The Vertices of Lower Degree in Contraction-Critical κ Connected Graphs *

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Abstract

Let G be a contraction-critical κ connected graph. It is known (see *Graphs and Combinatorics*, 7 (1991) 15-21) that the minimum degree of G is at most $\lfloor \frac{5\kappa}{4} \rfloor - 1$. In this paper we show that if G has at most one vertex of degree κ, then either G has a pair of adjacent vertices such that each of them has degree at most $\lfloor \frac{5\kappa}{4} \rfloor - 1$, or there is a vertex of degree κ whose neighborhood has a vertex of degree at most $\lfloor \frac{4\kappa}{3} \rfloor - 1$. Moreover, if the minimum degree of G equals to $\frac{5\kappa}{4} - 1$ (and thus $\kappa = 0$ *mod* 4), Su showed that G has κ vertices of degree $\frac{5\kappa}{4} - 1$, guessed that G has $\frac{3\kappa}{2}$ such vertices (see *Combinatorics Graph Theory Algorithms and Application* (Yousef Alavi et. al Eds.), World Scientific, 1993, 329-337). Here we verify that this is true.

Keywords Contraction-Critical Graph; Fragment; $N(B)$-fragment.

1 Introduction

We consider only finite and simple graphs. Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a vertex $x \in V$, we denote the neighborhood of x by $N_G(x)$, which is the set of vertices adjacent to x. $d_G(x) = |N_G(x)|$ denotes the degree of x. Let $\delta(G)$ denote the minimum degree of G. For $F \subseteq V$, let $N_G(F) = (\bigcup_{x \in F} N_G(x)) - F$ and $\overline{F} = V - (F \cup N_G(F))$. The set F or the subgraph induced by F is called a fragment of G if $\overline{F} \neq \emptyset$ and $|N_G(F)| = \kappa(G)$, where $\kappa(G)$ denotes the

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connectivity number of G. A fragment with minimum cardinality is called an atom of G, its cardinality is denoted by $a(G)$. An end of G is a fragment of G that contains no other fragment as a proper subset. We often omit the subscript G if it is clear from the context.

Let G be a κ connected non-complete graph, $\kappa \geq 2$. An edge of G is called κ-contractible if its contraction results still in a κ connected graph. The graph G is said to be contraction-critical if G has no κ-contractible edge. A set T of n vertices of G is called a n-vertex-cut if $G - T$ is disconnected. It is easy to see that a κ connected graph G is contraction-critical if and only if for each edge $e = xy$ of G, G has a κ-vertex-cut containing $\{x, y\}$. A κ connected graph G is called almost critical if for each fragment F of G, there is a κ-vertex-cut T such that $F \cap T \neq \emptyset$.

Let G be a contraction-critical κ connected graph. Egawa ([2]) proved that $a(G) \leq \lfloor \frac{\kappa}{4} \rfloor$, and hence $\delta(G) \leq \lfloor \frac{5\kappa}{4} \rfloor - 1$. Su [6]) generalized this result to that G has two disjoin fragments F_1, F_2 of G such that $|F_1| + |F_2| \leq \frac{\kappa}{2}$. We may ask the following question: how many vertices of degree at most $\lfloor \frac{5\kappa}{4} \rfloor - 1$ are there in a contraction-critical κ connected graph G?

Let A be an atom of G. Then $|A| \leq \lfloor \frac{\kappa}{4} \rfloor$. Clearly, if $|A| \geq 2$, as the subgraph induced by A is connected and every vertex in A has degree at most $\lfloor \frac{5\kappa}{4} \rfloor - 1$, then G has a pair of adjacent vertices of degree at most $\lfloor \frac{5\kappa}{4} \rfloor - 1$. If $|A| = 1$ (and thus $\delta(G) = \kappa$), the above result is not known. Recently, Kriesell ([3]) proved that a contraction-critical κ connected graph G has two vertices x, y of distance one or two such that $d(x) + d(y) \leq 2\lfloor \frac{5\kappa}{4} \rfloor - 2$. It was improved to ([7]) that a contraction-critical κ connected graph G has two adjacent vertices x, y such that $d(x) + d(y) \leq 2\lfloor \frac{5\kappa}{4} \rfloor - 2$. So we may expect that G has a pair of adjacent vertices x, y such that both of x, y has lower degree. In this direction, we show in this paper the following result.

Theorem 1 *Let G be a contraction-critical κ connected graph. Then, G has two vertices of degree κ, or G has two adjacent vertices x, y such that* $\max\{d(x), d(y)\} \leq \lfloor \frac{5k}{4} \rfloor - 1$ *or such that $d(x) = \kappa$ and $d(y) \leq \lfloor \frac{4k}{3} \rfloor - 1$.*

For the extremal case of the minimum degree, Su obtained the following result.

Theorem 2 *([6], Theorem 2 and Corollary 2) Let G be a contraction-*

critical κ connected graph. If $\delta(G) = \frac{5\kappa}{4} - 1$ (and thus $\kappa = 0$ mod 4), then G has four disjoint atoms of size $\frac{\kappa}{4}$. Hence G has at least κ vertices of degree $\frac{5\kappa}{4} - 1$.

Su guessed that the number 'four' in Theorem 2 can be replaced by 'six'. If so, the result is best possible as shown in [6]. We prove that this is true.

Theorem 3 *Let G be a contraction-critical κ connected graph. If $\delta(G) = \frac{5\kappa}{4} - 1$, then G has six disjoint atoms of size $\frac{\kappa}{4}$, and hence G has at least $\frac{3\kappa}{2}$ vertices of degree $\frac{5\kappa}{4} - 1$.*

We will give the proof of Theorem 1 and Theorem 3 in section 3.

2 Some Properties of Fragments

We have the following known properties of fragments.

Lemma 1 *([4]) Let F and F' be two fragments of G, $T = N(F), T' = N(F')$.*

(1) If $F \cap F' \neq \emptyset$, then $|F \cap T'| \geq |\overline{F'} \cap T|$, $|F' \cap T| \geq |\overline{F} \cap T'|$.

(2) If $F \cap F' \neq \emptyset$ and $|N(F \cap F')| > \kappa(G)$, then $|F \cap T'| > |\overline{F'} \cap T|$, $|F' \cap T| > |\overline{F} \cap T'|$.

(3) If $F \cap F' \neq \emptyset \neq \overline{F} \cap \overline{F'}$, then both $F \cap F'$ and $\overline{F} \cap \overline{F'}$ are fragments of G, and $N(F \cup F') = (T \cap T') \cup (T \cap \overline{F'}) \cup (\overline{F} \cap T') = N(\overline{F} \cap \overline{F'})$.

