Chromaticity of Complete Tripartite Graphs With Certain Star or Matching Deleted

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ABSTRACT

Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. We write $[G] = \{H \mid H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete triparite graphs G according to the number of 4-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, we obtain new families of chromatically unique complete tripartite graphs with certain star or matching deleted.

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1 Introduction

All graphs considered in this paper are finite and simple. For a graph G, we denote by $P(G;\lambda)$ (or $P(G)$), the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (simply *χ-equivalent*), denoted $G \sim H$ if $P(G) = P(H)$. A graph G is said to be *chromatically unique* (simply *χ-unique*), if $H \sim G$ implies that $H \cong G$. A family $\mathcal{G}$ of

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graphs is said to be chromatically-closed (simply χ-*closed*) if for any graph $G \in \mathcal{G}$, $P(H) = P(G)$ implies that $H \in \mathcal{G}$. Many families of χ-unique graphs are known (see [3, 4]).

For a graph G, let $e(G)$, $v(G)$, $t(G)$ and $\chi(G)$ respectively be the number of vertices, edges, triangles and chromatic number of G. Let O_n be an edgeless graph with n vertices. Also let $Q(G)$ and $K(G)$ be the number of induced subgraphs C_4 and complete subgraphs K_4 in G. Suppose S be a set of $s(\geq 1)$ edges of G. Denote by $G - S$ the graph obtained from G by deleting all edges in S, and by $\langle S \rangle$ the graph induced by S. For $t \geq 2$ and $1 \leq p_1 \leq p_2 \leq \cdots \leq p_t$, let $K(p_1, p_2, \ldots, p_t)$ be a complete t-partite graph with partition sets V_i such that $|V_i| = p_i$ for $i = 1, 2, \ldots, t$. In [7], Zhao proved that certain families of complete tripartite graphs with a matching or a star deleted are χ-unique. In this paper, we first characterize certain complete triparite graphs G according to the number of 4-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, we obtain new families of chromatically unique complete tripartite graphs with certain star or matching deleted.

2 Preliminary results and notations

Let $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ be the family $\{K(p_1, p_2, \ldots, p_t) - S \,|\, S \subset E(K(p_1, p_2, \ldots, p_t))$ and $|S| = s\}$. For $p_1 \geq s + 1$, we denote by $K_{i,j}^{-K(1,s)}(p_1, p_2, \ldots, p_t)$ (respectively, $K_{i,j}^{-sK_2}(p_1, p_2, \ldots, p_t)$) the graph in $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ where the s edges in S induced a $K(1, s)$ with center in V_i and all the end-vertices in V_j, (respectively, a matching with end-vertices in V_i and V_j).

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G, k)$ denote the number of k-independent partitions in G. If G is of order n, then $P(G, \lambda) = \sum_{k=1}^{n} \alpha(G, k)(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda - 1) \cdots (\lambda - k + 1)$ (see [5]). Therefore, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \ldots$, if $G \sim H$.

For a graph G with n vertices, the polynomial $\sigma(G, x) = \sum_{k=1}^{n} \alpha(G, k)x^k$ is called the *σ-polynomial* of G (see [1]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs G and H.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H; $G \vee H$ denotes the graph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{xy | x \in V(G)$ and $y \in V(H)\} \cup E(G) \cup E(H)$. Throughout this paper, all the t-partite graphs G under consideration are 2-connected with $\chi(G) = t$. For terms used but not defined here we refer to [6].

Lemma 2.1. (Koh and Teo [3]) *Let G and H be two graphs with $H \sim G$, then $v(G) = v(H)$, $e(G) = e(H)$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G,k) = \alpha(H,k)$ for each $k = 1, 2, \dots,$ and*

$$-Q(G) + 2K(G) = -Q(H) + 2K(H).$$

Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.

