectively. Then

$$\begin{aligned}\bigcup_{d|m, D|n} \alpha(d)\beta(D) &= \bigcup_{1 \leq i \leq s,\ 1 \leq j \leq t} \alpha(d_i)\beta(D_j) \\ &= \{\alpha(d_1)\beta(D_1), ..., \alpha(d_1)\beta(D_t), ..., \alpha(d_s)\beta(D_1), ..., \alpha(d_s)\beta(D_t)\} \\ &= \{\alpha(d_1), ..., \alpha(d_s)\} \bullet \{\beta(D_1), ..., \beta(D_t)\} \\ &= \bigcup_{d|m} \alpha(d) \bullet \bigcup_{D|n} \beta(D)\end{aligned}$$

□

Theorem 3.12. *Let $\alpha, \beta \in MF(G)$. Then $\alpha \circ \beta \in MF(G \circ G)$.*

Proof. Let m and n be relatively prime positive integers. We recall that any divisor d of mn can be uniquely written as $d = st$, where $s|m$, $t|n$, and $gcd(s,t) = 1$, hence we have $(s,t) = (\frac{m}{s}, \frac{n}{t}) = 1$. Thus,

$$\begin{aligned}(\alpha \circ \beta)(mn) &= \bigcup_{d|mn} \alpha(d)\beta(\frac{mn}{d}) \\ &= \bigcup_{s|m,t|n} \alpha(st)\beta(\frac{m}{s}.\frac{n}{t}).\end{aligned}$$

Since $\alpha, \beta \in MF(G)$, thus $\alpha(st) = \alpha(s)\alpha(t)$, $\beta(\frac{m}{s}.\frac{n}{t}) = \beta(\frac{m}{s})\beta(\frac{n}{t})$, and the above set is equal to

$$\bigcup_{s|m,t|n} \alpha(s)\beta(\frac{m}{s})\alpha(t)\beta(\frac{n}{t})$$

which, in turn, by Lemma 3.11, is equal to

$$\bigcup_{s|m} \alpha(s)\beta(\frac{m}{s}) \bullet \bigcup_{t|n} \alpha(t)\beta(\frac{n}{t})$$

Therefore,

$$\begin{aligned}(\alpha \circ \beta)(mn) &= \bigcup_{s|m,t|n} \alpha(st)\beta(\frac{m}{s}.\frac{n}{t}) \\ &= \bigcup_{s|m,t|n} \alpha(s)\beta(\frac{m}{s})\alpha(t)\beta(\frac{n}{t}) \\ &= \bigcup_{s|m} \alpha(s)\beta(\frac{m}{s}) \bullet \bigcup_{t|n} \alpha(t)\beta(\frac{n}{t}) \\ &= (\alpha \circ \beta)(m) \bullet (\alpha \circ \beta)(n)\end{aligned}$$

Then $\alpha \circ \beta \in MF(G \circ G)$. □

The following theorem establishes a connection between certain sets involving $\alpha \circ \beta$ and sets involving α and β.

Theorem 3.13. *Let α and β be arithmetic functions. Then*

$$\bigcup_{m=1}^{n} (\alpha \circ \beta)(m) = \bigcup_{d=1}^{n} \bigcup_{k=1}^{[\frac{n}{d}]} \alpha(d)\beta(k).$$

Proof. By definition, we have

$$\bigcup_{m=1}^{n} (\alpha \circ \beta)(m) = \bigcup_{m=1}^{n} \bigcup_{d|m} \alpha(d)\beta(\frac{m}{d}).$$

Each integer d in the second set is in the range from 1 to n, and each integer in that range occurs as such a d at least once. If we fix d, we will obtain a term $\alpha(d)\beta(\frac{m}{d})$ for every multiple m of d, $1 \leq m \leq n$; that is, a set

$$\left\{\alpha(d)\beta(1), \alpha(d)\beta(\frac{2d}{d}), \ldots, \alpha(d)\beta(\frac{kd}{d})\right\},$$

where $k = [\frac{n}{d}]$ and $[\frac{n}{d}]$ is the *integral part* of $\frac{n}{d}$. If we take the set over all such d, and, for fixed d, for every integer between 1 and $k = [\frac{n}{d}]$, we obtain the desired result. □

4. Inverse functions under hyperstructures $(G, \circ)$

In this section, we denote by G the set of all arithmetic functions.

Notation. Let $\alpha, \beta, \gamma \in G$, then we say $\alpha = \beta$ (respectively, $\alpha \in \beta \circ \gamma$), if $\alpha(n) = \beta(n)$ (respectively, $\alpha(n) \in (\beta \circ \gamma)(n)$), for all positive integer n.