Lemma 2 *([4]) Let A be an atom of G and T be a $\kappa(G)$-vertex-cut of G. If $A \cap T \neq \emptyset$, then $A \subseteq T$.*

Lemma 3 *([3]) Let G be a contraction-critical κ connected graph and A be an atom of G. Then $G - A$ is almost critical graph and $\kappa(G - A) = \kappa - |A|$.*

Lemma 4 *([5]) Let G be an almost critical κ connected graph and A be an atom of G. Then $|A| \leq \frac{\kappa}{2}$.*

Lemma 5 *([6]) Let B be an end and F a fragment of G. If $N(F) \cap B \neq \emptyset$, then $|F \cap N(B)| \geq a(G)$.*

Lemma 6 *([6]) Let B be an end and F a fragment of G. If $N(F) \cap B \neq \emptyset$, the one of following statements holds:*

(1) $F \subseteq N(B)$ (2) $\overline{F} \subseteq N(B)$, $|F \cap N(B)| \geq |\overline{F}|$, $|\overline{F}| < |B|$.

(3) $|\overline{B}| < |F \cap N(B)|$. (4) $|B| \leq |F \cap N(B)|$.

In order to study the contraction-critical graphs, Su introduced in [6] the $N(B)$-fragments. Let B be an end of G. A fragment of G is called an $N(B)$-fragment of G if $N(F) \cap B \neq \emptyset$. If F is an $N(B)$-fragment of G, as $N(\overline{F}) = N(F)$, then $\overline{F}$ is also an $N(B)$-fragment of G.

Lemma 7 *([6], Theorem 3) Let G be a contraction-critical κ connected graph and let B be an end of G. Then G has four $N(B)$-fragments F_1, F_2, F_3 and F_4 such that F_1, F_2, F_3 and $F_4 \cap N(B)$ are pairwise disjoint.*

From Lemma 7 we can deduce the following result.

Corollary 1 *Let G be a contraction-critical κ connected graph. Let B be an end of G such that $|B| > \frac{\kappa}{4}$ and $|\overline{B}| > \frac{\kappa}{4}$. Then $N(B)$ contains a fragment F of G such that $|F| \leq \frac{\kappa}{4}$.*

Proof By Lemma 7, G has four $N(B)$-fragments F_1, F_2, F_3, F_4 such that F_1, F_2, F_3 and $F_4 \cap N(B)$ are pairwise disjoint. So, $F_1 \cap N(B), F_2 \cap N(B), F_3 \cap N(B), F_4 \cap N(B)$ are pairwise disjoint. Suppose that $|F_1 \cap N(B)| \leq |F_2 \cap N(B)| \leq |F_3 \cap N(B)| \leq |F_4 \cap N(B)|$. By $\sum_{i=1}^{4} |F_i \cap N(B)| \leq |N(B)| = \kappa$, we have $|F_1 \cap N(B)| \leq \frac{\kappa}{4}$. By Lemma 6, we have that, either $F_1 \subseteq N(B)$ and $|F_1| = |F_1 \cap N(B)| \leq \frac{\kappa}{4}$, or $\overline{F_1} \subseteq N(B)$ and $|\overline{F_1}| \leq |F_1 \cap N(B)| \leq \frac{\kappa}{4}$.

3 Proofs

Proof of Theorem 1 Let G be a contraction-critical κ connected graph. As we described in section 1, we may assume that $a(G) = 1$, and thus $\delta(G) = \kappa$. For the purpose, assuming that G has only one vertex a such that $d(a) = \kappa$, and G has no a pair of adjacent vertices x, y such that $\max\{d(x), d(y)\} \leq \lfloor \frac{5k}{4} \rfloor - 1$, we deduce that a is adjacent to a vertex y such that $d(y) \leq \lfloor \frac{4\kappa}{3} \rfloor - 1$.

Let $W = \{v \in V | \ d(v) \leq \lfloor \frac{5\kappa}{4} \rfloor - 1\}$. Clearly, if G contains a fragment F such that $a \notin F$ and $|F| \leq \lfloor \frac{\kappa}{4} \rfloor$, by the assumption, F has at least one edge, and thus has two adjacent vertices of W. Also, if G has a fragment F such that $a \in F$ and $2 \le |F| \le \lfloor \frac{\kappa}{4} \rfloor$, then F contains two adjacent vertices of W. Otherwise, F has no edges, and then F contains two vertices of degree κ. So we may assume that for each fragment F of G, if $|F| \ge 2$, then $|F| > \lfloor \frac{\kappa}{4} \rfloor$.

Let $D = V - (N(a) \cup \{a\})$. Now we deduce that D is an end of G. Clearly, D is a fragment of G. If D is not an end of G, then there is an end B of G such that $B \subseteq D$ and $B \ne D$. As $B \cup N(B) \subseteq D \cup N(a)$, $a \in \overline{B}$. Moreover, $|\overline{B}| \ge 2$, and hence $|\overline{B}| > \lfloor \frac{\kappa}{4} \rfloor$ by the assumption. As $a \notin B$, we also have $|B| > \lfloor \frac{\kappa}{4} \rfloor$. Then, by Corollary 1, $N(B)$ contains a fragment F of G such that $|F| \le \lfloor \frac{\kappa}{4} \rfloor$. As $a \in \overline{B}$, $a \notin |F|$, a contradiction.

In the follow, let D be an end of G. We claim that for any κ-vertex-cut T of G, if $T \cap D \ne \emptyset$, then $a \in T$. For otherwise, let T be a κ-vertex-cut T of G such that $T \cap D \ne \emptyset$ and $a \notin T$. Let F be a fragment of G such that $N(F) = T$ and $a \in F$. As $a \in F$, $N(a) \cap \overline{F} = \emptyset$, and hence $\overline{F} \subseteq D$. Note that $T \cap D \ne \emptyset$, $\overline{F} \ne D$, a contradiction.

Let $G' = G - \{a\}$. Then G' is $\kappa - 1$ connected. For convenience, we denote $N_{G'}(F)$ by $N'(F)$ for any fragment F of G'. Clearly, F is a fragment of G' if and only if F is a fragment of G such that $a \in N(F)$, so $N(F) = N'(F) \cup \{a\}$. Hence $N(a) \cap F \ne \emptyset$ for any fragment F of G'. So $|N'(F) \cap N(a)| \le \kappa - 2$ for any fragment F of G' as $N(a) \cap F \ne \emptyset \ne N(a) \cap \overline{F}$, and thus $N'(F) \cap D \ne \emptyset$. Let A be an atom of G'. Let $x \in N'(A) \cap D$ and $y \in N(x) \cap A$. Since G is contraction-critical, G has a κ-vertex-cut T such that $\{x, y\} \subseteq T$. Then $a \in T$ as $x \in T \cap D$. Let F be a fragment of G' such that $N'(F) = T - \{a\}$. Note that $y \in N'(F) \cap A$, we have $A \subseteq N'(F)$ by Lemma 2. Now we consider the fragments F of G' such that $N'(F) \supseteq A$. We distinguish two cases to deduce that $|A| \le \frac{\kappa}{3}$.

