BINOMIAL IDENTITIES INVOLVING THE GENERALIZED FIBONACCI TYPE POLYNOMIALS

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ABSTRACT. We present some binomial identities for sums of the bivariate Fibonacci polynomials and for weighted sums of the usual Fibonacci polynomials with indices in arithmetic progression.

1. INTRODUCTION

The Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ are satisfied the recursion

$$u_{n+1}(x) = xu_n(x) + u_{n-1}(x)$$

with $F_0(x) = 0$, $F_1(x) = 1$, and, $L_0(x) = 2$, $L_1(x) = x$, for $n > 0$.

The bivariate Fibonacci $\{F_n(x,y)\}$ and Lucas $\{L_n(x,y)\}$ polynomial sequences are satisfied the recursion

$$U_{n+1}(x,y) = xU_n(x,y) + yU_{n-1}(x,y)$$

with $F_0(x,y) = 0$, $F_1(x,y) = 1$ and $L_0(x,y) = 2$, $L_1(x,y) = x$, for $n > 0$.

The sequences $\{F_n(x,y)\}$ and $\{L_n(x,y)\}$ can be expressed as

$$F_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n(x,y) = \alpha^n + \beta^n,$$

where α and β are roots of $\lambda^2 - x\lambda - y = 0$.

When $x = y = 1$, $L_n(1,1) = l_n$ (nth Lucas number) and $F_n(1,1) = f_n$ (nth Fibonacci number). General background material on Fibonacci type polynomials can be found in [2, 4].

In [3], the authors obtained various results for these polynomials sequences. Thus we recall that for any positive integer r, we have

$$F_{r(n+1)}(x,y) = L_r(x,y) F_{rn}(x,y) - z^r F_{r(n-1)}(x,y) \tag{1.1}$$

where $L_r(x,y) = \alpha^r + \beta^r$, $z^r = (\alpha\beta)^r = (-y)^r$.

In this paper we use F_n and L_n instead of $F_n(x,y)$ and $L_n(x,y)$, respectively.

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Let $H(t,x,y)=\sum_{n=0}^{\infty}F_{r(n+1)}t^n$. Clearly,

$$H(t,x,y)=\sum_{n=0}^{\infty}F_{r(n+1)}t^n=\frac{F_r}{1-L_rt+z^rt^2}. \tag{1.2}$$

and further

$$\begin{aligned}\frac{1}{1-L_rt+z^rt^2} &= \sum_{m=0}^{\infty}(L_r-z^rt)^m t^m=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{m}{n}(-z^r)^n L_r^{m-n}t^{m+n}\\ &= \sum_{m=0}^{\infty}\sum_{n=m}^{2m}\binom{m}{n-m}L_r^{2m-n}(-z^r)^{n-m}t^n\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n-m}{m}L_r^{n-2m}(-z^r)^m t^n. \end{aligned}\tag{1.3}$$

By (1.2-1.3), clearly

$$F_{r(n+1)}=F_r\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n-m}{m}L_r^{n-2m}y^{rm}(-1)^{m(r+1)}. \tag{1.4}$$

The following sums can be found in [10]:

$$\sum_{a_1+a_2+\cdots+a_k=n}\prod_{l=1}^{k}F_{a_l+1}(x)=\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}x^{n-2m}. \tag{1.5}$$

In [11], the author generalized the sums above as for $k,m>0$ and $n\geqslant 0$,

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n}f_{m(a_1+1)}\cdots f_{m(a_{k+1}+1)}=(-i)^{mn}\frac{f_m^{k+1}}{2^k k!}U_{n+k}^{(k)}\left(\frac{i^m}{2}l_m\right).$$

where $U_n^{(k)}(x)$ denotes the kth derivation of the Chebyshev polynomials of second kind $U_n(x)$ with respect to x.

Also in [9], the authors gave a generalization of (1.5) as

$$\sum_{a_1+a_2+\cdots+a_k=n}\prod_{l=1}^{k}F_{a_l+1}(x,y)=\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}x^{n-2m}y^m.$$

In similar meaning, some authors also derived various identities including Fibonacci and Lucas numbers (see [5, 8, 12]). In this paper, we consider both the Fibonacci polynomials and the bivariate generalized Fibonacci polynomials with indices in arithmetic progress. We generalize the results of [9].