Lemma 2.2. (Brenti [1]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G,x)\sigma(H,x).$$

In particular,

$$\sigma(K(n_1, n_2, \dots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x).$$

Lemma 2.3 *Let G be a connected t-partite graph. If $H \sim G$, then there exists a complete t-partite graph $F = K(x_1, x_2, \dots, x_t)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G)$.*

Proof. Since $V(G)$ has a t-independent partition, then $V(H)$ also has a t-independent partition with independent sets $V_1, V_2, \dots, V_t$ such that $|V_i| = x_i$. Hence, H is a t-partite graph and there exists a graph complete t-partite $F = K(x_1, x_2, \dots, x_t)$ such that $H = F - S'$. Since $H \sim G$, by Lemma 2.1, we have $s' = e(F) - e(G)$. □

Let $H = K(x_1, x_2, x_3, \dots, , x_t)$ and $H' = K(x_1, x_2, \dots, x_i + 1, \dots, x_j - 1, \dots, x_t)$. If $i < j$ and $x_j - x_i \geq 2$, then H' is called an *improvement* of H.

Lemma 2.4 *Suppose $H' = K(x_1, x_2, \dots, x_i + 1, \dots, x_j - 1, \dots, x_t)$ is an improvement of $H = K(x_1, x_2, x_3, \dots, , x_t)$, then $\alpha(H, t+1) > \alpha(H', t+1)$.*

Proof. Note that $\alpha(H', t+1) = \sum_{k=1}^{t} 2^{x_k - 1} + 2^{x_i - 1} - 2^{x_j - 2}$ and $\alpha(H, t+1) = \sum_{k=1}^{t} 2^{x_k - 1}$. Hence, $\alpha(H, t+1) - \alpha(H', t+1) = 2^{x_j - 2} - 2^{x_i - 1} \geq 2^{x_i - 1} > 0$.

Suppose $G = K(p_1, p_2, \dots, p_t)$ and $H = G - S$ for a set S of s edges of G. Define $\alpha_k(H) = \alpha(H,k) - \alpha(G,k)$ for $k \geq t+1$.

Lemma 2.5. (Zhao [7]) *Let $G = K(p_1, p_2, \dots, p_t)$ and $H = G - S$. If $p_1 \geq s + 1$, then*

$$s \leq \alpha_{t+1}(H) = \alpha(H, t+1) - \alpha(G, t+1) \leq 2^s - 1,$$

$\alpha_{t+1}(H) = s$ if and only if the subgraph induced by any $r \geq 2$ edges in S is not a complete multipartite graph, and $\alpha_{t+1}(H) = 2^s - 1$ if and only if $\langle S \rangle = K(1, s)$.

Lemma 2.6. (Dong et al. [2]) *Let p_1, p_2 and s be positive integers with $3 \le p_1 \le p_2$, then*

(i) $K_{1,2}^{-K(1,s)}(p_1, p_2)$ *is χ-unique for* $1 \le s \le p_2 - 2$,

(ii) $K_{2,1}^{-K(1,s)}(p_1, p_2)$ *is χ-unique for* $1 \le s \le p_1 - 2$, *and*

(iii) $K^{-sK_2}(p_1, p_2)$ *is χ-unique for* $1 \le s \le p_1 - 1$

For a graph $G \in \mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$, we say an induced subgraph C_4 of G is of Type 1 (respectively Type 2, and Type 3) if the vertices of the induced C_4 are in exactly two (respectively three, and four) partite sets of $V(G)$. An example of induced C_4 of Type 1, 2 and 3 are shown in Figure 1.

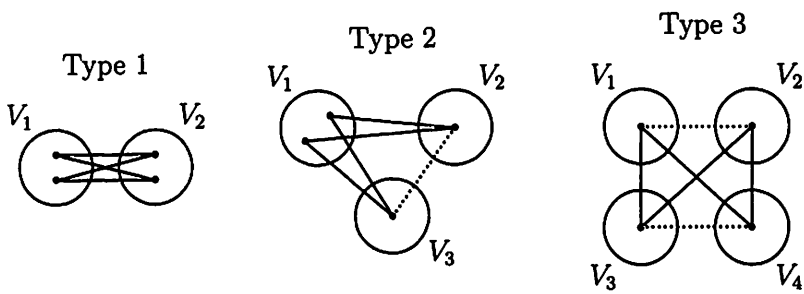

Figure 1: Three types of induced C_4

Suppose G is a graph in $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$. Let S_{ij} $(1 \le i \le t, 1 \le j \le t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \ge 0$.