Case 1 $\quad A \cap D \ne \emptyset$. Let $x \in A \cap D$ and let $y \in N'(x) \cap F$ for any fragment F of G' with $N'(F) \supseteq A$, we choose a κ-vertex-cut S of G such that $S \supseteq \{x, y\}$. By our choice, $S \cap D \ne \emptyset$, so we have $a \in S$ and $A \subseteq S - \{a\}$. Hence, $S_1 := S - (A \cup \{a\})$ is a vertex-cut of $\kappa - |A| - 1$ in $G' - A$. By regarding F as a fragment of $G' - A$, we have that $G' - A$ has a vertex-cut S_1 of $\kappa - |A| - 1$ such that $y \in S_1 \cap F, S_1 \cap F \ne \emptyset$. Note that $G' - A$ is $\kappa - |A| - 1$ connected and each fragment F of $G' - A$ is a fragment of G' such that $N'(F) \supseteq A$. So $G' - A$ is almost critical. Let C

be an atom of $G' - A$. Then $|C| \leq \frac{\kappa - |A| - 1}{2}$ by Lemma 4. As $|A| \leq |C|$, we obtain $|A| \leq \frac{\kappa}{3}$.

Case 2 $\quad A \cap D = \emptyset$. Then $A \subseteq N(a)$.

If any end B of $G' - A$ satisfies that $N'(A) \cap (B \cap D) = \emptyset$, then $N'(A) \cap B \subseteq N(a)$. If $B \cap D = \emptyset$, then $N(a) \cap B = B$. As B is a fragment of G', $|B| \geq |A|$, and thus $|N(a) \cap B| = |B| \geq |A|$. If $B \cap D \neq \emptyset$, as B is a fragment of G', then $(N(a) \cap B) \cup (N'(B) \cap D) \cup (N(a) \cap N'(B) - A)$ is a vertex-cut of G', and thus $|(N(a) \cap B)| + |(N'(B) \cap D)| + |N(a) \cap N'(B)| - |A| \geq N'(B)$. Hence, $|N(a) \cap B| \geq |A|$. Now we choose an end B of $G' - A$ and an end B' of $G' - A$ contained in $\overline{B}$. Then, by the assumption, we have $|N(a) \cap B| \geq |A|$ and $|N(a) \cap B'| \geq |A|$. So, $3|A| \leq |N(a) \cap B| + |A| + |N(a) \cap B'| \leq |N(a)| = \kappa$, and thus $|A| \leq \frac{\kappa}{3}$.

If there is an end B of $G' - A$ such that $N'(A) \cap (B \cap D) \neq \emptyset$, then we choose a vertex $z \in N'(A) \cap (B \cap D)$ and a vertex $t \in N'(z) \cap A$ and a κ-vertex-cut R of G such that $R \supseteq \{z, t\}$. Let U be a fragment of G such that $N(U) = R$. Note that $R \cap D \neq \emptyset$ and $R \cap A \neq \emptyset$, $R_1 := R - (A \cup \{a\})$ is a $\kappa - |A| - 1$-vertex-cut of $G' - A$. So, U is a fragment of $G' - A$ such that $N_{G'-A}(U) = R_1$. Since $z \in R_1 \cap B$, $R_1 \cap B \neq \emptyset$. By using Lemma 5 and Lemma 6, we can deduce that one of $B, \overline{B}, U, \overline{U}$ has at most $\frac{\kappa - |A| - 1}{2}$ vertices. Clearly, each of them is a fragment of G'. So we have $|A| \leq \frac{\kappa - |A| - 1}{2}$, and then $|A| \leq \frac{\kappa}{3}$.

As each vertex of A has degree at most $\lfloor \frac{\kappa}{3} \rfloor - 1$ and $N(a) \cap A \neq \emptyset$, there is a vertex $y \in N(a) \cap A$ such that $d(y) \leq \lfloor \frac{4\kappa}{3} \rfloor - 1$. ∎

The proof of Theorem 3 By contradiction. Since $\frac{5\kappa}{4} - 1 = \delta(G) \leq a(G) + \kappa - 1 \leq \frac{5\kappa}{4} - 1$, we have $a(G) = \frac{\kappa}{4}$. It follows that the subgraph of G induced by an atom is a complete graph. Suppose that G has at most five disjoint atoms of cardinality $\frac{\kappa}{4}$, we deduce a contradiction.

(3.1) Every end of G is an atom of G.

Proof Suppose that (3.1) is not true. Let B is an end of G such that $|B| > a(G) = \frac{\kappa}{4}$. By Lemma 7, G has four $N(B)$-fragments F_1, F_2, F_3, F_4 such that F_1, F_2, F_3 and $F_4 \cap N(B)$ are pairwise disjoint. Since F_i is $N(B)$-fragment for $i = 1, 2, 3, 4$, we have $N(F_i) \cap B \neq \emptyset$, by Lemma 5, $|F_i \cap N(B)| \geq a(G)$. So we have $\kappa = 4a(G) \leq \sum_{i=1}^{4} |F_i \cap N(B)| \leq |N(B)| = \kappa$, implying that $|F_i \cap N(B)| = a(G) = \frac{\kappa}{4}$, and hence $|B| > |F_i \cap N(B)|$ for $i = 1,2,3,4$. For the end B and the fragment F_i for each $i \in \{1,2,3,4\}$, by noting that $|F_i \cap N(B)| = a(G) < |B|$ and $|\overline{B}| \geq a(G)$ and $|\overline{F_i}| \geq a(G) = |F_i \cap N(B)|$, Lemma 6 implies that either $F_i \subseteq N(B)$, or $\overline{F_i} \subseteq N(B)$ and $|F_i \cap N(B)| \geq |\overline{F_i}|$ and $|\overline{F_i}| < |B|$. We claim that $F_i \subseteq N(B)$.

For otherwise, either $F_i \cap B \neq \emptyset$ or $F_i \cap \overline{B} \neq \emptyset$. Moreover, $\overline{F_i} \subseteq N(B)$ and $|F_i \cap N(B)| \geq |\overline{F_i}|$ and $|\overline{F_i}| < |B|$. Then $|F_i \cap N(B)| = |\overline{F_i}| = |\overline{F_i} \cap N(B)| = \frac{\kappa}{4}$. This follows that $|N(B) \cap N(F_i)| = \frac{\kappa}{2}$. If $F_i \cap B \neq \emptyset$, as B is an end of G, then $|N(F_i) \cap B| > |\overline{F_i} \cap N(B)| = \frac{\kappa}{4}$, and thus $|N(F_i) \cap \overline{B}| < \frac{\kappa}{4} = |\overline{F_i} \cap N(B)|$, implying $F_i \cap \overline{B} = \emptyset$ and $|\overline{B}| = |N(F_i) \cap \overline{B}| < \frac{\kappa}{4}$, a contradiction. Hence, $F_i \cap B = \emptyset$ and $F_i \cap \overline{B} \neq \emptyset$. Thus $B = N(F_i) \cap B$. As $|B| > |\overline{F_i}|$, we still have $|N(F_i) \cap B| > |\overline{F_i} \cap N(B)| = \frac{\kappa}{4}$, then, by using the same argument as above, we have $F_i \cap \overline{B} = \emptyset$, also a contradiction.

