A note on path factors in claw-free graphs*

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Abstract

A graph G is called claw-free if G has no induced subgraph isomorphic to $K_{1,3}$. Ando et al. obtained the result: a claw-free graph G with minimum degree at least d has a path-factor such that the order of each path is at least $d+1$; in particular G has a $\{P_3, P_4, P_5\}$-factor whenever $d \geq 2$. Kawarabayashi et al. proved that every 2-connected cubic graph has a $\{P_3, P_4\}$-factor. In this article, we show that if G is a connected claw-free graph with at least 6 vertices and minimum degree at least 2, then G has a $\{P_3, P_4\}$-factor. As an immediate consequence, it follows that every claw-free cubic graph (not necessarily connected) has a $\{P_3, P_4\}$-factor.

Keywords: Graph; Claw-free; Path factor; $\{P_3, P_4\}$-factor.
AMS Subject Classification: 05C70

1 Introduction

In this paper we consider finite graphs without loops nor multiple edges. A graph G is called *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$. Let $\mathcal{F}$ be a set of connected graphs. A spanning subgraph F of a graph G is called an $\mathcal{F}$-factor if every component of F is isomorphic to one member of $\mathcal{F}$. In particular, a *path factor* means an $\mathcal{F}$-factor such that each member of $\mathcal{F}$ is a path. If let P_d denote the path of order d and $\mathcal{P}_{\geq k} = \{P_i \mid i \geq k\}$ for a positive integer k, a $\mathcal{P}_{\geq k}$-factor of a graph G is a path factor in which each component has order at least k, and a $\{P_2\}$-factor of G is just its 1-factor.

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There are a number of results concerning the path factor in graphs. Analogous to Tutte's 1-factor theorem, Akiyama et al. [1] obtained a necessary and sufficient condition for a graph that has a $\mathcal{P}_{\geq 2}$-factor, and Kaneko [4] found a criterion for a graph to have a $\mathcal{P}_{\geq 3}$-factor. Afterward, Kano et al. [6] gave a simple proof to Kaneko's theorem. In fact, a graph has a $\mathcal{P}_{\geq 3}$-factor if and only if it has a $\{P_3, P_4, P_5\}$-factor. As a corollary, Kaneko [4] showed that every cubic graph has a $\{P_3, P_4, P_5\}$-factor. Along this line, the following results were obtained.

Theorem 1 (Kano et al. [5]) *Every connected cubic bipartite graph of order at least 8 has a $\mathcal{P}_{\geq 8}$-factor. Hence, it has a $\{P_3, P_4\}$-factor.*

Theorem 2 (Kawarabayashi et al. [7]) *Every 2-connected cubic graph has a $\{P_3, P_4\}$-factor.*

Moreover, Akiyama and Kano [2] proved that every 3-connected cubic graph of order $4n$ has a $\{P_4\}$-factor, and they posed the following conjecture, which is still open.

Conjecture 3 (Akiyama and Kano [2]) *Every 3-connected cubic graph of order $3n$ has a $\{P_3\}$-factor.*

For claw-free graphs Ando et al. [3] obtained the following result concerning the path factor.

Theorem 4 (Ando et al. [3]) *Let G be a claw-free graph with minimum degree $\delta(G) \geq d$ for a positive integer d. Then G has a $\mathcal{P}_{\geq d+1}$-factor.*

The theorem implies that a claw-free graph G has a $\{P_3, P_4, P_5\}$-factor if $\delta(G) \geq 2$, and a $\{P_3, P_4\}$-factor if $\delta(G) \geq 5$. Motivated by this observation, we wish to investigate whether a claw-free graph with $\delta(G) = 2$, 3, or 4 has a $\{P_3, P_4\}$-factor or not. Our main result is the following.

Theorem 5 *If G is a claw-free graph with $\delta(G) \geq 2$, then each component of order exactly 5 of G has a spanning path, and all other components have $\{P_3, P_4\}$-factors.*

The proof of the main Theorem is presented in next section. Finally as immediate consequences, we obtain that if G is a connected claw-free graph with at least 6 vertices and minimum degree at least 2, then G has a $\{P_3, P_4\}$-factor; and every claw-free cubic graph (not necessarily connected) has a $\{P_3, P_4\}$-factor.