Lemma 2.7. *Let $F = K(p_1, p_2, p_3)$ be a complete tripartite graph and $G = F - S$ for a set S of $s \ge 1$ edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) &= Q(F) - \sum_{1 \le i < j \le 3} (p_i - 1)(p_j - 1)s_{ij} + \binom{s}{2} - \\ & s_{12}s_{13} - s_{12}s_{23} - s_{13}s_{23} + \sum_{\substack{1 \le i < j \le 3 \\ k \notin \{i, j\}}} s_{ij} \binom{p_k}{2}. \end{aligned}$$

Moreover,

$$max\,\{Q(G)\} = Q(F) - s(p_1 - 1)(p_2 - 1) + \binom{s}{2} + s\binom{p_3}{2}$$

if and only if each edge in S joins vertices in the same two partite sets of smallest size.

Proof. Note that G has only induced C_4 of Type 1 or Type 2. Let $Q_1(G)$ (respectively, $Q_2(G)$) be the number of Type 1 (respectively, Type 2) induced C_4 in G. Observe that $S = \bigcup_{1\le i<j\le t} S_{ij}$ with $s_{ij} \ge 0$. Hence,

$$\begin{aligned} Q_1(G) &= \sum_{1\le i<j\le 3} \binom{p_i}{2}\binom{p_j}{2} - \sum_{1\le i<j\le 3} (p_i-1)(p_j-1)s_{ij} + \\ &\quad \sum_{1\le i<j\le 3} \binom{s_{ij}}{2} \\ &= Q(F) - \sum_{1\le i<j\le 3} (p_i-1)(p_j-1)s_{ij} + \binom{s}{2} - \\ &\quad s_{12}s_{13} - s_{12}s_{23} - s_{13}s_{23}. \end{aligned}$$

Hence,

$$Q_1(G) \le Q(F) - \sum_{1\le i<j\le 3} (p_i-1)(p_j-1)s_{ij} + \binom{s}{2}$$

with the equality holds if and only if $s = s_{ij}$ for $1 \le i < j \le 3$. Now, observe that $(p_1-1)(p_2-1)s \le (p_i-1)(p_j-1)s_{ij}$ for $1 \le i < j \le 3$ and the equality holds if and only if each edge in S joins vertices in the same two partite sets of smallest size.

We now find $Q_2(G)$. Since the number of 2-element subsets of V_k is $\binom{p_k}{2}$, we have

$$\begin{aligned} Q_2(G) &= \sum_{\substack{1\le i<j\le 3 \\ k\notin\{i,j\}}} s_{ij}\binom{p_k}{2} \\ &\le s\binom{p_3}{2}, \end{aligned}$$

with the equality holds if and only if each edge in S joins vertices in the same two partite sets of smallest size. Hence, $max\{Q(G)\} = Q(F) - s(p_1 - 1)(p_2-1) + \binom{s}{2} + s\binom{p_3}{2}$ if and only if each edge in S joins vertices in the same two partite sets of smallest size. □

3 Characterization

In this section, we shall characterize certain complete tripartite graphs $G = K(p_1, p_2, p_3)$ according to the number of 4-independent partitions of

G where $p_3 - p_1 \leq 6$.