Hence, $F_i \subseteq N(B)$ and thus $|F_i| = |F_i \cap N(B)| = \frac{\kappa}{4}$ for $i \in \{1,2,3,4\}$. This follows that F_i for $i \in \{1,2,3,4\}$ is an atom of G. So $N(B)$ contains four disjoint atoms F_1, F_2, F_3, F_4.

In the below, we assume that the four atoms contained in $N(B)$ are A_1, A_2, A_3, A_4, so $N(B) = \bigcup_{i=1}^{4} A_i$. Let T be a κ-vertex-cut T of G and F a fragment of G with $N(F) = T$. By Lemma 2, if $A_i \cap T \neq \emptyset$, then $A_i \subseteq T$ for each $i \in \{1,2,3,4\}$, and if $A_i \cap F \neq \emptyset$ (or $A_i \cap \overline{F} \neq \emptyset$), then $A_i \subseteq F$ ($A_i \subseteq \overline{F}$, respectively) for each $i \in \{1,2,3,4\}$.

Note that if an atom A of G contains a vertex x which is adjacent to a vertex x' of an atom A' of G, then $N(A) \cap A' \neq \emptyset$ and $N(A') \cap A \neq \emptyset$, by Lemma 2, $A' \subseteq N(A)$ and $A \subseteq N(A')$. We define that two atoms A, A' of G are adjacent, if $A' \subseteq N(A)$ and $A \subseteq N(A')$.

Pick any vertex $x \in N(A_1) \cap B$, $y \in N(x) \cap A_1$. Since G is contraction-critical, there is a κ-vertex-cut T of G such that $T \supseteq \{x, y\}$. Let F be a fragment of G with $N(F) = T$. So $A_1 \subseteq T$.

For the moment we assume that $N(B) \cap F = \emptyset$. As $x \in B \cap T$, then $F \cap \overline{B} = \emptyset$, hence $F \cap B$ is a fragment, contradicting that B is an end. Thus $N(B) \cap F \neq \emptyset$. Similarly we have $N(B) \cap \overline{F} \neq \emptyset$. Then, as we state above, we may assume $A_2 \subseteq N(B) \cap F$, $A_3 \subseteq N(B) \cap \overline{F}$. We consider A_4.

If $A_4 \subseteq N(B) \cap T$, then $|N(B) \cap T| = |A_1 \cup A_4| = \frac{\kappa}{2}$. If $|B \cap T| = |\overline{B} \cap T| = \frac{\kappa}{4}$, then $|(B \cap T) \cup (N(B) - \overline{F})| = \kappa$. As B is an end, $F \cap B = \emptyset$. Similarly we can deduce that $\overline{F} \cap B = \emptyset$, and thus $B = B \cap T$, implying that $|B| = a(G)$, a contradiction. If $|B \cap T| > |\overline{B} \cap T|$, then $|B \cap T| > \frac{\kappa}{4}$, $|\overline{B} \cap T| < \frac{\kappa}{4}$. Then $|(N(B) \cap F) \cup (T - B)| < \kappa$, and thus $\overline{B} \cap F = \emptyset$. By the same argument, we have $\overline{B} \cap \overline{F} = \emptyset$. Hence, $|\overline{B}| = |\overline{B} \cap T| < \frac{\kappa}{4}$, a contradiction. By symmetry, $|B \cap T| < |\overline{B} \cap T|$ is also impossible, implying that $A_4 \not\subseteq N(B) \cap T$. So $A_4 \subseteq F \cap N(B)$ or $A_4 \subseteq \overline{F} \cap N(B)$.

By symmetry, we may assume that $A_4 \subseteq N(B) \cap \overline{F}$. Now $N(B) \cap T = A_1$, $N(B) \cap F = A_2$, $N(B) \cap \overline{F} = A_3 \cup A_4$. If $B \cap F \neq \emptyset$, as B is an end, $|B \cap T| \geq \frac{\kappa}{2} + 1$, $|\overline{B} \cap T| \leq \frac{\kappa}{4} - 1$, and thus $|(A_3 \cup A_4 \cup A_1) \cup (\overline{B} \cap T)| \leq \kappa - 1$, and $|(A_1 \cup A_2) \cup (\overline{B} \cap T)| < \kappa - 1$, implying that $\overline{B} \cap F = \emptyset$ and $\overline{B} \cap \overline{F} = \emptyset$, and thus $|\overline{B}| = |\overline{B} \cap T| \leq \frac{\kappa}{4} - 1$, contradict $a(G) = \frac{\kappa}{4}$. Hence, $B \cap F = \emptyset$. On the other hand, if $\overline{B} \cap F \neq \emptyset$, then $|B \cap T| \leq |F \cap N(B)| = \frac{\kappa}{4}$, and so $B \cap \overline{F} \neq \emptyset$. From that we have $B \cap \overline{F}$ is a fragment of G. But B is an end, and $B \cap F \subseteq B - \{x\}$, a contradiction. So $\overline{B} \cap F = \emptyset$. So, $F = F \cap N(B) = A_2$, and $T = N(A_2) \supseteq A_1$. It follows that A_1, A_2 are adjacent and $N(A_1) \cap B \subseteq N(A_2) \cap B$. Note that $F = A_2$ and $A_3 \cup A_4 \subseteq \overline{F}$, A_2 is not adjacent to A_3, A_4.

Let $u \in B \cap T = B \cap N(A_2)$, $v \in N(u) \cap A_2$. Since G is contraction-critical and $uv \in E(G)$, there is a κ-vertex-cut T_1 of G such that $T_1 \supseteq \{u, v\}$. Then $A_2 \subseteq N(B) \cap T_1$ by Lemma 2. Moreover, we have $N(B) \cap T_1 = A_2$ by using the same argument as above. Let F_1 be a fragment of G such that $N(F_1) = T_1$ and $F_1 \supseteq A_1$. As A_2 is not adjacent to A_3, A_4, in view of the above proof we have $F_1 = N(B) \cap F_1 = A_1$ and $N(B) \cap \overline{F_1} = A_3 \cup A_4$. Hence $N(A_1) = T_1$, and thus $u \in T_1 \cap B = N(A_1) \cap B$. It follows that $N(A_2) \cap B \subseteq N(A_1) \cap B$. Combining the fact $N(A_1) \cap B \subseteq N(A_2) \cap B$, we have $N(A_1) \cap B = N(A_2) \cap B$. Note that $F_1 = A_1$ and $A_3 \cup A_4 \subseteq \overline{F_1}$, A_1 is not adjacent to A_3, A_4.