Proposition 4.1. *For a positive integer n, we define ι by the rules:*

$$\iota(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Then $\alpha \in \alpha \circ \iota$ for all $\alpha \in G$.

Proof. For any positive integer n, we have

$$(\alpha \circ \iota)(n) = \bigcup_{d|n} \alpha(d)\iota(\frac{n}{d}) = \{\alpha(n)\iota(1), 0\} = \{\alpha(n), 0\},$$

which implies that $\alpha(n) \in (\alpha \circ \iota)(n)$. Therefore $\alpha \in \alpha \circ \iota$. □

The above proposition shows that ι is to the role played by the number 1 for hypergroupoid $(G, \circ)$.

Definition 4.2. An element $\iota \in G$ is called an *identity* or *unit* of $(G, \circ)$, such that

$$(\alpha \circ \iota)(n) = \{\alpha(n), 0\},$$

for any positive integer n.

We are thus led to the following definition:

Definition 4.3. An element $\alpha^{-1} \in G$ is called an *inverse* of $\alpha \in G$, if

$$\iota = \alpha \circ \alpha^{-1}.$$

Proposition 4.4. *If α has an inverse, it is unique.*

Proof. Let β and γ be two inverse for α. For any positive integer n, we have:

$$\{\beta(n), 0\} = (\beta \circ \iota)(n) = [\beta \circ (\alpha \circ \gamma)](n) = [(\beta \circ \alpha) \circ \gamma](n) = (\iota \circ \gamma)(n) = \{\gamma(n), 0\},$$

which implies that $\beta(n) = \gamma(n)$. Thus $\beta = \gamma$. □

Proposition 4.5. *Let $\alpha \in G$ and $\alpha(1) \neq 0, \alpha(n) = 0$ for all positive integer $n > 1$. Then*

$$\beta(n) = \begin{cases} \frac{1}{\alpha(1)} & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

is the inverse of α.

Proof. For all positive integer $n > 1$, we have

$$(\alpha \circ \iota)(n) = \bigcup_{d|n} \alpha(d)\beta(\frac{n}{d}) = \{\alpha(1)\beta(n), 0\} = 0 = \iota(n).$$

On the other hand, $(\alpha \circ \beta)(1) = \alpha(1)\beta(1) = 1 = \iota(1)$. So $\beta = \alpha^{-1}$. □

Remark. If $\alpha \in G$ and $\alpha(1) = 0$, then α hasn't inverse. Since, if exists $\beta \in G$ such that $\alpha \circ \beta = \iota$. Then

$$(\alpha \circ \beta)(1) = \iota(1) = 1 \qquad (1)$$

In other works, we have

$$(\alpha \circ \beta)(1) = \alpha(1)\beta(1) = 0 \qquad (2)$$

So, we see that the relations (1) and (2) are contractions.

Theorem 4.6. *Let $\alpha \in MF(G)$, $\alpha(1) \neq 0$ and α^{-1} be the inverse of α. Then $\alpha^{-1} \in MF(G)$.*

Proof. Let $\alpha^{-1} \notin MF(G)$. Then there exist m, n, where $gcd(m, n) = 1$ such that

$$\alpha^{-1}(mn) \neq \alpha^{-1}(m)\alpha^{-1}(n). \qquad (\bullet)$$

Let mn be the least positive integer such that the relation $(\bullet)$ is holds, i.e., for all positive integer a and b, such that $gcd(a, b) = 1, ab < mn$, we have

$$\alpha^{-1}(ab) = \alpha^{-1}(a)\alpha^{-1}(b). \qquad (*)$$

It is obvious that $m \neq 1$ and $n \neq 1$, so $mn > 1$. In other words, implying that

$$\begin{aligned} \{\alpha^{-1}(m)\alpha^{-1}(n),\ 0\} &= \{\alpha^{-1}(m)\alpha^{-1}(n),\ \iota(mn)\} \\ &= \{\alpha^{-1}(m)\alpha^{-1}(n),\ (\alpha \circ \alpha^{-1})(mn)\} \\ &= \alpha^{-1}(m)\alpha^{-1}(n) \cup \bigcup_{d|mn} \alpha^{-1}(d)\alpha(\frac{mn}{d}). \end{aligned}$$

If $\alpha \neq 0$ and $\alpha \in MF(G)$, then $\alpha(1) = 1$, we have

$$\{\alpha^{-1}(m)\alpha^{-1}(n),\ 0\} = \alpha^{-1}(m)\alpha^{-1}(n) \cup \bigcup_{d|mn, d<mn} \alpha^{-1}(d)\alpha(\frac{mn}{d}) \cup \alpha^{-1}(mn).$$