Lemma 3.1. *Let $G = K(p_1, p_2, p_3)$ be a complete tripartite graph such that $p_1 + p_2 + p_3 = 3p$ and $p_3 - p_1 \leq 6$. Define $\theta(G) = (\alpha(G,4) - 2^{p-1} - 2^p + 3)/2^{p-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(p,p,p)$*;*

(ii) $\theta(G) = 1$ *if and only if* $G = K(p-1,p,p+1)$*;*

(iii) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(p-2,p+1,p+1)$*;*

(iv) $\theta(G) = 4$ *if and only if* $G = K(p-1,p-1,p+2)$*;*

(v) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(p-2,p,p+2)$*;*

(vi) $\theta(G) = 6\frac{1}{4}$ *if and only if* $G = K(p-3,p+1,p+2)$*;*

(vii) $\theta(G) = 10\frac{1}{8}$ *if and only if* $G = K(p-4,p+2,p+2)$*;*

(viii) $\theta(G) = 11\frac{1}{2}$ *if and only if* $G = K(p-2,p-1,p+3)$*;*

(ix) $\theta(G) = 12\frac{1}{4}$ *if and only if* $G = K(p-3,p,p+3)$*;*

(x) $\theta(G) = 27$ *if and only if* $G = K(p-2,p-2,p+4)$*.*

Proof. In order to complete the proof of the theorem, we first give a table about the θ-value of various complete tripartite graphs with $3p$ vertices as shown in Table 1.

By the definition of improvement, we have the followings.

(i) G_1 is the improvement of G_2 with $\theta(G_2) = 1$;

(ii) G_2 is the improvement of G_3, G_4 and G_5 with $\theta(G_3) = 2\frac{1}{2}, \theta(G_4) = 4\frac{1}{2}$ and $\theta(G_5) = 4$;

(iii) G_3 is the improvement of G_4 and G_6 with $\theta(G_4) = 4\frac{1}{2}$ and $\theta(G_6) = 6\frac{1}{4}$;

(iv) G_4 is the improvement of G_6, G_7 and G_8 with $\theta(G_6) = 6\frac{1}{4}, \theta(G_7) = 12\frac{1}{4}$ and $\theta(G_8) = 11\frac{1}{2}$;

(v) G_5 is the improvement of G_4 and G_8 with $\theta(G_4) = 4\frac{1}{2}$ and $\theta(G_8) = 11\frac{1}{2}$;

(vi) G_6 is the improvement of G_7, G_9 and G_{10} with $\theta(G_7) = 12\frac{1}{4}$, $\theta(G_9) = 10\frac{1}{8}$ and $\theta(G_{10}) = 14\frac{1}{8}$;

Table 1: Some complete tripartite graphs with $3p$ vertices and their θ-values

G	$\theta(G)$	G	$\theta(G)$
$G_1 = K(p,p,p)$	0	$G_{11} = K(p-4,p,p+4)$	$28\frac{1}{8}$
$G_2 = K(p-1,p,p+1)$	1	$G_{12} = K(p-3,p-1,p+4)$	$27\frac{1}{4}$
$G_3 = K(p-2,p+1,p+1)$	$2\frac{1}{2}$	$G_{13} = K(p-2,p-2,p+4)$	27
$G_4 = K(p-2,p,p+2)$	$4\frac{1}{2}$	$G_{14} = K(p-5,p+2,p+3)$	$18\frac{1}{16}$
$G_5 = K(p-1,p-1,p+2)$	4	$G_{15} = K(p-5,p+1,p+4)$	$30\frac{1}{16}$
$G_6 = K(p-3,p+1,p+2)$	$6\frac{1}{4}$	$G_{16} = K(p-3,p-2,p+5)$	$58\frac{3}{4}$
$G_7 = K(p-3,p,p+3)$	$12\frac{1}{4}$	$G_{17} = K(p-6,p+3,p+3)$	$26\frac{1}{32}$
$G_8 = K(p-2,p-1,p+3)$	$11\frac{1}{2}$	$G_{18} = K(p-6,p+2,p+4)$	$34\frac{1}{32}$
$G_9 = K(p-4,p+2,p+2)$	$10\frac{1}{8}$	$G_{19} = K(p-7,p+3,p+4)$	$42\frac{1}{64}$
$G_{10} = K(p-4,p+1,p+3)$	$14\frac{1}{8}$		

(vii) G_7 is the improvement of G_{10}, G_{11} and G_{12} with $\theta(G_{10}) = 14\frac{1}{8}$, $\theta(G_{11}) = 28\frac{1}{8}$ and $\theta(G_{12}) = 27\frac{1}{4}$;