Pick any vertex $x \in N(A_3) \cap B$ and $y \in N(x) \cap A_3$. By noting that both of A_1, A_2 are not adjacent to A_3, A_4, we can similarly deduce that $N(A_3) \cap B = N(A_4) \cap B$ and A_3, A_4 are adjacent.

Summarizing above results, we have that A_1, A_2 are adjacent, A_3, A_4 are adjacent, and both of A_1, A_2 are adjacent to neither of A_3, A_4. Moreover, $N(A_1) \cap B = N(A_2) \cap B$ and $N(A_3) \cap B = N(A_4) \cap B$.

Claim 1 $B \subseteq N(A_1), B \subseteq N(A_3)$. For otherwise, we may assume that $B - N(A_1) \neq \emptyset$, so $N(A_2) \cap (B - N(A_1)) = \emptyset$. Then, $(N(A_1) \cap B) \cup (A_3 \cup A_4)$ is a vertex-cut of G. It follows that $|N(A_1) \cap B| \geq \frac{\kappa}{2}$, and thus $|N(A_1) \cap \overline{B}| \leq$

$\frac{\kappa}{4}$. Clearly, $|N(A_1) \cap \overline{B}| \geq |A_1| = \frac{\kappa}{4}$ (for otherwise, $\overline{B} - N(A_1) \neq \emptyset$, and thus $(N(A_1) \cup \overline{B}) \cup A_2 \cup A_3 \cup A_4$ is a vertex-cut of G with less than κ vertices, a contradiction). So $|N(A_1) \cap \overline{B}| = \frac{\kappa}{4}$, and thus $|N(A_1) \cap B| = \frac{\kappa}{2}$. It follows that $(N(A_1) \cap B) \cup (A_3 \cup A_4)$ is a vertex-cut of G with κ vertices, and thus $B - N(A_1)$ is a fragment of G, contradict that B is an end of G.

By Claim 1, $\overline{A_1} \cap B = \emptyset$. As $A_2 \subseteq N(A_1)$ and that A_1 is not adjacent to A_3, A_4, $N(A_2) \cap \overline{A_1} \subseteq \overline{A_1} \cap \overline{B}$, and thus $N(A_2) \cap (\overline{A_1} \cap \overline{B}) \neq \emptyset$, implying that $N(A_2) \cap \overline{B} \not\subseteq N(A_1) \cap \overline{B}$. We can similarly deduce that $N(A_1) \cap \overline{B} \not\subseteq N(A_2) \cap \overline{B}$, $N(A_3) \cap \overline{B} \not\subseteq N(A_4) \cap \overline{B}$, $N(A_4) \cap \overline{B} \not\subseteq N(A_3) \cap \overline{B}$.

Pick a vertex $x \in N(A_1) \cap \overline{B} - A_2$, $y \in N(x) \cap A_1$, and let T be a κ-vertex-cut of G such that $T \supseteq \{x, y\}$. By Lemma 2, $A_1 \subseteq T$.

Claim 2 $T \cap B = \emptyset$. For otherwise, $T \cap B \neq \emptyset$. Let F be a fragment of G such that $N(F) = T$.

(i) If $F \cap B \neq \emptyset \neq \overline{F} \cap B$, as B is an end of G, then $\overline{F} \cap \overline{B} = \emptyset = F \cap \overline{B}$, and $|F \cap N(B)| > |T \cap \overline{B}| \geq \frac{\kappa}{4}$ and $|\overline{F} \cap N(B)| > |T \cap \overline{B}| \geq \frac{\kappa}{4}$. Note that $A_1 \subseteq T$, it implies that $F \cap N(B)$ contains at least two of A_2, A_3, A_4 and $\overline{F} \cap N(B)$ contains at least two of A_2, A_3, A_4, a contradiction.

(ii) If $F \cap B \neq \emptyset$ and $\overline{F} \cap B = \emptyset$, then $\overline{F} \cap \overline{B} = \emptyset$, and thus $\overline{F} = \overline{F} \cap N(B)$. If $\overline{F}$ contains at least two of A_2, A_3, A_4, then $|B \cap T| > |\overline{F}| \geq \frac{\kappa}{2}$, implying that $|\overline{B} \cap T| < \frac{\kappa}{4} < |\overline{F}|$, and thus $F \cap \overline{B} = \emptyset$, and hence $|\overline{B}| = |\overline{B} \cap T| < \frac{\kappa}{4}$, a contradiction. So $\overline{F}$ contains exact one of A_2, A_3, A_4. If $\overline{F} = A_2$, then $x \in T \cap \overline{B} = N(A_2) \cap \overline{B}$, contradict the choice of x. Hence, $\overline{F} = A_3$ or A_4. We may assume $\overline{F} = A_3$, then $T = N(A_3) \supseteq A_1$, contradict the fact that A_1 is not adjacent to A_3. We can similarly deduce a contradiction if $F \cap B = \emptyset$ and $\overline{F} \cap B \neq \emptyset$.

By (i) and (ii), we have $B \subseteq T$, and thus $|B \cap T| = |B| > \frac{\kappa}{4}$. Then we can similarly deduce that $|\overline{B}| < \frac{\kappa}{4}$ if we assume $T \cap N(B)$ contains two of A_1, A_2, A_3, A_4. So $T \cap N(B) = A_1$. We may assume that $A_2 \subseteq F \cap N(B)$. Now if $\overline{F} \cap N(B) = A_3$ or A_4, then $|\overline{F} \cap N(B)| < |T \cap B| = |B|$, implying that $\overline{F} = \overline{F} \cap N(B) = A_3$ or A_4, and thus that A_1 is adjacent to A_3 or A_4, a contradiction. If $\overline{F} \cap N(B) = A_3 \cup A_4$, then $F \cap N(B) = A_2$, similarly we can deduce that $F = F \cap N(B) = A_2$, and thus $x \in T \cap \overline{B} = N(A_2) \cap \overline{B}$, contradict the choice of x. This proves Claim 2.

By Claim 2, $T \cap B = \emptyset$. As B is an end of G, $B \subseteq F$ or $B \subseteq \overline{F}$. We may suppose $B \subseteq \overline{F}$, then $F = F \cap \overline{B}$. Let $B' \subseteq F \cap \overline{B}$ be an end of G. Note that $x \in T \cap \overline{B}$, we have that $\overline{F} \cap N(B) \neq \emptyset$. It follows that $\overline{F} \cap N(B)$ contains at least one of A_2, A_3, A_4.

Claim 3 B' is an atom of G. For otherwise, $|B'| > \frac{\kappa}{4}$. We use the same argument by substituting B by B', we obtain that $N(B')$ contains four disjoint atoms of G. By the assumption that G has at most five disjoint atoms, we may suppose that $N(B') = A_1 \cup A_2 \cup A_3 \cup A_5$ and $A_5 \subseteq \overline{B}$. Since A_1, A_2 are adjacent, A_3, A_5 are adjacent. Clearly, Claim 1 holds also for B', i.e. $B' \subseteq N(A_3)$. Hence, $|N(A_3)| \geq |B| + |A_4| + |A_5| + |B'| > \kappa$, a contradiction.