2. Sums of Bivariate Generalized Fibonacci Polynomials

Theorem 1. *For $k,n,r>0$,*

$$\begin{aligned}&\sum_{a_1+\cdots+a_k=n}\prod_{l=1}^{k}F_{r(a_l+1)}\\ =\ & F_r^k\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}L_r^{(n-2m)}y^{rm}(-1)^{m(r+1)}\end{aligned}$$

where $a_l\geqslant 0$, for $l=1,2,\ldots,k$.

Proof. Denote the $l_1 th$ derivation of $H(t,x,y)$ according to the bivariate Lucas polynomials L_r, and, the $l_2 th$ derivation of $H(t,x,y)$ according to $(-z)$ by $\frac{\partial^{l_1+l_2} H(t,x,y)}{\partial L^{l_1} \partial(-z)^{l_2}}$. By (1.2) and induction, we write

$$\begin{aligned}
\frac{\partial^{2l+1} H(t,x,y)}{\partial L_r^{l+1} \partial(-z^r)^l} &= \frac{\partial^{2l+1}}{\partial L_r^{l+1} \partial(-z^r)^l} \left(\frac{F_r}{1-L_r t+z^r t^2} \right) = \frac{F_r (2l+1)! t^{3l+1}}{(1-L_r t+z^r t^2)^{2l+2}} \\
&= F_r \sum_{n=0}^{\infty} F_{r(n+1)}^{(l+1,l)} t^n = F_r \sum_{n=l}^{\infty} F_{r(n+2l+2)}^{(l+1,l)} t^{n+2l+1} \\
&= F_r \sum_{j=0}^{\infty} F_{r(j+3l+2)}^{(l+1,l)} t^{j+3l+1}.
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial^{2l} H(t,x,y)}{\partial L_r^{l} \partial(-z^r)^l} &= \frac{\partial^{2l}}{\partial L_r^{l} \partial(-z^r)^l} \left(\frac{F_r}{1-L_r t+z^r t^2} \right) = \frac{F_r (2l+1)! t^{3l}}{(1-L_r t+z^r t^2)^{2l+1}} \\
&= F_r \sum_{n=0}^{\infty} F_{r(n+1)}^{(l,l)} t^n = F_r \sum_{n=l}^{\infty} F_{r(n+2l+1)}^{(l,l)} t^{n+2l} \\
&= F_r \sum_{j=0}^{\infty} F_{r(j+3l+1)}^{(l,l)} t^{j+3l}.
\end{aligned}$$

Let

$$\sum_{n=0}^{\infty} t^n \sum_{a_1+a_2+\cdots+a_k=n} \prod_{l=1}^{k} F_{r(a_l+1)} = \left(\sum_{n=0}^{\infty} F_{r(n+1)} t^n \right)^k = I. \tag{2.1}$$

For $k = 2l+2$, we obtain

$$I = \frac{F_r^{2l+1}}{(2l+1)! t^{l+3l}} \frac{\partial^{2l+1} H(t,x,y)}{\partial L_r^{(l+1)} \partial(-z^r)^l} = \frac{F_r^{2l+1}}{(2l+1)!} \sum_{n=0}^{\infty} F_{r(n+3l+2)}^{(l+1,l)} t^n. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\sum_{n=0}^{\infty} \sum_{a_1+a_2+\cdots+a_k=n} \prod_{l=1}^{k} F_{r(a_l+1)} t^n = \frac{F_r^{2l+1}}{(2l+1)!} \sum_{n=0}^{\infty} F_{r(n+3l+2)}^{(l+1,l)} t^n. \tag{2.3}$$