(viii) G_8 is the improvement of G_7, G_{12} and G_{13} with $\theta(G_7) = 12\frac{1}{4}$, $\theta(G_{12}) = 27\frac{1}{4}$ and $\theta(G_{13}) = 27$;

(ix) G_9 is the improvement of G_{10} and G_{14} with $\theta(G_{10}) = 14\frac{1}{8}$ and $\theta(G_{14}) = 18\frac{1}{16}$;

(x) G_{10} is the improvement of G_{11}, G_{14} and G_{15} with $\theta(G_{11}) = 28\frac{1}{8}$, $\theta(G_{14}) = 18\frac{1}{16}$ and $\theta(G_{15}) = 30\frac{1}{16}$.

(xi) G_{13} is the improvement of G_{12} and G_{16} with $\theta(G_{12}) = 27\frac{1}{4}$ and $\theta(G_{16}) = 58\frac{3}{4}$;

(xii) G_{14} is the improvement of G_{15}, G_{17} and G_{18} with $\theta(G_{15}) = 30\frac{1}{16}$, $\theta(G_{17}) = 26\frac{1}{32}$ and $\theta(G_{18}) = 34\frac{1}{32}$;

(xiii) G_{17} is the improvement of G_{18} and G_{19} with $\theta(G_{18}) = 34\frac{1}{32}$ and $\theta(G_{19}) = 42\frac{1}{64}$.

Hence, By Lemma 2.4 and the above arguments, we know Theorem 3.1 (i) to (x) hold. The proof is thus complete. □

Similar to the proof of Lemma 3.1, we can obtain Lemmas 3.2 and 3.3.

Lemma 3.2. *Let $G = K(p_1, p_2, p_3)$ be a complete tripartite graph such that $p_1 + p_2 + p_3 = 3p + 1$ and $p_3 - p_1 \leq 6$. Define $\theta(G) = (\alpha(G, 4) - 2^{p+1} + 3)/2^{p-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(p, p, p+1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(p-1, p+1, p+1)$;

(iii) $\theta(G) = 3$ *if and only if* $G = K(p-1, p, p+2)$;

(iv) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(p-2, p+1, p+2)$;

(v) $\theta(G) = 8\frac{1}{4}$ *if and only if* $G = K(p-3, p+2, p+2)$;

(vi) $\theta(G) = 10$ *if and only if* $G = K(p-1, p-1, p+3)$;

(vii) $\theta(G) = 10\frac{1}{2}$ *if and only if* $G = K(p-2, p, p+3)$;

(viii) $\theta(G) = 12\frac{1}{4}$ *if and only if* $G = K(p-3, p+1, p+3)$;

(ix) $\theta(G) = 25\frac{1}{2}$ *if and only if* $G = K(p-2, p-1, p+4)$;

Lemma 3.3. *Let $G = K(p_1, p_2, p_3)$ be a complete tripartite graph such that $p_1 + p_2 + p_3 = 3p + 1$ and $p_3 - p_1 \leq 6$. Define $\theta(G) = (\alpha(G, 4) - 2^{p-1} - 2^{p+1} + 3)/2^{p-1}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(p, p+1, p+1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(p, p, p+2)$;

(iii) $\theta(G) = 1\frac{1}{2}$ *if and only if* $G = K(p-1, p+1, p+2)$;

(iv) $\theta(G) = 3\frac{1}{4}$ *if and only if* $G = K(p-2, p+2, p+2)$;

(v) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(p-1, p, p+3)$;

(vi) $\theta(G) = 5\frac{1}{4}$ *if and only if* $G = K(p-2, p+1, p+3)$;

(vii) $\theta(G) = 7\frac{1}{8}$ *if and only if* $G = K(p-3, p+2, p+3)$;

(viii) $\theta(G) = 12$ *if and only if* $G = K(p-1, p-1, p+4)$;

(ix) $\theta(G) = 12\frac{1}{4}$ *if and only if* $G = K(p-2, p, p+4)$;

4 Chromatically closed tripartite graphs

We shall in this section obtain the χ-closed families of graphs obtained from the graphs in Lemma 3.1 to Lemma 3.3 with a set S of s edges deleted.