If $d|mn$, then exist positive integers s, t and $d = st$ such that $s|m$, $t|n$ and $gcd(s, t) = 1$. So

$$\{\alpha^{-1}(m)\alpha^{-1}(n),\ 0\} = \alpha^{-1}(m)\alpha^{-1}(n) \cup \bigcup_{s|m, t|n, st<mn} \alpha^{-1}(st)\alpha(\frac{mn}{d}) \cup \alpha^{-1}(mn).$$

Since $m \neq 1$, $n \neq 1$ and $\alpha(1) = 1$, according to (*), we can write:
$\{\alpha^{-1}(m)\alpha^{-1}(n),\ 0\} =$

$$= \alpha^{-1}(m)\alpha(1)\alpha^{-1}(n)\alpha(1) \cup \bigcup_{s|m,t|n,st<mn} \alpha^{-1}(s)\alpha(\frac{m}{t})\alpha^{-1}(t)\alpha(\frac{n}{t}) \cup \alpha^{-1}(mn)$$

$$= \bigcup_{s|m,t|n} \alpha^{-1}(s)\alpha(\frac{m}{t})\alpha^{-1}(t)\alpha(\frac{n}{t}) \cup \alpha^{-1}(mn).$$

By Lemma 3.11, we have

$$\begin{aligned}\{\alpha^{-1}(m)\alpha^{-1}(n),\ 0\} &= \left(\bigcup_{s|m} \alpha^{-1}(s)\alpha(\frac{m}{s}) \bullet \bigcup_{t|n} \alpha^{-1}(t)\alpha(\frac{n}{t})\right) \cup \alpha^{-1}(mn) \\ &= \left((\alpha^{-1} \circ \alpha)(m) \bullet (\alpha^{-1} \circ \alpha)(n)\right) \cup \alpha^{-1}(mn) \\ &= (\iota(m) \bullet \iota(n)) \cup \alpha^{-1}(mn) \\ &= \{\alpha^{-1}(mn),\ 0\}.\end{aligned}$$

Then, $\alpha^{-1}(mn) = \alpha^{-1}(m)\alpha^{-1}(n)$, which contradicts the choice of m and n. Thus, $\alpha^{-1}(ab) = \alpha^{-1}(a)\alpha^{-1}(b)$ for any pair of relatively prime positive integers, a, b; that is, $\alpha^{-1} \in MF(G)$. □

5. A PROPERTY OF FUZZY SET WITH ARITHMETIC FUNCTIONS AND HYPERSTRUCTURES

Let $(H, \circ)$ be a hypergroupoid. We can associate a membership function $\tilde{\mu} : H \longrightarrow [0,1]$, as in [2], and we obtain a join space ${}^1H$ as follows (also see [4]):

$$\forall (x,y) \in H^2, \quad x \circ_1 y = \{z \in H |\ \tilde{\mu}(x) \wedge \tilde{\mu}(y) \leq \tilde{\mu}(z) \leq \tilde{\mu}(x) \wedge \tilde{\mu}(y)\}.$$

Then, from $({}^1H, \circ_1)$ we obtain, in the same way, a membership function $\tilde{\mu_1}$ and then the join space ${}^2H$ and so on. A sequence of fuzzy sets and of join spaces $({}^rH, \tilde{\mu}_r)$ is determined. For any $(x,y) \in H^2$, and any $u \in H$, we consider:

$$\begin{aligned}\mu_{x,y}(u) &= 0 \ \text{ iff } \ u \notin x \circ y, \\ \mu_{x,y}(u) &= \frac{1}{|x \circ y|} \ \text{ iff } \ u \in x \circ y, \\ A(u) &= \sum_{(x,y)\in H^2} \mu_{x,y}(u), \qquad (w) \\ Q(u) &= \{(a,b) | (a,b) \in H^2, u \in a \circ b\}, \\ q(u) &= |Q(u)|, \\ \tilde{\mu}(u) &= \frac{A(u)}{q(u)}.\end{aligned}$$

If $Q(u) = \emptyset$, then we set $\tilde{\mu}(u) = 0$.

We denote $\tilde{\mu}_0 = \tilde{\mu}$, ${}^0H = H$. If two consecutive hypergroups of the obtained sequence are isomorphic, then the sequence stops.

For all positive integer $n > 1$, let $\phi(n)$ denote Euler's Phi-Function and

$$\tau(n) = \sum_{d|n} 1, \ \sigma(n) = \sum_{d|n} d, \ \sigma_k(n) = \sum_{d|n} d^k,$$

where $k \geq 2$ and d runs through the positive divisor of n.