By (1.4), we write

$$\begin{aligned}
& F_{r(n+3l+2)}^{(l+1,l)} \\
= {} & F_r \frac{\partial^{2l+1}}{\partial L_r^{(l+1)} \partial(-z^r)^l} \left(\sum_{m=0}^{\lfloor (n+3l+1)/2 \rfloor} \binom{n+3l+1-m}{m} L_r^{n+3l+1-2m} (-z^r)^m \right) \\
= {} & F_r \sum_{m=l}^{\lfloor (n+2l)/2 \rfloor} \frac{(n+3l+1-m)!}{(m-l)!(n+2l-2m)!} L_r^{(n+2l-2m)} (-z^r)^{m-l} \\
= {} & F_r \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(n+2l+1-m)! L_r^{(n-2m)} (-z^r)^m}{m!(n-2m)!} \\
= {} & F_r \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(n+k-m)! L_r^{(n-2m)} (-z^r)^m}{m!(n-2m)!}.
\end{aligned}$$

So

$$\frac{F_r^{2l+1}F_{r(n+3l+2)}^{(l+1,l)}}{(2l+1)!} = \frac{F_r^k}{k!}\sum_{m=0}^{\lfloor n/2\rfloor}\frac{(n+k-m)!L_r^{(n-2m)}(-z^r)^m}{m!(n-2m)!}$$

$$= F_r^k\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}L_r^{(n-2m)}z^{rm}(-1)^m. \quad (2.4)$$

For $k = 2l+1$, from (2.3-2.4), we have the conclusion. For $k = 2l+1$, the proof is similarly obtained. □

When $r = x = y = 1$ in Theorem 1, we obtain the result of [9].

3. Weighted Sums for the Fibonacci Polynomials

We derive some identities involving the terms of sequence $\{F_{rn}(x)\}$.

Theorem 2. *For $k, n > 0$,*

$$\sum_{a_1+a_2+\cdots+a_k=n}\prod_{l=1}^{k}(a_l+1)F_{r(a_l+1)}(x)$$

$$= F_r^k\sum_{m=0}^{\min\{\lfloor n/2\rfloor,k-1\}}\sum_{j=0}^{\lfloor(n-2m)/2\rfloor}\frac{n-2m+2k-1}{n-2m-j}\binom{k-1}{m}$$

$$\times\binom{n-2m-j}{j}\binom{n-2m+2k-2-j}{2k-1}(L_r(x))^{n-2m-2j}.$$

where $a_l \geqslant 0$, for $l = 1, 2, \ldots, k$.

Proof. Let $G(t,x) = \sum_{n=0}^{\infty}(n+1)F_{r(n+1)}t^n$. Thus

$$G(t,x) = \frac{F_r\left(1-(-1)^r t^2\right)}{(1-L_r(x)t+(-1)^r t^2)^2}. \quad (3.1)$$

By induction on (3.1), we obtain

$$\frac{\partial^l G(t,x)}{\partial L_r^l} = \frac{F_r\left(1+(-1)^{r+1}t^2\right)(l+1)!t^l}{(1-L_r(x)t+(-1)^r t^2)^{l+2}}. \quad (3.2)$$

Since

$$\left(1+(-1)^{r+1}t^2\right)^{k-1} = \sum_{m=0}^{k-1}\binom{k-1}{m}(-1)^{m(r+1)}t^{2m}$$

and from (3.1-3.2), we get

$$\sum_{n=0}^{\infty}t^n\sum_{a_1+a_2+\cdots+a_k=n}\prod_{l=1}^{k}(a_l+1)F_{a_l+1}(x)$$

$$= \frac{F_r^k\left(1+(-1)^{r+1}t^2\right)^k}{(1-L_r(x)t+(-1)^r t^2)^{2k}}$$

$$= \frac{F_r^{k-1}\left(1+(-1)^{r+1}t^2\right)^{k-1}}{(2k-1)!}\sum_{n=0}^{\infty}(n+2k-1)F_{r(n+2k-1)}^{(2k-2)}(x)t^n$$

$$= \frac{F_r^{k-1}}{(2k-1)!}\sum_{n=0}^{\infty}\sum_{m=0}^{\min\{\lfloor n/2\rfloor,k-1\}}(-1)^{m(r+1)}\binom{k-1}{m}$$