Lemma 4.1. *The family of graphs* $\mathcal{K}^{-s}(p_1,p_2,p_3)$ *where* $p_1+p_2+p_3=3p$, $p_3-p_1\leq 6$ *and* $p_1\geq s+2$ *is* χ*-closed.*

Proof. By Lemma 3.1, there are 10 cases to consider. Denote each graph in Lemma 3.1(i), (ii), ..., (x) by $G_1, G_2, \dots, G_{10}$, respectively. Suppose $H \sim G_i - S$. It suffices to show that $H \in \{G_i - S\}$. By Lemma 2.1, we know there exists a complete tripartite graph $F = K(x,y,z)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s \geq 0$.

Case (i). Let $G = G_1$ with $p \geq s+2$. In this case, $H \sim G - S \in \mathcal{K}^{-s}(p,p,p)$. By Lemma 2.5,

$$\alpha(G-S,4) = \alpha(G,4) + \alpha_4(G-S) \text{ with } s \leq \alpha_4(G-S) \leq 2^s - 1,$$

$$\alpha(F-S',4) = \alpha(F,4) + \alpha_4(F-S') \text{ with } 0 \leq s' \leq \alpha_4(F-S').$$

Hence,

$$\alpha(F-S',4) - \alpha(G-S,4) = \alpha(F,4) - \alpha(G,4) + \alpha_4(F-S') - \alpha_4(G-S).$$

By definition, $\alpha(F,4) - \alpha(G,4) = 2^{p-2}(\theta(F) - \theta(G))$. By Lemma 3.1, $\theta(F) \geq 0$. Suppose $\theta(F) > 0$, then

$$\begin{aligned}\alpha(F-S',4) - \alpha(G-S,4) &\geq 2^{p-2} + \alpha_4(F-S') - \alpha_4(G-S)\\ &\geq 2^s + \alpha_4(F-S') - 2^s + 1\\ &\geq 1,\end{aligned}$$

contradicting $\alpha(F-S',4) = \alpha(G-S,4)$. Hence, $\theta(F) = 0$ and so $F \cong G$ and $s = s'$. Therefore, $H \in \mathcal{K}^{-s}(p,p,p)$.

Case (ii). Let $G = G_2$ with $p \geq s+2$. In this case, $H \sim G - S \in \mathcal{K}^{-s}(p-1,p,p+1)$. By Lemma 2.5,

$$\alpha(G-S,4) = \alpha(G,4) + \alpha_4(G-S) \text{ with } s \leq \alpha_4(G-S) \leq 2^s - 1,$$

$$\alpha(F-S',4) = \alpha(F,4) + \alpha_4(F-S') \text{ with } 0 \leq s' \leq \alpha_4(F-S').$$

Hence,

$$\alpha(F-S',4) - \alpha(G-S,4) = \alpha(F,4) - \alpha(G,4) + \alpha_4(F-S') - \alpha_4(G-S).$$

By definition, $\alpha(F,4)-\alpha(G,4) = 2^{p-2}(\theta(F)-\theta(G))$. Suppose $\theta(F) \neq \theta(G)$. We consider two subcases.

Subcase (a). $\theta(F) < \theta(G)$. By Lemma 3.1, $F = G_1$ and so $H = G_1 - S' \in \{G_1 - S'\}$. However, $G - S \notin \{G_1 - S'\}$ since $\{G_1 - S'\}$ is χ-closed, a contradiction.

Subcase (b). $\theta(F) > \theta(G)$. By Lemma 3.1, $\alpha(F,4) - \alpha(G,4) \geq \frac{3}{2}(2^{p-2})$. So,

$$\begin{aligned} \alpha(F-S',4) - \alpha(G-S,4) &\geq \frac{3}{2}(2^{p-2}) + \alpha_4(F-S') - \alpha_4(G-S) \\ &\geq 2^s + \alpha_4(F-S') - 2^s + 1 \\ &\geq 1, \end{aligned}$$

contradicting $\alpha(F-S',4) = \alpha(G-S,4)$. Hence, $\theta(F) - \theta(G) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathcal{K}^{-s}(p-1,p,p+1)$.