2 Proof of Theorem 5

To prove the main theorem we need some further notations. Let G be a graph with edge-set $E(G)$ and vertex-set $V(G)$. For a vertex x of G, a neighbor of x means a vertex adjacent to x, and the degree of x is the number of neighbors of x. Let $\delta(G)$ be the minimum degree of G. Given a subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. Let P_d be the path of order d. Sometimes we add some superscripts to P_d to distinguish different paths of order d, for example P_d^i, P_d^j, $\cdots$. In this paper, the notation "$\cup$" always stands for the disjoint union of two graphs, and kP_d denotes the disjoint union of k copies of P_d. For two graphs H and G, $H + G$ stands for the union of H and G which allows $H \cap G \neq \emptyset$.

Proof of Theorem 5. By Theorem 4, G has a $\{P_3, P_4, P_5\}$-factor. Choose a $\{P_3, P_4, P_5\}$-factor $\mathcal{P}$ such that the number of P_5 components is as minimum as possible. We shall show that the $\{P_3, P_4, P_5\}$-factor $\mathcal{P}$ is the required factor: If a component of $\mathcal{P}$ is P_5, then it spans a component of G. We first obtain the following claims.

Claim 1. *Let P_5 be a component of $\mathcal{P}$. Then only the middle vertex of P_5 may be adjacent to an end-vertex of a P_4 component of $\mathcal{P}$.*

Proof. Let $P_5 = v_1...v_5$ and $P_4 = v_1'...v_4'$ be components of $\mathcal{P}$. If v_1 and v_1' are adjacent in G, $P_5 + v_1v_1' + P_4$ forms a path of order 9 of G, which has a spanning subgraph $3P_3$ (see the bold lines in Fig. 1 (left)). Then $3P_3$ can replace the $P_5 \cup P_4$ in $\mathcal{P}$ to obtain another $\{P_3, P_4, P_5\}$-factor. Hence the number of P_5 components reduces by one, which contradicts the choice of $\mathcal{P}$. If v_2 is adjacent to v_1', $P_5 + v_2v_1' + P_4$ also has a spanning subgraph $3P_3$, i.e. $v_1v_2v_1'$, $v_3v_4v_5$ and $v_2'v_3'v_4'$ (see Fig. 1 (right)). So a similar contradiction appears. □

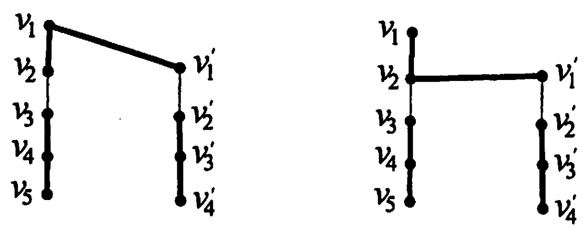

Figure 1. Illustration for the proof of Claim 1.

Claim 2. *Suppose $\mathcal{P}$ has a sequence of distinct components $P_5^0, P_4^1, P_4^2, \ldots,$ $P_4^k (k \geq 1)$ such that for each $0 \leq i < k$, v_1^i is adjacent to both vertices*

v_2^{i+1} *and* v_3^{i+1}*, where* $P_5^0 = v_1^0 ... v_5^0$ *and* $P_4^i = v_1^i ... v_4^i$*. Then the following statements hold.*

(i) Neither of the end-vertices of P_4^k *is adjacent to an end-vertex of another* P_4 *component of* $\mathcal{P}$*,*

(ii) neither of the middle vertices of P_4^k *is adjacent to an end-vertex of another* P_4 *component of* $\mathcal{P}$*, and*

(iii) neither of the end-vertices of P_5^0 *and* P_4^k *is adjacent to a vertex of a* P_3 *component of* $\mathcal{P}$*.*