By Claim 3, B' is an atom of G, we denote $B' = A_5$. By Lemma 3, $G - A_5$ is almost critical graph. By Lemma 4, $G - A_5$ has a fragment F such that $|F| \leq \frac{\kappa - |A|}{2} = \frac{3\kappa}{8}$. Since F is also a fragment of G such that $N(F) = N_{G-A}(F) \cup A$, there is an end B_1 of G such that $B_1 \subseteq F$. As $|B_1| \geq \frac{\kappa}{4}$, then $|F - B_1| \leq \frac{3\kappa}{8} - \frac{\kappa}{4} = \frac{\kappa}{8}$. If $N(B_1) \cap A_5 = \emptyset$, then $N(B_1) \subseteq (F - B_1) \cup (N(F) - A_5)$, and thus $|N(B_1) \leq \frac{\kappa}{8} + (\kappa - \frac{\kappa}{4}) < \kappa$, a contradiction. Hence, $N(B_1) \cap A_5 \neq \emptyset$, and thus $A_5 \subseteq N(B_1)$. If B_1 is an atom of G, then, by the assumption, B_1 is one of A_1, A_2, A_3, A_4, and hence A_5 is adjacent to one of A_1, A_2, A_3, A_4. If $|B_1| > \frac{\kappa}{4}$, then we use the same argument by substituting B by B_1, we obtain that $N(B_1)$ contains four disjoint atoms of G, and that $A_5 \subseteq B_1$ is adjacent to another atom contained in B_1. It follows that A_5 is also adjacent to one of A_1, A_2, A_3, A_4. We may assume that A_5 is adjacent to A_1.

Claim 4 $A_5 \subseteq N(A_2)$. Since we assume that A_1, A_5 are adjacent, we have a κ-vertex-cut T of G such that $T \supseteq A_1 \cup A_5$. Then, $T \cap B \neq \emptyset$ (for otherwise, we can deduce, as in the proof of Claim 3, that one fragment F of G with $N(F) = T$ satisfies that $F \subseteq \overline{B}$, and thus $\overline{B}$ has an atom which is disjoint with A_5, contradict our assumption). If $|T \cap N(B)| = \frac{\kappa}{2}$, then $|T \cap B| = |F \cap N(B)| = |\overline{F} \cap N(B)| = \frac{\kappa}{4}$, by assuming $F \cap B \neq \emptyset$, then we have that $F \cap B$ is a fragment of G, a contradiction. So $|T \cap N(B)| < \frac{\kappa}{4}$, and thus $T \cap N(B) = A_1$. We may assume that $|F \cap N(B)| = \frac{\kappa}{4}$ and $|\overline{F} \cap N(B)| = \frac{\kappa}{2}$. Then, $F \cap B = \emptyset$ as $|F \cap N(B)| = |A_5| \leq |T \cap \overline{B}$ and that B is an end of G. On the other hand, if $B \cap \overline{F} \neq \emptyset$, then we have $F \cap \overline{B} = \emptyset$; if $B \cap \overline{F} = \emptyset$, then $|B \cap T| = |B| > \frac{\kappa}{4} = |F \cap N(B)|$, and thus $F \cap \overline{B} = \emptyset$.

So $F \cap B = \emptyset = F \cap \overline{B}$, it follows that $F \subseteq N(B)$. As A_1 is not adjacent to A_3, A_4 and $A_1 \subseteq T$, $F \cap N(B) = A_2$, implying that $A_5 \subseteq N(A_2)$. This proves Claim 4.

By Claim 4 we have that $A_1 \cup A_2 \subseteq N(A_5)$. Pick a vertex $x_3 \in N(A_3) \cap \overline{B} - N(A_4)$ and a vertex $y_3 \in N(x_3) \cap A_3$. Then there is a κ-vertex-cut T_3 of G such that $T_3 \supseteq \{x_3, y_3\}$. As Claim 2 shows, $T_3 \cap B = \emptyset$. We suppose $B \subseteq \overline{F_3}$ for a fragment F_3 of G with $N(F_3) = T_3$. Then, $A_5 \subseteq F_3, A_3 \subseteq T_3$ and $\overline{F_3} \cap N(B) \neq \emptyset$ by Claim 2 and Claim 3. Similarly, we pick a vertex $x_4 \in N(A_4) \cap \overline{B} - N(A_3)$ and a vertex $y_4 \in N(x_4) \cap A_4$. Then there is a κ-vertex-cut T_4 of G such that $T_4 \supseteq \{x_4, y_4\}$. Then $T_4 \cap B = \emptyset$ by Claim 2. By supposing $B \subseteq \overline{F_4}$ for a fragment F_4 of G with $N(F_4) = T_4$. Then, $A_5 \subseteq F_4, A_4 \subseteq T_4$ and $\overline{F_4} \cap N(B) \neq \emptyset$. We consider the fragments F_3, F_4.

Since $A_5 \subseteq F_3 \subseteq \overline{B}$ and $A_1 \cup A_2 \subseteq N(A_5)$, we have $T_3 \supseteq A_1 \cup A_2 \cup A_3$, and thus $\overline{F_3} \cap N(B) = A_4$. Similarly, we have that $T_4 \supseteq A_1 \cup A_2 \cup A_4$ and $\overline{F_4} \cap N(B) = A_3$. As $x_3 \in N(A_3)$ and $A_3 \subseteq \overline{F_4}$, $x_3 \in T_3 - F_4$. Similarly, $x_4 \in T_4 - F_3$.

Note that $F_3 \cap F_4 \subseteq A_5 \neq \emptyset$ and $\overline{F_3} \cap \overline{F_4} \supseteq B \neq \emptyset$. So, $\overline{F_3} \cap \overline{F_4}$ is a fragment of G and $N(\overline{F_3} \cap \overline{F_4}) = (T_3 - F_4) \cup (T_4 - F_3)$. Let $T^* = (T_3 - F_4) \cup (T_4 - F_3)$ and $F^* = F_3 \cup F_4$. Then $N(F^*) = T^*$ and $\overline{F^*} = \overline{F_3} \cap \overline{F_4}$. Clearly, $A_5 \subseteq F^*$ and $B \subseteq \overline{F^*}$. Moreover, $(\bigcup_{i=1}^4 A_i) \cup \{x_3, x_4\} \subseteq T^*$. So $\overline{F^*} \cap N(B) = \overline{F^*} \cap (\bigcup_{i=1}^4 A_i) = \emptyset$. On the other hand, as $B \cap \overline{F^*} = B$, $|\overline{F^*} \cap N(B)| \geq |T^* \cap \overline{B}| \geq |\{x_3, x_4\}|$, a contradiction. This proves (3.1).

(3.2) Every atom of G is adjacent to another atom of G.