Using Table 1, we can prove Cases (iii) to (x) in a similar way. This completes the proof. □

Similar to the proofs of Lemma 4.1, we can prove Lemmas 4.2 and 4.3.

Lemma 4.2. *The family of graphs $\mathcal{K}^{-s}(p_1,p_2,p_3)$ where $p_1+p_2+p_3 = 3p+1$, $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$ is χ-closed.*

Lemma 4.3. *The family of graphs $\mathcal{K}^{-s}(p_1,p_2,p_3)$ where $p_1+p_2+p_3 = 3p+2$, $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$ is χ-closed.*

5 Chromatically unique tripartite graphs

The following two Lemmas give several families of chromatically unique complete tripartite graphs having $3p$ vertices with a set S of s edges deleted where the deleted edges induce a star $K(1,s)$ and a matching sK_2, respectively.

Lemma 5.1. *The graphs $K_{i,j}^{-K(1,s)}(p_1,p_2,p_3)$ where $p_1+p_2+p_3 = 3p$, $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$ are χ-unique for $1 \leq i \neq j \leq 3$.*

Proof. By Lemma 3.1, there are 10 cases to consider. Denote each graph in Lemma 3.1 (i), (ii), ..., (x) by $G_1, G_2, \ldots, G_{10}$, respectively. The proof for each graph obtained from G_i ($i = 1,2,\ldots,10$) are similar, so we only give the detail proof of the graphs obtained from G_3 as follow.

By Lemma 2.5 and Lemma 4.1 Case (iii), we know that $\mathcal{K}_{i,j}^{-K(1,s)}(p-2,p+1,p+1) = \{K_{i,j}^{-K(1,s)}(p-2,p+1,p+1) \mid (i,j) \in \{(1,2),(2,1),(2,3)\}\}$ is χ-closed for $p \geq s+3$. Note that

$$\begin{aligned} t(K_{i,j}^{-K(1,s)}(p-2,p+1,p+1)) &= (p-2)(p+1)^2 - p - 1 \\ &\quad \text{for } (i,j) \in \{(1,2),(2,1)\}, \\ t(K_{2,3}^{-K(1,s)}(p-2,p+1,p+1)) &= (p-2)(p+1)^2 - p + 2. \end{aligned}$$

By Lemmas 2.2 and 2.6, we conclude that $\sigma(K_{1,2}^{-K(1,s)}(p-2,p+1,p+1)) \neq \sigma(K_{2,1}^{-K(1,s)}(p-2,p+1,p+1))$. Hence, by Lemma 2.1, $K_{i,j}^{-K(1,s)}(p-2,p+1,p+1)$ where $p \geq s+3$ is χ-unique for $1 \leq i \neq j \leq 3$.

The proof is thus complete. □

Lemma 5.2. *The graphs $K_{1,2}^{-sK_2}(p_1,p_2,p_3)$ where $p_1+p_2+p_3 = 3p$, $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$ are χ-unique.*

Proof. By Lemma 3.1, there are 10 cases to consider. Denote each graph in Lemma 3.1 (i), (ii), ..., (x) by $G_1, G_2, \ldots, G_{10}$, respectively. For a graph $K(x,y,z)$, let $S = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_s\}$ be a set of s edges in $E(K(x,y,z))$ and let $t(\epsilon_i)$ denote the number of triangles containing ϵ_i in $K(x,y,z)$. The proof for each graph obtained from G_i $(i = 1,2,\ldots,10)$ are similar, so we only give the proofs of the graphs obtained from G_2 and G_3 as follows.