Proof. (i) If v_1^k is adjacent to an end-vertex u_1 of a component P_4 of $\mathcal{P}$, where $P_4 := u_1u_2u_3u_4 \neq P_4^i$ for all $i \leq k$, then $P_5^0 \cup P_4^1 \cup P_4^2 \cup \cdots \cup P_4^k \cup P_4$ plus some edges $v_1^0v_2^1$, $v_1^1v_2^2$, $\cdots$, $v_1^{k-2}v_2^{k-1}$, $v_1^{k-1}v_3^k$, $v_1^ku_1$ has a spanning subgraph $kP_4 \cup 3P_3$, which is $\{v_2^0v_3^0v_4^0v_5^0, v_1^0v_2^1v_3^1v_4^1, \ldots, v_1^{k-2}v_2^{k-1}v_3^{k-1}v_4^{k-1}, v_1^{k-1}v_3^kv_4^k, v_2^kv_1^ku_1, u_2u_3u_4\}$ (see the bold lines in Fig. 2 (left)). Then $P_5 \cup (k+1)P_4$ in $\mathcal{P}$ can be replaced with $kP_4 \cup 3P_3$, giving another $\{P_3, P_4, P_5\}$-factor with less number of P_5 components than $\mathcal{P}$, a contradiction.

(ii) If a middle vertex v_2^k or v_3^k of P_4^k (say v_2^k) is adjacent to an end-vertex u_1 of a component P_4 of $\mathcal{P}$, where $P_4 := u_1u_2u_3u_4 \neq P_4^i$ for all $i \leq k$, then $P_5^0 \cup P_4^1 \cup P_4^2 \cup \cdots \cup P_4^k \cup P_4$ can be replaced with $kP_4 \cup 3P_3$: $\{v_2^0v_3^0v_4^0v_5^0, v_1^0v_2^1v_3^1v_4^1, \ldots, v_1^{k-2}v_2^{k-1}v_3^{k-1}v_4^{k-1}, v_1^{k-1}v_3^kv_4^k, v_1^kv_2^ku_1, u_2u_3u_4\}$ (see Fig. 2 (middle)), giving another $\{P_3, P_4, P_5\}$-factor which reduces the number of P_5 components by one, a contradiction.

(iii) If v_1^k is adjacent to an end-vertex (say u_1) of a component P_3 of $\mathcal{P}$, where $P_3 := u_1u_2u_3$, then $G[V(P_5^0 \cup P_4^1 \cup P_4^2 \cup \cdots \cup P_4^k \cup P_3)]$ has a spanning subgraph $(k+2)P_4$: $\{v_2^0v_3^0v_4^0v_5^0, v_1^0v_2^1v_3^1v_4^1, \ldots, v_1^{k-1}v_2^kv_3^kv_4^k, v_1^ku_1u_2u_3\}$ (see Fig. 2 (right)). A similar contradiction follows. If v_1^k is only adjacent to the middle vertex u_2 of P_3, u_1 and u_3 are adjacent since G is claw-free. Let $P_3' = u_2u_3u_1$. Then $\mathcal{P}' := (\mathcal{P} \setminus \{P_3\}) \cup \{P_3'\}$ is another $\{P_3, P_4, P_5\}$-factor of G with the same number of P_5 components as $\mathcal{P}$. So v_1^k is adjacent to an end-vertex u_2 of a component P_3' of $\mathcal{P}'$. So this case is reduced to the above case. □

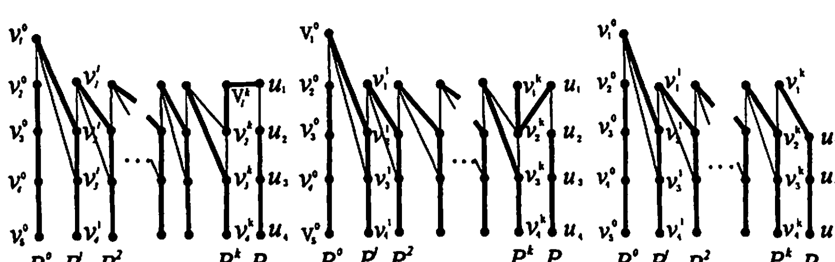

Figure 2. Illustration for the proof of Claim 2.