Proof Let A be an atom of G, then $G - A$ is almost critical graph by Lemma 3. By Lemma 4, $G - A$ has a fragment F such that $|F| \leq \frac{\kappa - |A|}{2} = \frac{3\kappa}{8}$. Since F is also a fragment of G such that $N(F) = N_{G-A}(F) \cup A$, by (3.1), there is an atom A' of G such that $A' \subseteq F$. If A' is not adjacent to A, then $N(A') \cap A = \emptyset$, and thus $N(A') \subseteq (N(F) - A) \cup (F - A')$, hence $|N(A')| \leq (\kappa - \frac{\kappa}{4}) + (\frac{3\kappa}{8} - \frac{\kappa}{4}) = \frac{7\kappa}{8} < \kappa$, which contradicts $|N(A')| = \kappa$. Hence, $N(A') \cap A \neq \emptyset$, implying that A is adjacent to A'.

For the moment, we assume that G has only four disjoint atoms A_1, A_2, A_3, A_4. By (3.2), A_1, A_2, A_3 and A_4 satisfy one of the following properties:

(a) There is an atom which is adjacent to all other three atoms.

(b) There are two pairs of atoms such that every pair of atoms are adjacent.

In the case of (a), we may assume that A_1 is adjacent to A_2, A_3, A_4. We obtain $N(A_1) \supseteq A_2 \cup A_3 \cup A_4$, then $\overline{A_1}$ contains an end which is also an atom by (3.1), a contradiction.

In the case of (b), we may assume that A_1 is adjacent to A_2, A_3 is adjacent to A_4. We consider an edge $xy \in E(G)$ such that $x \in A_1$ and $y \in A_2$. Since G is contraction-critical, G has an κ-vertex-cut $T \supseteq \{x, y\}$, then $T \supseteq A_1 \cup A_2$ by Lemma 2. Since A_3 is adjacent to A_4, there is a fragment F of G such that $N(F) = T$ and $F \cap (A_3 \cup A_4) = \emptyset$. Then F contains an atom of G by (3.1), a contradiction.

So G has at least five disjoint atoms. By our assumption, we suppose that A_1, A_2, A_3, A_4, A_5 are all atoms of G. By using the reason of (a), each atom of G is adjacent to at most three of the other atoms of G.

Claim 5 Each atom of G is adjacent to at most two of the other atoms of G.

For otherwise, we may assume that A_1 is adjacent to A_2, A_3, A_4, and $A_5 \subseteq \overline{A_1}$ is adjacent to A_4. By Lemma 7, G has four $N(A_1)$-fragments F_1, F_2, F_3, F_4 such that $F_1 \cap N(A_1), F_2 \cap N(A_1), F_3 \cap N(A_1), F_4 \cap N(A_1)$ are pairwise disjoint. As in the proof of first paragraph of (3.1), we can deduce that $|F_i \cap N(A_1)| = \frac{\kappa}{4}$ for each $i \in \{1, 2, 3, 4\}$. Clearly, there is a $i \in \{1, 2, 3, 4\}$ that $F_i \cap N(A_1)$ does not contains any of A_2, A_3, A_4. We may suppose that $(F_1 \cap N(A_1)) \cap (A_2 \cup A_3 \cup A_4) = \emptyset$. Then, $F_1 \cap \overline{A_1} \neq \emptyset$ (for otherwise, $F_1 = F_1 \cap N(A_1)$ is an atom of G which is different to A_2, A_3, A_4, a contradiction). It follows that $F_1 \cap \overline{A_1}$ is a fragment of G. As this fragment contains one end of G, by (3.1) and our assumption, $A_5 \subseteq F_1 \cap \overline{A_1}$. As we assume that A_5 is adjacent to A_4, $A_4 \subseteq N(F_1) \cap N(A_1)$.

Note that $N(A_1) - F_1 = A_2 \cup A_3 \cup A_4$. If $N(F_1) \cap N(A_1) = A_3 \cup A_4$, then $\overline{F_1} \cap N(A_1) = A_2$. Thus $\overline{F_1} \cap \overline{A_1} = \emptyset$ (for otherwise, by the same reasoning as in last paragraph, we can deduce that $A_5 \subseteq \overline{F_1} \cap \overline{A_1}$, a contradiction). So, $\overline{F_1} = \overline{F_1} \cap N(A_1) = A_2$. It follows that A_2 is adjacent to A_3, A_4. Now we pick a κ-vertex-cut T' such that $T' \supseteq A_4 \cup A_5$. Note that A_1, A_2, A_3 are pairwise adjacent, there is a fragment F' of G such that $N(F') = T'$ and $F' \cap (A_1 \cup A_2 \cup A_3) = \emptyset$, and thus F' contains an atom of G by (3.1), a contradiction. If $N(F_1) \cap N(A_1) = A_2 \cup A_4$, then we similarly deduce a contradiction.

Hence $N(F_1) \cap N(A_1) = A_4$. It follows that $\overline{F_1} \cap N(A_1) = A_2 \cup A_3$ and A_5 is not adjacent to A_2, A_3. Denote $F_1 \cap N(A_1) = D$.

We still pick a κ-vertex-cut T' of G such that $T' \supseteq A_4 \cup A_5$. We claim that $A_1 \subseteq T'$. For otherwise, we may assume that A_1 is contained in a fragment F' of G with $N(F') = T'$. Then we can deduce that $\overline{F'} \cap \overline{A_1}$ is a fragment of G, which contains an atom of G by (3.1). This contradict $A_5 \subseteq T'$. Let F' be a fragment of G with $N(F') = T'$, then $F' \cap N(A_1) \neq \emptyset$ as $A_1 \subseteq T'$. We claim that $\frac{\kappa}{4} < |F' \cap N(A_1)| < \frac{\kappa}{2}$. For otherwise, we have $|F' \cap N(A_1)| \leq \frac{\kappa}{4}$ or $|F' \cap N(A_1)| \geq \frac{\kappa}{2}$. If $|F' \cap N(A_1)| \geq \frac{\kappa}{2}$, as $N(F') \cap N(A_1) = A_4$, then $|\overline{F'} \cap N(A_1)| \leq \frac{\kappa}{4}$. So we always have that either $|F' \cap N(A_1)| \leq \frac{\kappa}{4}$ or $|\overline{F'} \cap N(A_1)| \leq \frac{\kappa}{4}$. Without loss of the generality, we assume that $|F' \cap N(A_1)| \leq \frac{\kappa}{4}$. Then $F' \cap \overline{A_1} = \emptyset$ (for otherwise, we can deduce that $F' \cap \overline{A_1}$ is a fragment of G, which contains an atom of G by (3.1), by noting that $A_5 \subseteq T'$, a contradiction). Hence, $|F'| = |F' \cap N(A_1)| \leq \frac{\kappa}{4}$, implying that $F' \subseteq N(A_1)$ is an atom of G. So $F' \cap N(A_1) = A_2$ or A_3 as $A_4 \subseteq T'$, it follows that A_2 or A_3 is adjacent to A_5 as $A_5 \subseteq T' = N(F')$, a contradiction. This shows that $\frac{\kappa}{4} < |F' \cap N(A_1)| < \frac{\kappa}{2}$. Clearly, for $\overline{F'}$ we also have that $\frac{\kappa}{4} < |\overline{F'} \cap N(A_1)| < \frac{\kappa}{2}$. From that we have that both of $F' \cap N(A_1)$, $\overline{F'} \cap N(A_1)$ can not contain $A_2 \cup A_3$, but contain at least one of A_2, A_3.