Suppose $H \sim G = K_{1,2}^{-sK_2}(p-1,p,p+1)$ for $p \geq s+2$. By Lemma 4.1 and Lemma 2.1, $H \in \mathcal{K}^{-s}(p-1,p,p+1)$ and $\alpha_4(H) = \alpha_4(G) = s$. Let $H = F - S$ where $F = K(p-1,p,p+1)$. Clearly, $t(\epsilon_i) \leq p+1$ for each $\epsilon_i \in S$. So,

$$t(H) \geq t(F) - s(p+1) \qquad (1)$$

with equality holds only if $t(\epsilon_i) = p+1$ for all $\epsilon_i \in S$. Since $t(H) = t(G) = t(F) - s(p+1)$, equality in (1) holds with $t(\epsilon_i) = p+1$ for all $\epsilon_i \in S$. Therefore, each edge in S has an end-vertex in V_1 and another end-vertex in V_2. Moreover, S must induce a matching in F. Otherwise, $\alpha_4(H) > s$. Hence, $\langle S \rangle \cong sK_2$ and $H \cong G$.

Now, suppose $H \sim G = K_{1,2}^{-sK_2}(p-2,p+1,p+1)$ for $p \geq s+3$. By Lemma 4.1 and Lemma 2.1, $H \in \mathcal{K}^{-s}(p-2,p+1,p+1)$ and $\alpha_4(H) = \alpha_4(G) = s$. Let $H = F - S$ where $F = K(p-2,p+1,p+1)$. Clearly, $t(\epsilon_i) \leq p+1$ for each $\epsilon_i \in S$. So,

$$t(H) \geq t(F) - s(p+1) \qquad (2)$$

with equality holds only if $t(\epsilon_i) = p+1$ for all $\epsilon_i \in S$. Since $t(H) = t(G) = t(F) - s(p+1)$, equality in (2) holds with $t(\epsilon_i) = p+1$ for all $\epsilon_i \in S$. Therefore, each edge in S has an end-vertex in V_1, and another end-vertex in V_2 or in V_3. Moreover, S must induce a matching in F. Otherwise, equality in (2) does not hold or $\alpha_4(H) > s$. By Lemma 2.7, $Q(G) = Q(F) - sp(p-3) + \binom{s}{2} + s\binom{p+1}{2} \geq Q(H)$ and the equality holds if and only if each edge in S joins vertices in the same two partite sets of smallest size. Therefore, $\langle S \rangle \cong sK_2$ with $H \cong G$.

The proof is thus complete. □

Similar to the proofs of Lemmas 5.1 and 5.2, we can prove Lemmas 5.3 to 5.6.

Lemma 5.3. *The graphs $K_{i,j}^{-K(1,s)}(p_1,p_2,p_3)$ where $p_1+p_2+p_3 = 3p+1$, $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$ are χ-unique for $1 \leq i \neq j \leq 3$.*

Lemma 5.4. *The graphs $K_{1,2}^{-sK_2}(p_1,p_2,p_3)$ where $p_1+p_2+p_3 = 3p+1$, $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$ are χ-unique.*

Lemma 5.5. *The graphs $K_{i,j}^{-K(1,s)}(p_1,p_2,p_3)$ where $p_1+p_2+p_3 = 3p+2$, $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$ are χ-unique for $1 \leq i \neq j \leq 3$.*

Lemma 5.6. *The graphs $K_{1,2}^{-sK_2}(p_1,p_2,p_3)$ where $p_1+p_2+p_3 = 3p+2$, $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$ are χ-unique.*

We thus have our main theorem as follow.

Theorem 5.1. *For integers $p_3 - p_1 \leq 6$ and $p_1 \geq s+2$, the tripartite graphs $K_{i,j}^{-K(1,s)}(p_1,p_2,p_3)$ where $1 \leq i \neq j \leq 3$ and $K_{1,2}^{-sK_2}(p_1,p_2,p_3)$ are χ-unique.*

Remark. Our main theorem improves the condition of Theorems 6.4.2 to 6.4.4 in [7] significantly especially when s is "sufficiently" large. We also obtained a similar result for 4-partite graphs which will appear in other journal.

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