Let p be a prime positive integer, then we can write:

$$\phi(p) = p - 1, \ \tau(p) = 2, \ \sigma(p) = p + 1, \ \sigma_k(p) = p^k + 1.$$

Let $H = \{\phi, \tau, \sigma, \sigma_k\}$, by definition hyperstructure " $\circ$ ", we have

$$(\alpha \circ \beta)(p) = \bigcup_{d|p} \alpha(d)\beta(\frac{p}{d}) = \{\alpha(p), \ \beta(p)\},$$

where $(\alpha, \beta) \in H^2$. Thus

$\circ$	ϕ	τ	σ	σ_k
ϕ	$p-1$	$2, p-1$	$p-1, p+1$	$p-1, p^k+1$
τ		2	$2, p+1$	$2, p^k+1$
σ			$p+1$	$p+1, p^k+1$
σ_k				p^k+1

Let $H = \{\phi, \tau, \sigma, \sigma_k\}$, for all $(\alpha, \beta) \in H^2$, we can define a hyperstructure on H, as follows:

$$\alpha \circ_0 \beta = \{\gamma \in H | \min\{\alpha(p), \beta(p)\} \leq \gamma(p) \leq \max\{\alpha(p), \beta(p)\}\}. \qquad (\star)$$

Now, we have the main result:

Theorem 5.1. *Let $H = \{\phi, \tau, \sigma, \sigma_k\}$ and $p \geq 5$. By definition (w), we have:*

$$\tilde{\mu}_1(\phi) = \tilde{\mu}_1(\sigma) < \tilde{\mu}_1(\tau) = \tilde{\mu}_1(\sigma_k).$$

Proof. By definition $(\star)$, we can obtain

${}^1H$	ϕ	τ	σ	σ_k
ϕ	ϕ	ϕ, τ	ϕ, σ	$H - \tau$
τ		τ	$H - \sigma_k$	H
σ			σ	σ, σ_k
σ_k				σ_k

Therefore, by definition (w), one obtains:

$$A(\phi) = \sum_{(\alpha,\beta)\in H^2} \mu_{\alpha,\beta}(\phi) = \frac{1}{|\phi \circ_0 \phi|} + \frac{2}{|\phi \circ_0 \tau|} + \frac{2}{|\phi \circ_0 \sigma|} + \frac{2}{|\tau \circ_0 \sigma|} + \frac{2}{|\phi \circ_0 \sigma_k|} + \frac{2}{|\tau \circ_0 \sigma_k|}$$

$$= 1 + 1 + 1 + \frac{2}{3} + \frac{2}{3} + \frac{2}{4} = \frac{29}{6}$$

similarly, $A(\sigma) = \frac{29}{6}$ and $A(\tau) = A(\sigma_k) = \frac{19}{6}$. Also, one obtains:

$$\tilde{\mu}_1(\phi) = \tilde{\mu}_1(\sigma) = \frac{29/6}{11} = 0/4\overline{39} < \tilde{\mu}_1(\tau) = \tilde{\mu}_1(\sigma_k) = \frac{19/6}{7} = 0/4\overline{523809}.$$

□

The following result show that the sequence stops.

Theorem 5.2. *Let $H = \{\phi, \tau, \sigma, \sigma_k\}$ and $p \geq 5$. By definition (w), we have:*

$$\tilde{\mu}_2(\phi) = \tilde{\mu}_2(\sigma) = \tilde{\mu}_2(\tau) = \tilde{\mu}_2(\sigma_k).$$

Proof. By definition (w), the join space ${}^2H$ associated is

${}^2H$	ϕ	τ	σ	σ_k
ϕ	ϕ, σ	H	ϕ, σ	H
τ		τ, σ_k	H	τ, σ_k
σ			ϕ, σ	H
σ_k				τ, σ_k

Again, one obtains that

$$\tilde{\mu}_2(\phi) = \tilde{\mu}_2(\sigma) = \tilde{\mu}_2(\tau) = \tilde{\mu}_2(\sigma_k) = \frac{1}{3} = 0/\overline{3}.$$

□

6. Conclusions and future work

After the introduction of fuzzy sets by Zadeh [10], there have been a number of generalizations of this fundamental concept. On the other hand, the concept of hyperstructure first was introduced by F. Marty [7]. This paper is intended to build up a connection between hyperstructure, fuzzy sets and arithmetic functions. We have presented the definition of the hyperoperation associated to the set of all arithmetic functions and have studied the properties of this new hyperoperation.

Our future work on this definition will be focused on the properties of hypergroups and canonical hypergroups.

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