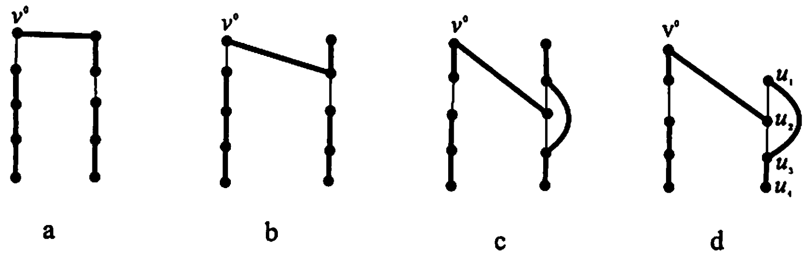

Figure 3.

Let P_5^0 be any component of $\mathcal{P}$ (if exists). We now show that neither of the end-vertices of P_5^0 is adjacent to vertices outside $V(P_5^0)$. Otherwise, we would have the following claim, which contradicts the finiteness of graph G.

Claim 3. If an end-vertex of P_5^0 is adjacent to vertices outside $V(P_5^0)$, then *$\mathcal{P}$ has a sequence of distinct components $P_5^0, P_4^1, P_4^2, ..., P_4^n, ...$ such that*
(i) an end-vertex v^0 of P_5^0 must be adjacent to both middle vertices of P_4^1,
(ii) for each $n \geq 1$, an end-vertex v^n of P_4^n must be adjacent to both middle vertices of P_4^{n+1}, and
(iii) for all $i < n$, the end-vertex v^n of P_4^n has no neighbors in P_4^i or P_5^0.

Proof. We shall show by induction the existence of such a sequence of components of $\mathcal{P}$ as in the Claim. First, by the above converse supposition we have that P_5^0 exists and has an end-vertex v^0 that is adjacent to a vertex u of other components of $\mathcal{P}$. By Claim 2 (iii) u lies in no P_3 components of $\mathcal{P}$. We also show that u lies in no P_5 components of $\mathcal{P}$. Suppose that u belongs to a component P_5 of $\mathcal{P}$. If u is an end-vertex (Fig. 3 (a)), or the second or the fourth vertex (Fig. 3 (b)) of P_5, $2P_3 \cup P_4$ can replace $P_5^0 \cup P_5$. This contradicts the choice of $\mathcal{P}$ similarly. So u can be only the middle vertex of P_5. As G is claw-free, the second and the fourth vertices of P_5 must be adjacent. Likewise $2P_3 \cup P_4$ can replace $P_5^0 \cup P_5$ (see Fig. 3 (c)), a contradiction. The remaining case is that u lies in a P_4 component of $\mathcal{P}$, denoted by $P_4^1 = u_1u_2u_3u_4$. By Claim 1 u must be either u_2 or u_3. If v^0 is adjacent to exactly one of u_2 and u_3 (say u_2), u_1 must be adjacent to u_3 since G is claw-free. $3P_3$ can also replace $P_5^0 \cup P_4^1$ (see Fig. 3 (d)). This also produces a contradiction. Hence P_5^0 and P_4^1 exist and satisfy Property (i).

We now suppose that $\mathcal{P}$ has a sequence of distinct components $P_5^0, P_4^1, P_4^2, ..., P_4^k (k \geq 1)$ which satisfies all Properties (i)-(iii) for each $0 \leq n < k$, where $P_5^0 =: v_1^0v_2^0v_3^0v_4^0v_5^0$ and $P_4^i =: v_1^iv_2^iv_3^iv_4^i$ for all $1 \leq i \leq k$. By Property (i) or (ii) an end-vertex v_1^{k-1} of P_4^{k-1} (P_5^0 if $k = 1$) is adjacent to both middle vertices of P_4^k. It is sufficient to prove that v_1^k is only adjacent to both middle vertices of other P_4 components of $\mathcal{P}$ except for v_2^k. By Claim 2 (iii) v_1^k is not adjacent to any vertex of P_3 components of $\mathcal{P}$.

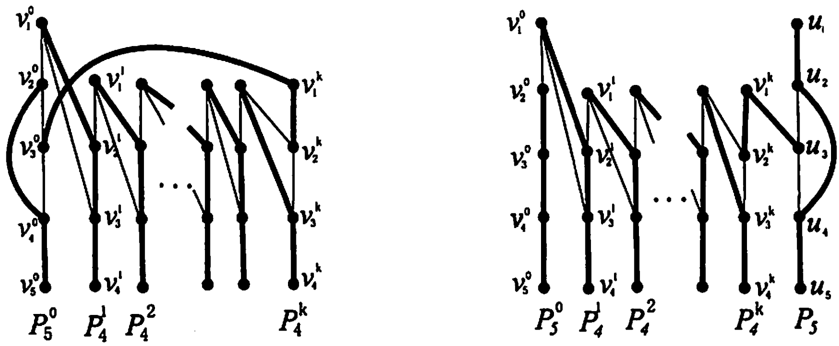

Figure 4.