$$\times(n-2m+2k-1)F_{r(n-2m+2k-1)}^{(2k-2)}(x)t^n. \quad (3.3)$$

From (3.3) and (2.4), we derive

$$\sum_{a_1+a_2+\cdots+a_k=n} \prod_{l=1}^{k} (a_l+1)\, F_{r(a_l+1)}(x)$$

$$= \frac{F_r^{k-1}}{(2k-1)!} \sum_{m=0}^{\min\{\lfloor n/2\rfloor, k-1\}} (-1)^{m(r+1)} \binom{k-1}{m} (n-2m+2k-1)$$

$$\times F_{r(n-2m+2k-1)}^{(2k-2)}(x)$$

$$= F_r^k \sum_{m=0}^{\min\{\lfloor n/2\rfloor, k-1\}} (-1)^{m(r+1)} \binom{k-1}{m} \frac{(n-2m+2k-1)}{(2k-1)!} F_{r(n-2m+2k-1)}^{(2k-2)}(x)$$

$$= F_r \sum_{m=0}^{\min\{\lfloor n/2\rfloor, k-1\}} \sum_{j=0}^{\lfloor \frac{n-2m}{2}\rfloor} \frac{n-2m+2k-1}{n-2m-j} \binom{k-1}{m}$$

$$\times \binom{n-2m-j}{j} \binom{n-2m+2k-2-j}{2k-1} (L_r(x))^{n-2m-2j}$$

□

Theorem 3. *For $k, n, r > 0$,*

$$\sum_{a_1+a_2+\cdots+a_k=n} \prod_{l=1}^{k} (a_l+2)\, F_{r(a_l+1)}(x)$$

$$= F_r^k \sum_{j=0}^{\min\{k,n\}} \sum_{i=0}^{\lfloor (n-j)/2\rfloor} \sum_{m=0}^{\lfloor (n-j-2i)/2\rfloor} (-1)^{i+j} \frac{n-j-2(i-k)-1}{n-j-2i-m} 2^{k-j} \binom{k}{j}$$

$$\times \binom{n-j-2i-m}{m} \binom{n-j-2i+2k-2-m}{2k-1} (L_r(x))^{n-2i-m} (-1)^{m(r+1)}.$$

Proof. By taking $y=1$ in (1.2), we write

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} F_{r(n+1)}(x)\, t^{n+2}\right) = \frac{\partial}{\partial t}\left(\frac{F_r t^2}{1-L_r(x)t+(-1)^r t^2}\right).$$

Then

$$\sum_{n=0}^{\infty} (n+2)\, F_{r(n+1)}(x)\, t^n = \frac{F_r(2-L_r(x)t)}{(1-L_r(x)t+(-1)^r t^2)^2}.$$

Since

$$(2-L_r(x)\,t)^k = \sum_{j=0}^{k} \binom{k}{j} (-L_r(x)\,t)^j\, 2^{k-j} \text{ and } \frac{1}{1+t^2} = \sum_{i=0}^{\infty} (-t^2)^i,$$

using the same method given in the previous theorem gives us

$$\sum_{n=0}^{\infty} t^n \sum_{a_1+\cdots+a_k=n} \prod_{l=1}^{k} (a_l+2)\, F_{r(a_l+1)}(x) = \frac{F_r^k(2-L_r(x)t)^k}{(1-L_r(x)t+(-1)^r t^2)^{2k}}$$

$$= \frac{F_r^k(2-L_r(x)t)^k}{(1+t^2)(2k-1)!} \sum_{n=0}^{\infty} (n+2k-1)\, F_{r(n+2k-1)}^{(2k-2)}(x)\, t^n$$

$$= \frac{F_r^k}{(2k-1)!} \sum_{n=0}^{\infty} \sum_{j=0}^{\min\{k,n\}} \sum_{i=0}^{\lfloor (n-j)/2\rfloor} (-1)^{i+j} (n-j-2(i+k)-1)$$

$$\times F_{r(n-j-2i+2k-1)}^{(2k-2)}(x)\, t^n.$$

Considering the proof of previous theorem, we have the conclusion. □

Identities for the bivariate Fibonacci polynomials $F_n(x, y)$ generate identities for specially multiplicative functions at prime powers $f_{p^{n-1}}$ with $x = f_p$ and $y = -f_{p^2}^{-1}$, and vice versa, see [1, 6, 7].

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