We may assume that $A_2 \subseteq F' \cap N(A_1)$, $A_3 \subseteq \overline{F'} \cap N(A_1)$. Let $D_1 = F' \cap N(A_1) - A_2 \subseteq D$ and $D_2 = \overline{F'} \cap N(A_1) - A_3 \subseteq D$. By above results we have $D_1 \neq \emptyset \neq D_2$. Then, $D_2 \subseteq F_1 \cap \overline{F'}$ and $A_2 \subseteq F' \cap \overline{F_1}$. Since $F_1 \cap \overline{F'} \neq \emptyset \neq F' \cap \overline{F_1}$, $F_1 \cap \overline{F'}$ is a fragment of G. Let $F^* = F_1 \cap \overline{F'}$, then $T^* := N(F^*) = (T' - \overline{F_1}) \cup (N(F_1) \cap \overline{F'})$. So $A_1 \cup A_4 \cup A_5 \subseteq T^*$. Note that $D_1 \subseteq F' \cap F_1$, $A_2 \subseteq F' \cap \overline{F_1}$ and $A_3 \subseteq \overline{F'} \cap \overline{F_1}$, we have that $D_1 \cup A_2 \cup A_3 \subseteq F' \cup \overline{F_1} = \overline{F^*}$. By (3.1), F^* contains an atom of G and $F^* \cap (\bigcup_{i=1}^{5} A_i) = \emptyset$, a contradiction. This proves Claim 5.

By (3.2) and Claim 5, we may assume that A_2 is adjacent to A_1, A_3, but not adjacent to A_4, A_5, and A_4, A_5 are adjacent.

Pick a κ-vertex-cut T_1 of G such that $T \supseteq A_4 \cup A_5$, let F_1 be a fragment of G with $N(F_1) = T_1$. By (3.1), $F_1 \cap (A_1 \cup A_2 \cup A_3) \neq \emptyset$ and $\overline{F_1} \cap (A_1 \cup A_2 \cup A_3) \neq \emptyset$. As A_2 is adjacent to both of A_1, A_3, we have $A_2 \subseteq T_1$. We may assume that $A_1 \subseteq F_1$, $A_3 \subseteq \overline{F_1}$. Pick a κ-vertex-cut T_2 of G such that $T_2 \supseteq A_1 \cup A_2$. let F_2 be a fragment of G with $N(F_2) = T_2$. By (3.1), we also have that $F_2 \cap (A_3 \cup A_4 \cup A_5) \neq \emptyset$ and $\overline{F_2} \cap (A_3 \cup A_4 \cup A_5) \neq \emptyset$. As A_4, A_5 are adjacent, either $F_2 \cap (A_4 \cup A_5) = \emptyset$ or $\overline{F_2} \cap (A_4 \cup A_5) = \emptyset$. We may suppose that $\overline{F_2} \cap (A_4 \cup A_5) = \emptyset$. Then, $A_3 \subseteq \overline{F_2}$ and at least one of A_4, A_5 is contained in F_2. Suppose that $A_4 \subseteq F_2$. Then $A_5 \subseteq F_2 \cup T_2$. We distinguish two cases.

(i) $A_5 \subseteq F_2$. Then $F_1 \cap T_2 \supseteq A_1$, $F_2 \cap T_1 \supseteq A_4 \cup A_5$ and $T_1 \cap T_2 \supseteq A_2$. Note that $A_3 \subseteq \overline{F_1} \cap \overline{F_2}$, we have $\overline{F_1} \cap \overline{F_2} \neq \emptyset$. So $F_1 \cap F_2 = \emptyset$ (for otherwise, $F_1 \cap F_2$ is a fragment of G which is disjoint to $\bigcup_{i=1}^{5} A_i$, a contradiction). Moreover, $|T_2 \cap \overline{F_1}| \geq |F_2 \cap T_1| \geq \frac{\kappa}{2}$. Hence, $|F_1 \cap T_2| = |A_1| = \frac{\kappa}{4} = |A_2| = |T_1 \cap T_2$ and $|T_2 \cap \overline{F_1}| = \frac{\kappa}{2}$. Then $|\overline{F_2} \cap T_1| = |T_1 - (F_2 \cup T_2)| \leq \frac{\kappa}{4} < |T_2 \cap \overline{F_1}|$, implying that $F_1 \cap \overline{F_2} = \emptyset$. Thus $F_1 = F_1 \cap T_2 = A_1$. It follows that A_1 is adjacent to A_2, A_4, A_5 as $N(A_1) = T_1 \supseteq (A_2 \cup A_4 \cup A_5)$, contradict Claim 5.

(ii) $A_5 \subseteq T_2$. Then we have $F_1 \cap T_2 \supseteq A_1$, $F_2 \cap T_1 \supseteq A_4$ and $T_1 \cap T_2 \supseteq A_2 \cup A_5$. So $|\overline{F_1} \cap T_2| = |T_2 - (F_1 \cup T_1)| \leq \frac{\kappa}{4}$. As $A_3 \subseteq \overline{F_1} \cap \overline{F_2} \neq \emptyset$, we have $F_1 \cap F_2 = \emptyset$. Moreover, $\frac{\kappa}{4} \geq |\overline{F_1} \cap T_2| \geq |F_2 \cap T_1| \geq |A_4|$. It follows that $|\overline{F_1} \cap T_2| = \frac{\kappa}{4}$ and $F_2 \cap T_1 = A_4$. Hence $T_1 \cap T_2 = A_2 \cup A_5$, and thus $|(T_1 - \overline{F_2}) \cup (T_2 \cap \overline{F_1})| = \kappa$. If $\overline{F_1} \cap F_2 \neq \emptyset$, then $\overline{F_1} \cap F_2$ is a fragment of G with $N(\overline{F_1} \cap F_2) = (T_1 - \overline{F_2}) \cup (T_2 \cap \overline{F_1})$, and $\overline{F_1} \cap F_2$ is disjoint to $\bigcup_{i=1}^{5} A_i$, a contradiction. So $\overline{F_1} \cap F_2 = \emptyset$, and thus $F_2 = F_2 \cap T_1 = A_4$, implying that A_4 is adjacent to A_1, A_2, A_5 as $N(A_4) = T_2 \supseteq (A_1 \cup A_2 \cup A_5)$, contradicting Claim 5. This proves Theorem 3.

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