If v_1^k is adjacent to a vertex of a component P_5 of $\mathcal{P}$, by Claim 1 v_1^k is only adjacent to the middle vertex of P_5. Since G is claw-free, the second vertex of P_5 is adjacent to the fourth vertex. If $P_5 = P_5^0$, by the induction hypothesis $P_5^0 \cup P_4^1 \cup P_4^2 \cup \cdots \cup P_4^k$ can be replaced with $3P_3 \cup (k-1)P_4$: $\{v_2^0v_4^0v_5^0, v_3^0v_1^kv_2^k, v_1^{k-1}v_3^kv_4^k, v_1^0v_2^1v_3^1v_4^1, v_1^1v_2^2v_3^2v_4^2, \ldots, v_1^{k-2}v_2^{k-1}v_3^{k-1}v_4^{k-1}\}$ (see the bold lines in Fig. 4 (left)). Otherwise, let $P_5 =: u_1u_2\cdots u_5$. Similarly $P_5^0 \cup P_4^1 \cup P_4^2 \cup \cdots \cup P_4^k \cup P_5$ can be replaced with $2P_3 \cup (k+1)P_4$: $\{u_3v_1^kv_2^k, v_1^{k-1}v_3^kv_4^k, u_1u_2u_4u_5, v_2^0v_3^0v_4^0v_5^0,$ $v_1^0v_2^1v_3^1v_4^1, v_1^1v_2^2v_3^2v_4^2, \ldots, v_1^{k-2}v_2^{k-1}v_3^{k-1}v_4^{k-1}\}$ (see Fig. 4 (right)). Both contradict the choice of $\mathcal{P}$.

By Claim 2 (i) and (ii) none of the vertices of P_4^i is adjacent to v_1^k for $i < k$, considering a subsequence $P_5^0, P_4^1, P_4^2, \ldots, P_4^i$. Hence Property (iii) holds for $n = k$. If v_1^k is adjacent to v_3^k or v_4^k (say v_3^k), v_1^{k-1} is adjacent to an end-vertex v_2^k of $P_4' := v_2^kv_1^kv_3^kv_4^k$. But it is impossible by applying Claim 2 (i) to $\mathcal{P}' := (\mathcal{P} \setminus \{P_4^k\}) \cup \{P_4'\}$ and a subsequence $P_5^0, P_4^1, P_4^2, \ldots, P_4^{k-1}$ and P_4'. Since $\delta(G) \geq 2$, v_1^k is thus adjacent to a vertex of another P_4 component, denoted by $P_4^{k+1} := v_1^{k+1}v_2^{k+1}v_3^{k+1}v_4^{k+1}$, of $\mathcal{P}$ that is different from any P_4^i for all $i \leq k$. By Claim 2 (i) v_1^k is not adjacent to end-vertices v_1^{k+1} and v_4^{k+1}. Similarly, since G is claw-free v_1^k must be adjacent to both middle vertices v_2^{k+1} and v_3^{k+1} of P_4^{k+1}, i.e. Property (ii) holds for $n = k$. Therefore, $\mathcal{P}$ has a sequence of distinct components $P_5^0, P_4^1, P_4^2, \ldots, P_4^k, P_4^{k+1}$ which satisfies all Properties (i)-(iii) for each $0 \leq n < k+1$. The induction shows that the claim follows. □

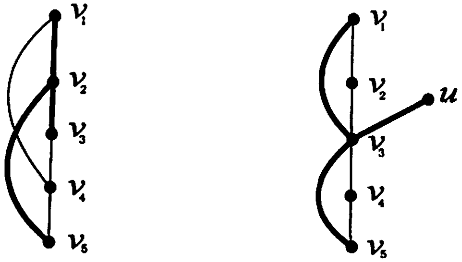

Figure 5.

We have already shown that neither of the end-vertices of any component P_5 of $\mathcal{P}$ is adjacent to vertices outside P_5. The remainder is to prove that any internal vertices of P_5 have no neighbors outside P_5 itself. Let $P_5 := v_1v_2v_3v_4v_5$ be a component of $\mathcal{P}$. Then all neighbors of v_1 and v_5 belong to $V(P_5)$. If the induced graph $G[V(P_5)]$ has a Hamiltonian cycle, each vertex v_i of P_5 can be an end-vertex of a path of order 5 in G, which can replace P_5 to obtain another $\{P_3, P_4, P_5\}$-factor $\mathcal{P}'$. Since $\mathcal{P}'$ has the same number of P_5 components as $\mathcal{P}$, the above proof already shows that all neighbors of v_i are in P_5. Hence $V(P_5)$ induces a component of G. Suppose that $G[V(P_5)]$ has no Hamiltonian cycle. Since $\delta(G) \geq 2$, it follows that each of both v_1 and v_5 has exactly two neighbors and either $\{v_1v_4, v_2v_5\} \subset E(G)$ or $\{v_1v_3, v_5v_3\} \subset E(G)$. If $\{v_1v_4, v_2v_5\} \subset E(G)$, $G[\{v_1, v_2, v_3, v_5\}]$ is a claw of G, i.e. $K_{1,3}$ (see the bold lines in Fig. 5 (left)), a contradiction. If v_1 and v_5 are both adjacent to v_3, each v_i for $i = 1, 2, 4, 5$ is an end-vertex of a path of order 5 in $G[V(P_5)]$ and their neighbors are all in P_5. If v_3 has a neighbor $u \notin V(P_5)$, $G[\{v_1, v_3, v_5, u\}]$ is a claw of G (see Fig. 5 (right)), a contradiction. Hence all neighbors of v_3 are also in P_5 as v_1, v_2, v_4 and v_5. So $G[V(P_5)]$ is a component of G. The entire proof is completed. □

3 Conclusions

For connected claw-free graphs we can derive a simpler result from Theorem 5:

Corollary 6 *If G is a connected claw-free graph with $\delta(G) \geq 2$ and at least 6 vertices, G has a $\{P_3, P_4\}$-factor.*

Proof. Since G is connected and has at least 6 vertices, G has no component with 5 vertices. Theorem 5 implies that G has a $\{P_3, P_4\}$-factor. □

We now remark that Theorem 5 and Corollary 6 are best possible. For example, the claw-free graph with vertices of degree 1 illustrated in Fig. 6 has neither $\{P_3, P_4\}$-factor nor component of order 5.

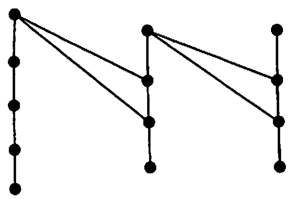

Figure 6. A claw-free graph without $\{P_3, P_4\}$-factor.

Theorem 2 shows that every 2-connected cubic graph has a $\{P_3, P_4\}$-factor. However, Kawarabayashi et al. [7] pointed out that the theorem fails when violating the connection condition by presenting a counter-example of a cubic graph with a cut-vertex. In fact this cubic graph contains many claws, i.e. induced subgraphs $K_{1,3}$. For a claw-free graph with cut-vertices illustrated in Fig. 7, however, it has a $\{P_3, P_4\}$-factor. Applying Theorem 5 or Corollary 6, we have the following general result.

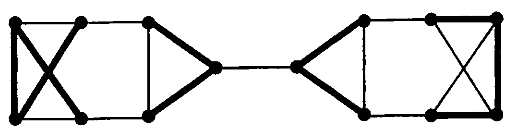

Figure 7. A claw-free cubic graph with cut-vertices and a $\{P_3, P_4\}$-factor (in bold lines).

Corollary 7 *Let G be a claw-free cubic graph. Then G has a $\{P_3, P_4\}$-factor.*

Proof. That G is a cubic graph implies that it has no component of order 5. So by Theorem 5 or Corollary 6 G has a $\{P_3, P_4\}$-factor. □

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